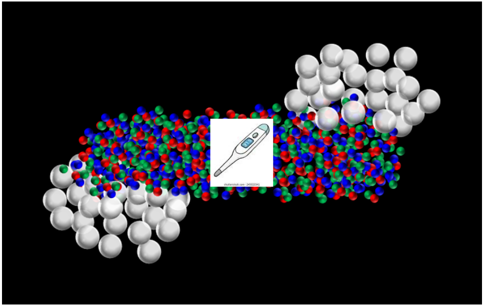
Lecture 5
Thermodynamics: equation of state (part I)



Assumptions

Particles are:

- Not interacting (ideal gas)
- Relativistic: $E = \sqrt{p^2 + m^2}$

Some results from quantum statistical physics and thermodynamics

Probability density for occupation of state:

$$N(E) = \frac{g}{(2\pi)^3} \frac{1}{e^{(E-\mu)/T} \pm 1}$$

$+$ for fermions (no two particles of the same type can be in the same state, half-integral spin)

$-$ for bosons (arbitrarily many bosons can be in the same state, integral spin)

μ chemical potential, g degeneracy

[Reference: Griffiths "Introduction to quantum physics"]

Number density: $n = \int N(E) d^3p \quad E = \sqrt{p^2 + m^2}$

Energy density: $\epsilon = \int N(E) E d^3p$

Pressure: $p = \int N(E) \frac{p^2}{3E} d^3p$

[Reference: Ollitrault https://arxiv.org/pdf/0708.2433.pdf]

Entropy density: $s = (\epsilon + p - \mu\, n)/T$

[First law]

What comes next:

- Massless bosons: $m = 0$ and $\mu = 0$
- Massless fermions: 1) $m = 0$ and $\mu = 0$, 2) $m = 0$ and $\mu \neq 0$
- Massive mesons with $\mu = 0$
- Massive baryons with $\mu \neq 0$
- Boltzmann (= non quantum) limit

"Now, in the *second* law of thermodynamics . . ."

Massless bosons

Suppose $m = 0$ and $\mu = 0$
Use $\int dz \frac{z^{x-1}}{e^z - 1} = \zeta(x)\Gamma(x)$
where $\zeta(x) = \sum 1/n^x$ Riemann zeta function, $\zeta(2) = \pi^2/6 \sim 1.645$, $\zeta(3) \sim 1.202$, $\zeta(4) = \pi^4/90 \sim 1.082$
$\Gamma(x) = \int_0^\infty dt\, e^{-t}\, t^{x-1}$ Euler gamma function, $\Gamma(n) = (n-1)!$ for positive integer.

$$n_B = \frac{g}{2\pi^2} T^3 \int_0^\infty dz \frac{z^2}{e^z - 1} = \frac{g}{2\pi^2} T^3 \Gamma(3)\zeta(3) = \frac{g}{\pi^2} 1.202\, T^3$$

$$\epsilon_B = \frac{g}{2\pi^2} T^4 \int_0^\infty dz \frac{z^3}{e^z - 1} = \frac{g}{2\pi^2} T^4 \Gamma(4)\zeta(4) = \frac{g\pi^2}{30} T^4$$

$$p_B = \frac{1}{3} \frac{g}{2\pi^2} T^4 \int_0^\infty dz \frac{z^3}{e^z - 1} = \epsilon_B/3$$

$$s_B = (4/3)\epsilon_B/T$$

This can be applied to gluons with $g_g = 16$ (to account for 8 color states and 2 spin polarization states) and massless pions with $g_\pi = 3$ (to account for π^+, π^0, π^-)

Massless pions at $\mu = 0$ is not a very good approximation since $m_\pi = 140$ MeV $\sim T_{deconf}$

Massless fermions

- Suppose $m = 0$ and $\mu = 0$
  Use $\int dz \frac{z^{x-1}}{e^z+1} = (1-2^{1-x})\zeta(x)\Gamma(x)$

$$
\begin{aligned}
n_F = n_{\bar{F}} &= \frac{g}{2\pi^2} T^3 \int_0^\infty dz \frac{z^2}{e^z+1} = \frac{g}{\pi^2} 0.9 T^3 \\
\epsilon_F = \epsilon_{\bar{F}} &= \frac{g}{2\pi^2} T^4 \int_0^\infty dz \frac{z^3}{e^z+1} = \frac{g 7\pi^2}{240} T^4 \\
p_F = p_{\bar{F}} &= \frac{1}{3} \frac{g}{2\pi^2} T^4 \int_0^\infty dz \frac{z^3}{e^z-1} = \epsilon_F/3 \\
s_F = s_{\bar{F}} &= (4/3)\epsilon_F/T
\end{aligned}
$$

This can be applied to quarks with $g = 12 N_f$ with N_f number of massless quarks flavors (to account for quarks and antiquaks, 3 color states and 2 spin states)

Summary for $m = 0$ and $\mu = 0$:
$n_F = (3/4) n_B \propto T^3, \epsilon_F, p_F = (7/8)\epsilon_B, p_B \propto T^4, s_B = (7/8) s_B \propto T^3$,
all $\propto g$

Exercise:

Compute the energy density, pressure and entropy for the quark-gluon plasma when $\mu = 0$ and the temperature $T >> m_u, m_d, m_s$

For very high temperatures, the QGP can be treated as ideal (no interactions), so we use the previous results. (Close to the crossover, this cannot be done.)

Define
$g_{QGP} = g_g + (7/8)g_q = 2 \times 8(\textit{gluon colors} \times \textit{polarization states}) + (7/8)2 \times 3 \times 2 \times N_f(\textit{quarks and antiquarks} \times \textit{color} \times \textit{spin} \times \textit{flavor})$
=37 for $N_f = 2$ and 47.5 for $N_f = 3$.

$$\epsilon_{QGP} = g_{QGP}\frac{\pi^2}{30}T^4$$
$$p_{QGP} = g_{QGP}\frac{\pi^2}{90}T^4$$
$$s_{QGP} = g_{QGP}\frac{4\pi^2}{90}T^3$$

Massless fermions/cont'd

- Suppose $m = 0$ and $\mu \neq 0$
  There is an additional trick (combined integrals give analytical formula)

$$
\begin{aligned}
\epsilon_F + \epsilon_{\bar{F}} &= \frac{g}{2\pi^2}\left(\int_0^\infty dp \frac{p^3}{e^{(p-\mu)/T}+1} + \int_0^\infty dp \frac{p^3}{e^{(p+\mu)/T}+1}\right) \\
&= \frac{g}{2\pi^2} T^4 \left(\int_0^\infty dx \frac{x^3}{e^{x-y}+1} + \int_0^\infty dx \frac{x^3}{e^{x+y}+1}\right) \\
&= \frac{g}{2\pi^2} T^4 \left(\int_0^\infty dX \frac{(X+y)^3}{e^X+1} + \int_0^\infty dX \frac{(X-y)^3}{e^X+1} + \int_0^y dX\,(y-X)^3\right) \\
&= g\left(\frac{7\pi^2}{120} T^4 + \frac{\mu^2}{4} T^2 + \frac{\mu^4}{8\pi^2}\right)
\end{aligned}
$$

$$
p_F + p_{\bar{F}} = (\epsilon_F + \epsilon_{\bar{F}})/3
$$

$$
n_F - n_{\bar{F}} = g\left(\frac{\mu}{6} T^2 + \frac{\mu^3}{6\pi^2}\right)
$$

$$
s_F + s_{\bar{F}} = \left[\epsilon_F + \epsilon_{\bar{F}} + p_F + p_{\bar{F}} - \mu(n_F - n_{\bar{F}})\right]/T
$$

Exercise:

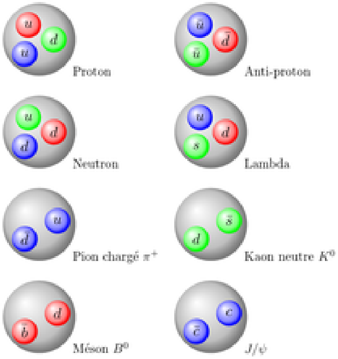
Compute the energy density, pressure, entropy, baryon density for the quark-gluon plasma with massless u and d quarks at $T = 0$

$$\epsilon_{QGP} = \frac{3}{2\pi^2}\mu_q^4$$
$$p_{QGP} = \frac{1}{2\pi^2}\mu_q^4$$
$$n_{QGP} = \frac{2}{\pi^2}\mu_q^3$$
$$s_{QGP} = 0 \ (from\ dp = sdT + nd\mu)$$

Watch out to compute g without double counting

Hadrons

Particles made of quarks are hadrons.
$q\bar{q}$ is a meson and behaves as a boson
qqq or $\bar{q}\bar{q}\bar{q}$ is a baryon and behaves as a fermion
(Exotic hadrons made of more than three quarks have also been discovered)



Massive mesons (bosons)

Use integer-order modified Bessel functions of the second kind:
$K_n(x) = \frac{2^n n!}{(2n)!} x^{-n} \int_x^\infty dt (t^2 - x^2)^{n-1/2} e^{-t}$
as well as $\frac{1}{e^y - 1} = \sum_{l=0}^\infty e^{-(l+1)y}$.
Introduce $y = \sqrt{p^2 + m^2}/T$ and $t = (n+1)y$

$$
\begin{aligned}
p_{mes} &= \frac{g_{mes}}{6\pi^2} \int dp \frac{p}{E} \frac{p^2}{e^{E/T} - 1} \\
&= \frac{g_{mes}}{6\pi^2} T^4 \sum_{n=0}^\infty \int_{m/T}^\infty dy \left(y^2 - \frac{m^2}{T^2} \right)^{3/2} e^{-(n+1)y} \\
&= \frac{g_{mes} m^2 T^2}{2\pi^2} \sum_{n=1}^\infty \frac{1}{n^2} K_2\left(\frac{nm}{T}\right)
\end{aligned}
$$

$$
\epsilon_{mes} = \ldots = 3p_{mes} + \frac{g_{mes} m^3 T}{2\pi^2} \sum_{n=1}^\infty \frac{1}{n} K_1\left(\frac{nm}{T}\right)
$$

$$
s_{mes} = \frac{3p_{mes}}{T} + \frac{g_{mes} m^3}{2\pi^2} \sum_{n=1}^\infty \frac{1}{n} K_1\left(\frac{nm}{T}\right)
$$

$$
n_{mes} = \frac{g_{mes}}{2\pi^2} m^2 T \sum_{n=1}^\infty \frac{1}{n} K_2\left(\frac{nm}{T}\right)
$$

Massive baryons (fermions)

Use $\frac{1}{e^{(E\mp\mu)/T}+1} = \sum_{l=0}^{\infty}(-1)^l e^{-(l+1)E/T} e^{\pm(l+1)\mu/T}$.

$$p_{bar} + p_{\bar{bar}} = \frac{g_{bar} m^2 T^2}{2\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} K_2(\frac{nm}{T}) \left[e^{n\mu/T} + e^{-n\mu/T} \right]$$

$$\epsilon_{bar} + \epsilon_{\bar{bar}} = 3(p_{bar} + p_{\bar{bar}}) + \frac{g_{bar} m^3 T}{2\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} K_1(\frac{nm}{T}) \left[e^{n\mu/T} + e^{-n\mu/T} \right]$$

$$n_{bar} - n_{\bar{bar}} = \frac{g_{bar} m^2 T}{2\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} K_2(\frac{nm}{T}) \left[e^{n\mu/T} - e^{-n\mu/T} \right]$$

$$s_{bar} + s_{\bar{bar}} = \frac{3(p_{bar} + p_{\bar{bar}})}{T} + \frac{g_{bar} m^3}{2\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} K_1(\frac{nm}{T}) \left[e^{n\mu/T} + e^{-n\mu/T} \right]$$

Boltzmann Limit

For $x \longrightarrow \infty, K_n(x) \sim \sqrt{\pi/2}\, e^{-x}/\sqrt{x}$. It is often the case that $m - \mu >> T$, so:

$$p_{mes} \longrightarrow \frac{g_{mes} m^2 T^2}{2\pi^2} K_2(\frac{m}{T})$$

$$p_{bar} + p_{\bar{bar}} \longrightarrow \frac{g_{bar} m^2 T^2}{2\pi^2} K_2(\frac{m}{T}) \left[e^{\mu/T} + e^{-\mu/T}\right]$$

$$\epsilon_{mes} \longrightarrow 3p_{mes} + \frac{g_{mes} m^3 T}{2\pi^2} K_1(\frac{m}{T})$$

$$\epsilon_{bar} + \epsilon_{\bar{bar}} \longrightarrow 3(p_{bar} + p_{\bar{bar}}) + \frac{g_{bar} m^3 T}{2\pi^2} K_1(\frac{m}{T}) \left[e^{\mu/T} + e^{-\mu/T}\right]$$

$$n_{mes} \longrightarrow \frac{g_{mes}}{2\pi^2} m^2 T K_2(\frac{m}{T})$$

$$n_{bar} - n_{\bar{bar}} \longrightarrow \frac{g_{bar} m^2 T}{2\pi^2} K_2(\frac{m}{T}) \left[e^{\mu/T} - e^{-\mu/T}\right]$$

$$s_{mes} \longrightarrow \frac{3p_{mes}}{T} + \frac{g_{mes} m^3}{2\pi^2} K_1(\frac{m}{T})$$

$$s_{bar} + s_{\bar{bar}} \longrightarrow \frac{3(p_{bar} + p_{\bar{bar}})}{T} + \frac{g_{bar} m^3}{2\pi^2} K_1(\frac{m}{T}) \left[e^{\mu/T} + e^{-\mu/T}\right]$$

Exercise:

Suppose a hadronic gas consists of pions and nucleons at $\mu = 0$ and $T = 130\,\text{MeV}$. Write its pressure, energy density, entropy, baryon density.

We must account for $\pi^{+,0,-}$, p, $\bar{p}$, n, $\bar{n}$

Only the pion is light and must be treated as a boson, the others can be treated in the Boltzmann limit. $g_\pi = 3$ and $g_N = 2 \times 4$

$$
\begin{aligned}
p &= \frac{3m_\pi^2 T^2}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} K_2(\frac{nm_\pi}{T}) + \frac{8m^2 T^2}{2\pi^2} K_2(\frac{m_N}{T}) \\
\epsilon &= 3p \\
n_{bar} - n_{\bar{bar}} &= 0 \\
s &= \frac{3}{T} \times \frac{3m_\pi^2 T^2}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} K_2(\frac{nm_\pi}{T}) + \frac{3m^3}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n} K_1(\frac{nm}{T}) \\
&+ \frac{3}{T} \times \frac{8m^2 T^2}{2\pi^2} K_2(\frac{m_N}{T}) + \frac{8m^3}{2\pi^2} K_1(\frac{m}{T})
\end{aligned}
$$

Challenge

Write the expressions for the energy density, pressure, baryon density, entropy density, for a gas of quarks (two massless flavors, one massive) and gluons at $T \neq 0$ and $\mu \neq 0$. Assume an equal number of quarks s and $\bar{s}$ (so you do need to introduce another chemical potential).

Homework

a) Compute the energy density and pressure in $GeV\ fm^{-3}$, the baryon density in fm^{-3}, entropy density in GeV^3, for a gas of quarks (two massless flavors) and gluons at $T = 200\,\text{MeV}$ and $\mu = 0$.

b) Write the expressions for the energy density, pressure, baryon density, entropy density, for a gas of quarks (three massless flavors) and gluons at $T \neq 0$ and $\mu = 0$.

Other references on this topic

- R. Vogt, Ultrarelativistic Heavy-ion Collisions, Elsevier, 2007
- W. Florkowski, Phenomenology of Ultra-Relativistic Heavy-Ion Collisions, World Scientific, 2010
- C.Y. Wong, Introduction to High-Energy Heavy-Ion Collisions, World Scientific, 1994