Lecture 7
Hydrodynamics

Part I

The description we are about to see is called the “Heavy-Ion Standard Model”

Context

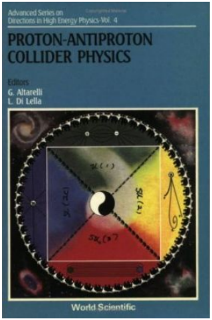
It did not start well:

(Some) particle physicists (ca. 90’s): “performing relativistic heavy ion collisions is like colliding fancy sport cars ou swiss watches” (you’ll get lots of pieces and won’t learn anything)

But I believe the origin is this:

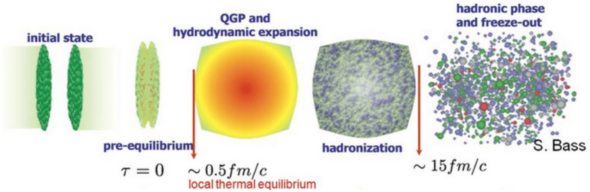
In the late seventies, colliding-beam reactions were classified as 'clean' and 'dirty'. Proton-proton collisions and *a fortiori* also proton–antiproton collisions were 'dirty'. Electron–positrons, instead, were 'clean'. The adjectives were coined after the spectacular nature of the successes of the

The naïve explanation for these problems was ordinarily blamed on the 'complexity' of the hadron collisions. Dick Feynman used to say: 'What will one ever learn colliding Swiss(!) watches against Swiss watches?'



Since then, physicists have had lots of successes colliding protons: Tevatron ($p-\bar{p}$) top quark (1995), SPS ($p-\bar{p}$) W and Z bosons (1983), LHC ($p-p$) Higgs bosons. How about ion collisions?

Heavy-Ion Standard Model



Fluctuating initial conditions + rapid thermalization + hydrodynamic expansion of low viscosity sQGP + transformation in hadron phase + particle emission

U. Heinz "Towards the Little Bang Standard Model" J. Phys.: Conf. Ser. 455 (2013) 012044

J. Schukraft "Results from the first heavy ion run at the LHC" J. Phys.: Conf. Ser. 381 (2012) 012011

Standard thermodynamics: p, T, μ constant over the entire volume
Hydrodynamics: local thermodynamic equilibrium $p(x^\mu)$, $T(x^\mu)$, $\mu(x^\mu)$

Baryon number conservation (w/o diffusion)

Mass conservation in nonrelativistic hydrodynamics

$\frac{\partial\rho}{\partial t} + \vec{\nabla}(\rho\vec{v}) = 0$ [nonrelativistic continuity equation]

Lorentz contraction in the relativistic case: $n \longrightarrow n\gamma = nu^0$

[Conserved quantity: baryon number]

Relativistic continuity equation: $\boxed{\frac{\partial nu^0}{\partial t} + \vec{\nabla}(n\vec{u}) = 0}$

nu^0 baryon density, $n\gamma\vec{v} = n\vec{u}$ baryon flux, u^μ 4-velocity of a fluid element

More compactly: $\boxed{\partial_\mu(nu^\mu) = 0}$

Notation: $x^\mu = (x^0, x^1, x^2, x^3) = (t, x, y, z,) = (t, \vec{x})$ [Contravariant vector]
$\partial_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial t}, \vec{\nabla})$, $\partial^\mu = \frac{\partial}{\partial x_\mu} = \frac{\partial}{\partial t}, -\vec{\nabla})$ [resp. cov. and contrav. derivatives]
$\partial_\mu a^\mu = (\frac{\partial}{\partial t}, \vec{\nabla}) \cdot (a^0, \vec{a}) = \frac{\partial a^0}{\partial t} + \vec{\nabla} \cdot \vec{a}$ [Summation convention]

Energy and momentum conservation (w/o viscosity)

Analogously to the contravariant 4-vector $j^\mu = nu^\mu$, one can define conserved currents for the energy and the three momentum components. These can be written as a contravariant tensor:

$T^{\mu\nu}$ [ν: component of 4-momentum, μ: component of the associated current]

So $T^{\mu\nu} = \begin{pmatrix} \text{energy density} & \text{momentum density} \\ \text{energy flux density} & \text{momentum flux density} \end{pmatrix}$

T^{00}: energy density
T^{0j}: density of j-th component of momentum, j=1,2,3
T^{i0}: energy flux along axis i
T^{ij}: flux along axis i of j-th component of momentum.

$T^{\mu\nu}$ is called energy-momentum tensor [pressure comes from momentum flux]

The energy-momentum tensor in the fluid rest frame (pressure is the same in all directions) is:

$$T_R^{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

In rest frame, $T^{\mu\nu}$ reduces to its static form:
- no energy flux i.e. $T^{i0} = 0$
- no momentum accumulation $T^{0j} = 0$
- In direction x, pressure is

$\frac{\Delta p_x}{\Delta t}\frac{1}{\Delta y \Delta z} = \Delta p_x \frac{\Delta x}{\Delta t}\frac{1}{\Delta x \Delta y \delta z} = \frac{\Delta p_x\, v_x}{V} \longrightarrow \int d^3pN(E)p_x\, v_x = T^{xx} = \int d^3pN(E)\frac{p\, v}{3} = \int d^3pN(E)\frac{p^2}{3E}$ so $T^{ii} = p$ but $T^{ij} = 0$ if $i \neq j$

Using the general form of a Lorentz transformation, one gets:

e.g. Denicol & Rischke " Microscipic Foundations of Relativistic Fluid Dynamics" Springer 2021

$$\boxed{T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu - g^{\mu\nu} p}$$

Exercise:

Start from the rest frame expressions of the baryon current and energy-momentum tensor and obtain their expressions in a moving frame.

[This is an alternative derivation to that of Denicol & Rischke or Ollitrault, close to Vogt's]

- In the moving frame, the fluid velocity becomes:

$u^\mu = \Lambda^\mu_\nu u^\nu_R$ with $u^\nu_R = (1,0,0,0)$ so $u^\mu = \Lambda^\mu_0 u^0_R = \Lambda^\mu_0$: the velocity in the boosted frame defined the $\nu = 0$ component of Λ^μ_ν.

The baryon current in the rest frame is $n^\mu_R = (n,0,0,0)$ and in the moving frame:

$n^\mu = \Lambda^\mu_\nu n^\nu_R = \Lambda^\mu_0 n = n u^\mu$ as expected.

- To obtain the energy-momentum tensor in the moving frame, note:

$$g^{\rho\sigma} = g^{\mu\nu}\Lambda^\rho_\mu \Lambda^\sigma_\nu = g^{00}\Lambda^\rho_0\Lambda^\sigma_0 + g^{ii}\Lambda^\rho_i\Lambda^\sigma_i = u^\rho u^\sigma - \Lambda^\rho_i\Lambda^\sigma_i$$

$g^{\mu\nu} = g_{\mu\nu} = diag(1,-1,-1,-1)$: metric tensor, all its matrix elements are zero except on diagonal . It is used to lower or rise indices: $a_\mu = g_{\mu\nu}a^\nu = (g_{0\nu}a^\nu, g_{1\nu}a^\nu, \ldots) = (a^0, -a^x, -a^y, -a^z)$

Therefore $T^{\rho\sigma} = \Lambda^\rho_\mu\Lambda^\sigma_\nu T^{\mu\nu}_R = \Lambda^\rho_0\Lambda^\sigma_0\epsilon + \Lambda^\rho_i\Lambda^\sigma_i p =$

$u^\rho u^\sigma\epsilon + (u^\rho u^\sigma - g^{\rho\sigma})p = (\epsilon + p)u^\rho u^\sigma - g^{\rho\sigma}p$ as expected

Equations of non-viscous hydrodynamics

The conservation equations are then:

$$\boxed{\partial_\mu T^{\mu\nu} = 0, \quad \partial_\mu (n\, u^\mu) = 0}$$

= 5 equations with 6 unknowns: $u^x, u^y, u^z, \epsilon, p, n_b$,
so we need an equation of state such as $p(\epsilon, n_b)$ to close this system.

Note that the conservation equations are differential equations, so we have to choose initial conditions to solve them.

Exercise:

What is the relationship with the usual fluid mechanics equations?

• One equation can be recast as an entropy conservation equation.

$u_\nu \partial_\mu T^{\mu\nu} = 0 = u^\mu \partial_\mu \epsilon + (\epsilon + p)\partial_\mu u^\mu = u^\mu T \partial_\mu s + Tsu^\mu \partial_\mu \Rightarrow$
$\partial_\mu (su^\mu) = 0$

where we used the thermodynamic relations $\epsilon + p = Ts$ and $dp = sdT$. In the non-relativistic limit $\gamma \longrightarrow 1$ and $\frac{\partial s}{\partial t} + \vec{\nabla}(s\vec{v}) = 0$

• The three equations $\partial_\mu T^{\mu i} = 0$ for $i = 1, 2, 3$ can be recast in a Euler equation form.

Note $\partial_\mu T^{\mu 0} = 0 \longrightarrow \partial_\mu [(\epsilon + p)u^\mu u^0] = \partial^0 p$

$\partial_\mu T^{\mu i} = 0 = v^i \partial_\mu [(\epsilon + p)u^\mu u^0] + (\epsilon + p)u^\mu u^0 \partial_\mu v^i - \partial^i p =$
$v^i \partial^0 p + (\epsilon + p)u^\mu u^0 \partial_\mu v^i - \partial^i p$

$\Rightarrow \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{(1-v^2)}{(\epsilon+p)} \left[\vec{\nabla} p + \vec{v} \frac{\partial p}{\partial t} \right]$

In the non-relativistic limit v small, $p \ll \epsilon$, $\epsilon \sim \rho$, this equation reduces to the classical Euler equation.

A simple example of solution: the Bjorken model

- We ignore transverse expansion (as expected initially), so:

$$\begin{cases} \partial_t T^{00} + \partial_z T^{0z} &= 0 \\ \partial_t T^{0z} + \partial_z T^{zz} &= 0. \end{cases}$$

We introduce $u^\mu = (\cosh y, \sinh y)$ in the equations above:

$$\begin{cases} \cosh y \partial_t \epsilon + \sinh y \partial_z \epsilon + (\epsilon + p)(\sinh y \partial_t y + \cosh y \partial_z y) &= 0 \\ \sinh y \partial_t p + \cosh y \partial_z p + (\epsilon + p)(\cosh y \partial_t y + \sinh y \partial_z y) &= 0 \end{cases}$$

We change variables: $(t, z) \longrightarrow (\tau, \eta_s)$
where $\tau \equiv \sqrt{t^2 - z^2}$ and $\eta_s = atanh(z/t)$ (spacetime rapidity)
and use of the relations

$$\begin{cases} \dfrac{\partial}{\partial t} = \dfrac{\partial \tau}{\partial t}\dfrac{\partial}{\partial \tau} + \dfrac{\partial \eta_s}{\partial t}\dfrac{\partial}{\partial \eta_s} = \cosh \eta_s \dfrac{\partial}{\partial \tau} - \dfrac{\sinh \eta_s}{\tau}\dfrac{\partial}{\partial \eta_s} \\ \dfrac{\partial}{\partial z} = \dfrac{\partial \tau}{\partial z}\dfrac{\partial}{\partial \tau} + \dfrac{\partial_s}{\partial z}\dfrac{\partial}{\partial \eta_s} = -\sinh \eta_s \dfrac{\partial}{\partial \tau} + \dfrac{\cosh \eta_s}{\tau}\dfrac{\partial}{\partial \eta_s} \end{cases}$$

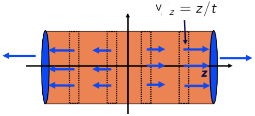
to obtain

$$\begin{cases} \tau\, \partial_\tau \epsilon + \tanh(y - \eta_s)\, \partial_\eta \epsilon + (\epsilon + p)\left[\tau \tanh(y - \eta_s)\, \partial_\tau y + \partial_{\eta_s} y\right] = 0 \\ \tau \tanh(y - \eta_s)\, \partial_\tau p + \partial_{\eta_s} p + (\epsilon + p)\left[\tau\, \partial_\tau y + \tanh(y - \eta_s)\, \partial_{\eta_s} y\right] = 0. \end{cases}$$

• So far the discussion has been quite general.

We know make the Bjorken hypothesis:

Fluid velocity is $v_z = z/t \Leftrightarrow y = \eta_s$ (fluid rapidity equals spacetime rapidity), initially at proper time τ_0 and at all proper times



[This comes from the expectation that dN/dy has a plateau around mid-rapidity, i.e. boost-invariance. The only flow boost invariant (=0 at z=0) is $v_z = z/t$ cf. Florkowski §2.7 and ch.21]

Using $y = \eta_s$, the hydro equations become:

$$\begin{cases} \tau \left.\dfrac{\partial \epsilon}{\partial \tau}\right|_{\eta_s} + (\epsilon + p) = 0 \\ \left.\dfrac{\partial p}{\partial \eta_s}\right|_{\tau} = 0. \end{cases}$$

So the thermodynamic quantities do not depend on η_s and $\epsilon(\tau) = \epsilon(\tau_0)(\tau_0/\tau)^{4/3}$ (for $p = \epsilon/3$).

Exercise:

For a gas of gluons and two flavors of massless quarks, compute $T(\tau)$ in the Bjorken model.

In this case: $p = \epsilon/3$ and $\epsilon = g_{qgp}\frac{\pi^2}{30}T^4$
So using the Bjorken solution:

$$T(\tau)^4 = T(\tau_0)^4(\tau_0/\tau)^{4/3} \Leftrightarrow \boxed{T(\tau) = T(\tau_0)(\tau_0/\tau)^{1/3}}$$

Challenge

Redo the calculation on slide 9 (non-relativistic limit) for the case $n_b \neq 0$.

Homework

Show that if $p = c_s^2 \epsilon$ with c_s constant, the solution of the Bjorken model is $\epsilon(\tau) = \epsilon(\tau_0)(\tau_0/\tau)^{1+c_s^2}$

Other references on this topic

- J.-Y.ollitrault Relativistic hydrodynamics for heavy-ion collisions Eur.J.Phys.29 (2008) 275, arXiv:0708.2433
- R. Vogt, Ultrarelativistic Heavy-ion Collisions, Elsevier, 2007
- W. Florkowski, Phenomenology of Ultra-Relativistic Heavy-Ion Collisions, World Scientific, 2010