2. MATRIX ALGEBRA

Here we consider a matrix $A_{M\times N}$ as an $M\times N$ array of scalars drawn from a field $\mathbb{F}$, i.e., $A \in \mathbb{F}^{M\times N}$. For now, $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.

2.1. Addition & Scalar Multiplication

Usual entrywise addition of matrices and scalar multiplication Details Meyer Ch3

2.2. Matrix Multiplication

Arthur Cayley composition of lin functions

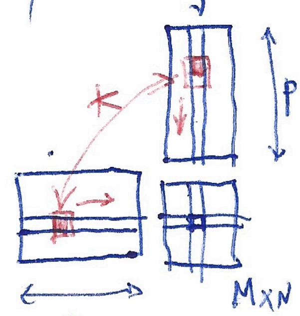
Matrices A and B in ~~…~~ AB must be conformable, i.e., # cols of A = # rows of B $A_{M\times P}\, B_{P\times N}$

- Non-commutative : $AB \neq BA$
- Distributive : $A(B+C) = AB + AC$, $(A+B)C = AC + BC$
- Associative : $(AB)C = A(BC)$

When matrices are conformable, their product can be carried out in different ways

a) INNER PRODUCT

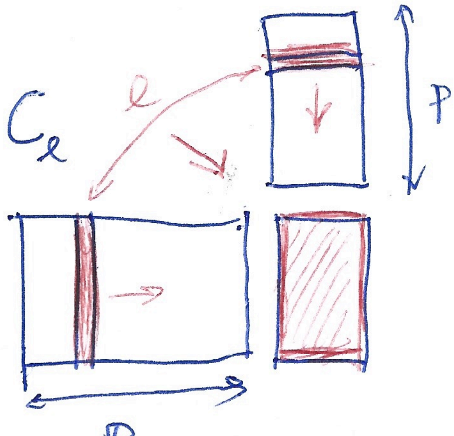
$$[C]_{ij} = [AB]_{ij} = \sum_{k=1}^{P} a_{ik}\, b_{kj} = A_{i*}\, B_{*j}$$



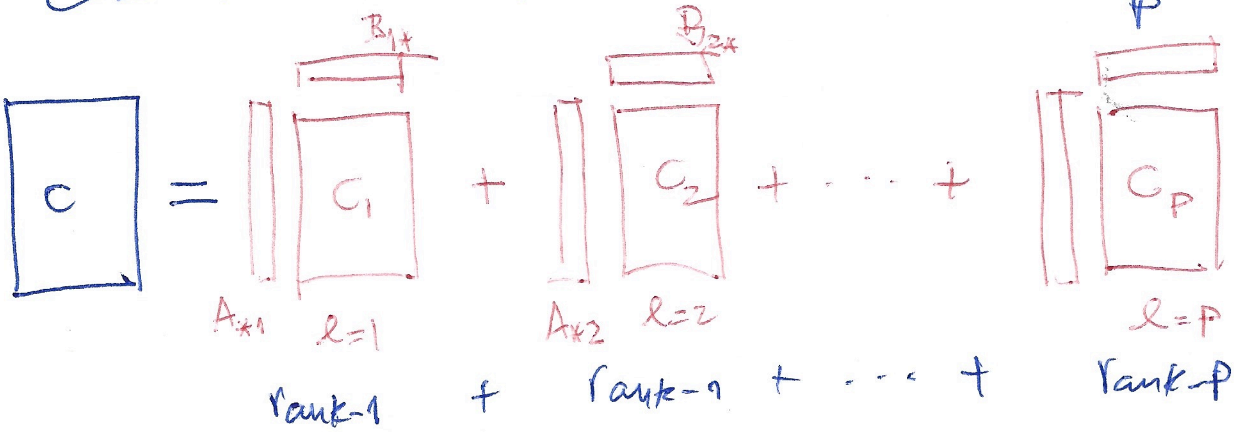
For each pair ij, c_{ij} is directly available right after the sum over k is finished

b) Outer Product

$$C = AB = \sum_{l=1}^{p} A_{*l} B_{l*} = \sum_{l=1}^{p} C_l$$



$$C = C_1 + C_2 + C_3 + \cdots + C_p$$



the entries c_{ij} are available after all the p outer products are carried out and summed

<u>example</u>: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$

$$C = \begin{array}{c|cc} & 2 & -1 \\ \hline 1 & 2 & -1 \\ 0 & 0 & 0 \\ 2 & 4 & -2 \end{array} + \begin{array}{c|cc} & 0 & 2 \\ \hline 2 & 0 & 4 \\ -1 & 0 & -2 \\ 1 & 0 & 2 \end{array} + \begin{array}{c|cc} & 1 & 1 \\ \hline 3 & 3 & 3 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{array}$$

$$C = \begin{bmatrix} 5 & 6 \\ 2 & 0 \\ 4 & 0 \end{bmatrix}$$

Pitfalls in Matrix products

a) $(A+B)^2 \neq A^2 + 2AB + B^2 \quad (AB \neq BA)$

b) If A is singular ($\nexists A^{-1}$)

$$AB = 0 \not\Rightarrow A = 0 \text{ or } B = 0$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

On the other hand, if either A or B is nonsingular, $AB = 0$ implies either A or B is zero

take cols $b_\ell \in N(A)$ or rows $a_k \in N(B^T)$ to form A or B

$\underset{\text{col}}{Ab_\ell} = 0$ for $b_\ell \neq 0$ if $b_\ell \in N(A)$ or $\underset{\text{row}}{a_k B} = 0 \Rightarrow B^T a_k^T = 0$ for $a_k^T \neq 0$ if $a_k^T \in N(B^T)$

c) If A is singular

$$AB = AC \not\Rightarrow B = C$$

If A is NONsingular $AB = AC \Leftrightarrow B = C$

$AB - AC = 0$

$A(B-C) = 0$

$AD = 0$

same case as before

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} = AC \text{ but } B \neq C$$

2.3. Block Matrices & Partitioning

For conformable matrices A and B and a conformable partitioning of both

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1Q} \\ A_{21} & A_{22} & & \\ \vdots & & & \\ A_{P1} & & \cdots & A_{PQ} \end{bmatrix}, \qquad B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1R} \\ B_{21} & B_{22} & & \\ \vdots & & \ddots & \\ B_{Q1} & & & B_{QR} \end{bmatrix}$$

the product $C = AB$ may be performed Blockwise

Perform product as if blocks were scalars

$$C_{ij} = \sum_{k=1}^{Q} A_{ik} B_{kj}, \qquad C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1R} \\ C_{21} & & & \\ \vdots & & & \\ C_{P1} & & \cdots & C_{PR} \end{bmatrix}$$

Conformable matrices A and B means their entrywise product must be defined (otherwise AB $\nexists$)

Conformable partitioning means the ~~..~~ inner blocks must be pairwise conformable, i.e., the blocks must be entrywise compatible

Example:

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & -1 & 2 & 1 \\ \hline 3 & -1 & 2 & 1 \\ 0 & 1 & 2 & 0 \end{array}\right], \qquad B = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \\ \hline 1 & 1 & 2 \end{array}\right]$$

$$C = AB = \left[\begin{array}{cc|c} 15 & 3 & 6 \\ 5 & -1 & 3 \\ \hline 8 & 0 & 7 \\ 8 & 1 & 2 \end{array}\right] = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

Block matrices are very useful in many scenarios. Sometimes they appear naturally (Kronecker product). Deliberate partitioning may ~~be~~ reduce computations, or optimize processing somehow. E.g., reduced memory allocation, ~~allow~~ real time operation (Not always the entire $C = AB$ is required at any given moment)

Entrywise product $\underset{NN}{A}\underset{NN}{B} \sim \Theta(N^3)$

Blockwise product $AB \sim \Theta(N^{2,37})$

(Coppersmith Alg)

$AB \sim \Theta(N^{2,81})$

(Strassen Alg)

Blocks and partitioning extend to special structures as well: Block diagonal, Block triangular, etc.

2.4. Transposition and Symmetries

If $A_{M\times N} = [a_{ij}]$, $\quad A \in \mathbb{C}^{M\times N}$

a) <u>transpose of A</u> $\quad A^T_{N\times M} = [a_{ji}]$

b) <u>Hermitian transpose of A</u> $\quad A^*_{N\times M} = [\bar{a}_{ji}] \quad A^H$

Trivially, $(A^T)^T = A$ and $(A^*)^* = A$

Symmetries

Symmetric: $A^T = A$ $\quad a_{ij} = a_{ji}$

Herm. Sym: $A^* = A$ $\quad a_{ij} = \bar{a}_{ji}$

Skew Sym: $A^T = -A$ $\quad a_{ij} = -a_{ji}$

Skew Herm Sym: $A^* = -A$ $\quad a_{ij} = -\bar{a}_{ji}$

A complex matrix can be symmetric without being Hermitian Sym.

A Hermitian Sym matrix is Not Sym.

transposition is not only for real mats

<u>examples</u>

$$A_1 = \begin{bmatrix} 1 & 1-j \\ 1+j & 2 \end{bmatrix} = A_1^* \text{ (Herm sym)}, \text{ but } A_1^T \neq A_1$$

$$A_2 = \begin{bmatrix} 2+j & j \\ j & 1 \end{bmatrix} = A_2^T \text{ (Sym)}, \text{ but } A_2^* \neq A_2$$

BACK

Note that the diagonal entries of a Herm Sym matrix must be real! ~~a_{ii}~~ $a_{ii} = \bar{a}_{ii}$ iff $a_{ii} \in \mathbb{R}$

Syms have impact in matrix properties, as eigs, and may be explored to optimize processing (e.g., ↓ Complexity) the same procedure shows that $(A^T)^{-1} = (A^{-1})^T$

$\downarrow$

can we switch the order $(\cdot)^* \leftrightarrow (\cdot)^{-1}$?

$$A^*(A^*)^{-1} = I$$

$( )^* = ( )^*$

$$\left[(A^*)^{-1}\right]^* A = I$$

$$\left[(A^*)^{-1}\right]^* A A^{-1} = A^{-1}$$

$$\left[(A^*)^{-1}\right]^* = A^{-1}$$

$( )^* = ( )^*$

$$\boxed{(A^*)^{-1} = (A^{-1})^*}$$

$$(A^{-1})^* \left[(A^{-1})^*\right]^{-1} = I$$

$( )^* = ( )^*$

$$\left[\left((A^{-1})^*\right)^{-1}\right]^* A^{-1} = I$$

$$(\ \cdot\ )A^{-1}A = A$$

$$\left[\left((A^{-1})^*\right)^{-1}\right]^* = A$$

$( )^* = ( )^*$

$$\left((A^{-1})^*\right)^{-1} = A^*$$

$$(A^{-1})^*\left[(A^{-1})^*\right]^{-1} = (A^{-1})^* A^*$$

$$I = (A^{-1})^* A^*$$

$$I(A^*)^{-1} = (A^{-1})^* A^*(A^*)^{-1}$$

$$\boxed{\cancel{A}(A^*)^{-1} = (A^{-1})^*}$$

EASIER: $(A^{-1})^*$

$( )^*$: $\left[(A^{-1})^*\right]^* = A^{-1}$

$$A\left[(A^{-1})^*\right]^* = AA^{-1}$$

$$A\left[(A^{-1})^*\right]^* = I \qquad ( )^* = ( )^*$$

$$\cancel{A}(A^{-1})^* A^* = I$$

$$(A^{-1})^* A^*(A^*)^{-1} = (A^*)^{-1}$$

$$\boxed{(A^{-1})^* = (A^*)^{-1}}$$

2.5. Diagonal, Trace and Vec Operators

Some operators are very common and useful in matrix analysis.

a) The diag(·) operator (square matrices $A_{N\times N}$)

Defined in two "directions"

1) $a_{N\times 1} = \text{diag}(A_{N\times N})$: extracts main diagonal into a vec col $a_{N\times 1}$

2) $A_{N\times N} = \text{Diag}(a_{N\times 1})$: places $a_{N\times 1}$ into the main diagonal of A

b) the trace Tr(·) operator (square A_{NN})

$$\text{Tr}(A) = \sum_k a_{kk} \quad \text{(a scalar)}$$

SUM of diag entries only

Properties

$\text{Tr}(A+B) = \text{Tr}(A) + \text{Tr}(B)$

$\text{Tr}(AB) = \text{Tr}(BA)$

$\text{Tr}(A^T) = \text{Tr}(A)$

$\text{Tr}(A^TA) = 0 \iff A = 0$

$\text{Tr}(A^TB) = \sum_{ij} (A \circ B)_{ij}$, for $A, B \in \mathbb{R}^{N\times N}$

$A \circ B =$ HADAMARD prod

$C = A \circ B =$ entrywise product

$[C]_{ij} = [A]_{ij} [B]_{ij}$

c) The Vec(·) Operator (generic matrix $A_{M\times N}$)

For a matrix $A_{M\times N}$, it stacks the N cols of A into a $MN\times 1$ col ~~(row)~~ ~~vec~~ vec a.

1) $$\underset{MN\times 1}{a} = \text{vec}(A) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}, \quad a_\ell = A_{*\ell}$$

stacks A into a

2) $$\underset{M\times N}{A} = \text{Vec}(a_{MN\times 1}) = [a_1 \; a_2 \cdots a_N]$$

restores A from a

sometimes unvec(A) = = Vec(A)

Properties

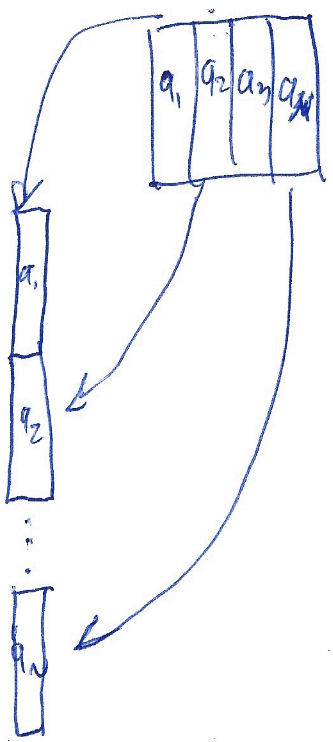
$$\text{vec}(BXA) = \underbrace{(A^T \otimes B)}_{\text{matrix}} \underbrace{\text{vec}(\overset{\text{matrix}}{X})}_{\text{col vec}}$$

$$\text{vec}^T(A^T)\,\text{vec}(B) = \text{Tr}(AB)$$

$$\text{Vec}(A+B) = \text{vec}(A) + \text{Vec}(B)$$

~~Rex~~

$\otimes \triangleq$ kronecker product



2.6. Kronecker Product (tensor product)

For any matrices A_{MN} and B_{PQ}, the Kronecker product is defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1N}B \\ a_{21}B & a_{22}B & & \\ \vdots & & \ddots & \\ a_{M1}B & & & a_{MN}B \end{bmatrix}_{MP \times NQ}$$

A and B do not need to be conformable

The resulting matrix $A \otimes B$ is $MP \times NQ$ and naturally has block structure

Properties

$(A \otimes B)(C \otimes D) = AC \otimes BD$ A, C and B, D must be conformable

$\boxed{\text{If } \exists A^{-1}, B^{-1}}$ ~~then~~ $(A \otimes B)^{-1} = (A^{-1} \otimes B^{-1})$

$(A \otimes B) \otimes C = A \otimes (B \otimes C)$

$(A \otimes B) \neq (B \otimes A)$

$(A+B) \otimes C = A \otimes C + B \otimes C$, $A \otimes (B+C) = A \otimes B + A \otimes C$

$(A \otimes B)^T = A^T \otimes B^T$

Applications 1) Solving Sylvester (Lyapunov) eqs $AX + XB = C$, or more general linear matrix systems of the form $\sum_k A_k X B_k = C$, via $\mathrm{vec}(\cdot)$ operator

2) Reduce complexity; for instance, if the coeffs matrix ~~has~~ in a lin sys of eqs has Kronecker structure, computations can be greatly reduced $(A \otimes B)x = \mathrm{vec}(C)$, A_{NN}, B_{PP} Direct sol: $\mathcal{O}(N^3P^3)$

Considering Kronecker sol: $\mathcal{O}(N^3+P^3)$

In working with Kronecker products, matrices are sometimes unfolded as vectors and vectors are sometimes made into matrices. We now introduce an operator that makes this precise.

Definition 1.8.2 Given a matrix $C = (c_1, c_2, \dots, c_n) \in \mathbb{R}^{m\times n}$ we define

$$\operatorname{vec}(C) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^{mn}, \tag{1.8.7}$$

i.e., $\operatorname{vec}(C)$ is the vector formed by stacking the columns of C into one long vector.

We now state an important result that shows how the vec operator is related to the Kronecker product.

Lemma 1.8.2 *If $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$, and $X \in \mathbb{R}^{q\times n}$, then*

$$(A \otimes B)\operatorname{vec}(X) = \operatorname{vec}(BXA^T). \tag{1.8.8}$$

Proof Denote the kth column of a matrix M by M_k. Then

$$\begin{aligned}(BXA^T)_k &= BX(A^T)_k = B\sum_{i=1}^{n} a_{ki}X_i \\ &= \begin{pmatrix} a_{k1}B & a_{k2}B & a_{kn}B \end{pmatrix} \operatorname{vec}(X),\end{aligned}$$

where $A = (a_{ij})$. But this means that $\operatorname{vec}(BXA^T) = (A \otimes B)\operatorname{vec}(X)$. □

Linear systems for which the matrix is a Kronecker product are ubiquitous in applications. Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{p\times p}$ be nonsingular, and $C \in \mathbb{R}^{p\times n}$. Consider the Kronecker linear system

$$(A \otimes B)x = c, \quad c = \operatorname{vec}(C), \tag{1.8.9}$$

which is of order np. Solving this by LU factorization would require $O(n^3p^3)$ flops. Using (1.8.4) the solution can be written as

$$x = (A \otimes B)^{-1}\operatorname{vec}(C) = (A^{-1} \otimes B^{-1})\operatorname{vec}(C). \tag{1.8.10}$$

Lemma 1.8.2 shows that this is equivalent to $X = B^{-1}CA^{-T}$, where $x = \operatorname{vec}(X)$. Here X can be computed by solving the two matrix equations

$$BY = C, \qquad A^TX = Y. \tag{1.8.11}$$

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A consequence of this result is that linear systems of the form (1.8.9) can be solved fast. The operation count is reduced from $O(n^3p^3)$ to $O(n^3 + p^3)$ flops. Similar savings can be made by using the Kronecker structure in many other problems.

The Kronecker product and its relation to linear matrix equations such as Lyapunov's equation are treated in Horn and Johnson [131, 1991], Chap. 4. See also Henderson and Searle [124, 1981] and Van Loan [198, 2000].

Send Åre 183 page to them

Examples

1) $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$.

$$A \otimes B = \begin{bmatrix} B & 2B & 3B \\ 3B & 2B & B \end{bmatrix} = \left[\begin{array}{cc|cc|cc} 2 & 1 & 4 & 2 & 6 & 3 \\ 2 & 3 & 4 & 6 & 6 & 9 \\ \hline 6 & 3 & 4 & 2 & 2 & 1 \\ 6 & 9 & 4 & 6 & 2 & 3 \end{array}\right].$$

2) Let B be an arbitrary 2×2 matrix and $I_2 = 2\times 2$ identity matrix

$$B \otimes I_2 = \left[\begin{array}{cc|cc} b_{11} & 0 & b_{12} & 0 \\ 0 & b_{11} & 0 & b_{12} \\ \hline b_{21} & 0 & b_{22} & 0 \\ 0 & b_{21} & 0 & b_{22} \end{array}\right]_{4\times 4} \neq I_2 \otimes B = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}_{4\times 4}$$

block diagonal, but <u>Not</u> diagonal

2.7. MATRIX INVERSION

Consider $A, B \in \mathbb{F}^{N \times N}$. The matrix $\underline{B}$ that satisfies

$$BA = I_N \quad \text{and} \quad AB = I_N$$

is called the inverse of A and denoted $B \triangleq A^{-1}$.

An invertible matrix is also called nonsingular.

CALCULATING A^{-1}

Rarely the inverse of a matrix needs to be explictly calculated

a) Lin Sys Approach

Let $e_\ell \triangleq I_{*\ell}$ be the ℓ^{th} canonical vector. Then solve the simultaneous N lin vec systems

$$Ax_\ell = e_\ell \qquad \ell = 1, \dots, N$$

for instance, via GE

Or in matrix form

$$A[x_1 \; x_2 \; \cdots \; x_N] = I$$

$$AX = I$$

(Linear Matrix System)

b) Another possibility is GJ directly

$$[A | I] \longrightarrow [I | A^{-1}] \quad \left\{ \begin{array}{l} A|e_1 \to I|b_1 \\ A|e_2 \to I|b_2 \\ \vdots \\ A|e_N \to I|b_N \end{array} \right.$$

Where $B \triangleq A^{-1}$ and $b_\ell = B_{*\ell}$.

b) Block Matrices : For a nonsingular matrix F_{NN} and blocks of arbitrary size $A_{PP}, B_{PQ}, C_{QP}, D_{QQ}$, partition F as

$$F_{NN} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad N = P+Q$$

so that, for nonsingular A and Δ*,

$$F^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \left[\begin{array}{c|c} A^{-1} - A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ \hline -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{array}\right]$$

the term $D-CA^{-1}B$ is the Schur Complement of A in F

$$F/A \triangleq D - CA^{-1}B,$$

likewise, the Schur complement of D in F is

$$F/D \triangleq A - CD^{-1}B.$$

c) MATRIX INVERSION LEMMA : For arbitrary matrices A, B, C, D of compatible dimensions and nonsingular A and C

$$\boxed{(A+BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1}}.$$

Useful in many scenarios. For instance, to derive the recursive least-squares algorithm (RLS). RLS is intimately related to Kalman Filtering.

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* $\exists\, D^{-1}$ is a sufficient condition

EXTRA CTs (1)

$F = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. We want F^{-1} by exploring submatrices. If we solve the system $FX = I$ via block GE techniques, we find that

$$F^{-1} = \begin{bmatrix} F_{/D}^{-1} & -F_{/D}^{-1} B D^{-1} \\ -D^{-1} C F_{/D}^{-1} & D^{-1} + D^{-1} C F_{/D}^{-1} B D^{-1} \end{bmatrix}, \quad \boxed{F_{/D} \triangleq A - BD^{-1}C}$$

Schur Complement

or

$$F^{-1} = \begin{bmatrix} A^{-1} + A^{-1} B F_{/A}^{-1} C A^{-1} & -A^{-1} B F_{/A}^{-1} \\ -F_{/A} C A^{-1} & F_{/A} \end{bmatrix} \quad \boxed{F_{/A} \triangleq D - CA^{-1}B}$$

Schur Complement

If $\exists F^{-1}$, then $\det F \neq 0$. We can show that

$$\boxed{\det F = \det A \cdot \det F_{/A}} \quad \underline{\text{OR}}$$
$$\boxed{\det F = \det D \cdot \det F_{/D}}$$

then, for nonsingular F ~~we~~ must have either

$$\det A \neq 0 \quad \& \quad \det F_{/A} \neq 0$$

<u>OR</u>

$$\det D \neq 0 \; \& \; \det F_{/D} \neq 0.$$

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 \\ 8 & \boxed{1} \end{bmatrix}$$

$\exists F^{-1}$, $\exists F_{/A}^{-1}$

$\nexists D^{-1}$

$$C = \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

In this case, F is nonsingular, A is nonsingular and D is singular. However, if we turn 1 into <u>0</u> in C, F becomes singular since $F_{/A} = D - CA^{-1}B$ is singular, even though $\exists A^{-1}$.

If we ~~to~~ keep C as it is, but turn $\boxed{1}$ into, say, <u>16</u> in B, then F is again singular, as $F_{/A}$ is singular.

That is, for A nonsingular and D singular, we may find B and/or C that make F singular. Note that $F_{/D}$ <u>does not exist</u> in this case.

If we manage to have A and D nonsingular, we <u>always</u> have a nonsingular F, regardless of B and C. Take, for instance, the worst case scenario:

$$F = \begin{bmatrix} A & O \\ O & D \end{bmatrix}$$

If A and D are nonsingular, F is <u>always</u> inversible. The blocks B and C <u>don't need</u> to be square, as long as A and D are square.

Another test: $F = \begin{bmatrix} 1 & 2 & 2 & 4 \\ -1 & 2 & 8 & 1 \\ -1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \end{bmatrix}$

A B

C D

A B

C D

$\exists A^{-1}, \exists F^{-1}, \exists F/A^{-1}, \nexists D^{-1}$

$\exists A^{-1}, \exists F^{-1}, \exists F/A^{-1}, \nexists D^{-1}$

HOMEWORK LEC 02 (MATRIX ALGEBRA)

(1) Solve the Lin vec sys $Ax=b$ with

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 4 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ -2 \\ 8 \\ -1 \end{bmatrix}$$

a) Via elimination techniques directly;

b) Partition the system and ~~solve~~ solve blockwise

(2) Solve the following Matrix Lin Sys

$$AX + XB = C$$

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 6 & -2 \\ 5 & 1 \end{bmatrix}$$

(3) Find the F_i's blocks in terms of A, B, C, D

$$\begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1}$$

(4) Create a nontrivial Lin Sys of 4×4 elements with Kronecker structure then solve it, counting the flops:

a) Directly (elimination)

b) the fast Method (low complexity)

BACK

(5) show that

$$\mathrm{Tr}(AB) = \mathrm{vec}^T(A^T)\,\mathrm{vec}(B)$$