CHAPTER 19

The Theory of Multiple Regression

This chapter provides an introduction to the theory of multiple regression analysis. The chapter has four objectives. The first is to present the multiple regression model in matrix form, which leads to compact formulas for the ordinary least squares (OLS) estimator and test statistics. The second objective is to characterize the sampling distribution of the OLS estimator, both in large samples (using asymptotic theory) and in small samples (if the errors are homoskedastic and normally distributed). The third objective is to study the theory of efficient estimation of the coefficients of the multiple regression model and to describe generalized least squares (GLS), a method for estimating the regression coefficients efficiently when the errors are heteroskedastic and/or correlated across observations. The fourth objective is to provide a concise treatment of the asymptotic distribution theory of instrumental variables (IV) regression in the linear model, including an introduction to generalized method of moments (GMM) estimation in the linear IV regression model with heteroskedastic errors.

The chapter begins by laying out the multiple regression model and the OLS estimator in matrix form in Section 19.1. This section also presents the extended least squares assumptions for the multiple regression model. The first four of these assumptions are the same as the least squares assumptions of Key Concept 6.4 and underlie the asymptotic distributions used to justify the procedures described in Chapters 6 and 7. The remaining two extended least squares assumptions are stronger and permit us to explore in more detail the theoretical properties of the OLS estimator in the multiple regression model.

The next three sections examine the sampling distribution of the OLS estimator and test statistics. Section 19.2 presents the asymptotic distributions of the OLS estimator and t-statistic under the least squares assumptions of Key Concept 6.4. Section 19.3 unifies and generalizes the tests of hypotheses involving multiple coefficients presented in Sections 7.2 and 7.3 and provides the asymptotic distribution of the resulting F-statistic. In Section 19.4, we examine the exact sampling distributions of the OLS estimator and test statistics in the special case that the errors are homoskedastic and normally distributed. Although the assumption of homoskedastic normal errors is implausible in most econometric applications, the exact sampling distributions are of theoretical interest, and p-values computed using these distributions often appear in the output of regression software.

The next two sections turn to the theory of efficient estimation of the coefficients of the multiple regression model. Section 19.5 generalizes the Gauss–Markov theorem to multiple regression. Section 19.6 develops the method of generalized least squares (GLS).

The final section takes up IV estimation in the general IV regression model when the instruments are valid and strong. This section derives the asymptotic distribution of the two stage least squares (TSLS) estimator when the errors are heteroskedastic and provides expressions for the standard error of the TSLS estimator. The TSLS estimator is one of many possible GMM estimators, and this section provides an introduction to GMM estimation in the linear IV regression model. It is shown that the TSLS estimator is the efficient GMM estimator if the errors are homoskedastic.

Mathematical prerequisite. The treatment of the linear model in this chapter uses matrix notation and the basic tools of linear algebra and assumes that the reader has taken an introductory course in linear algebra. Appendix 19.1 reviews vectors, matrices, and the matrix operations used in this chapter. In addition, multivariate calculus is used in Section 19.1 to derive the OLS estimator.

19.1 The Linear Multiple Regression Model and OLS Estimator in Matrix Form

The linear multiple regression model and the OLS estimator can each be represented compactly using matrix notation.

The Multiple Regression Model in Matrix Notation

The population multiple regression model (Key Concept 6.2) is

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \cdots + \beta_k X_{ki} + u_i, i = 1, \ldots, n. \tag{19.1}$$

To write the multiple regression model in matrix form, define the following vectors and matrices:

$$\boldsymbol{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}, \boldsymbol{U} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \boldsymbol{X} = \begin{pmatrix} 1 & X_{11} & \cdots & X_{k1} \\ 1 & X_{12} & \cdots & X_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_{1n} & \cdots & X_{kn} \end{pmatrix} = \begin{pmatrix} \boldsymbol{X}_1' \\ \boldsymbol{X}_2' \\ \vdots \\ \boldsymbol{X}_n' \end{pmatrix}, \text{ and } \boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{pmatrix}, \tag{19.2}$$

so $\boldsymbol{Y}$ is $n \times 1$, $\boldsymbol{X}$ is $n \times (k + 1)$, $\boldsymbol{U}$ is $n \times 1$, and $\boldsymbol{\beta}$ is $(k + 1) \times 1$. Throughout we denote matrices and vectors by bold type. In this notation,

- $\boldsymbol{Y}$ is the $n \times 1$ dimensional vector of n observations on the dependent variable.
- $\boldsymbol{X}$ is the $n \times (k + 1)$ dimensional matrix of n observations on the $k + 1$ regressors (including the "constant" regressor for the intercept).
- The $(k + 1) \times 1$ dimensional column vector $\boldsymbol{X}_i$ is the i^{th} observation on the $k + 1$ regressors; that is, $\boldsymbol{X}_i' = (1 \; X_{1i} \ldots X_{ki})$, where $\boldsymbol{X}_i'$ denotes the transpose of $\boldsymbol{X}_i$.

The Extended Least Squares Assumptions in the Multiple Regression Model

KEY CONCEPT 19.1

$$Y_i = \boldsymbol{X}_i'\boldsymbol{\beta} + u_i, i = 1, \ldots, n., \tag{19.3}$$

where $\boldsymbol{\beta}$ is the vector of causal effects and

1. $E(u_i \mid \boldsymbol{X}_i) = 0$ (u_i has conditional mean 0);
2. $(\boldsymbol{X}_i, Y_i)$, $i = 1, \ldots, n$, are independently and identically distributed (i.i.d.) draws from their joint distribution;
3. $\boldsymbol{X}_i$ and u_i have nonzero finite fourth moments;
4. $\boldsymbol{X}$ has full column rank (there is no perfect multicollinearity);
5. $\text{var}(u_i \mid \boldsymbol{X}_i) = \sigma_u^2$ (homoskedasticity); and
6. The conditional distribution of u_i given $\boldsymbol{X}_i$ is normal (normal errors).

- $\boldsymbol{U}$ is the $n \times 1$ dimensional vector of the n error terms.
- $\boldsymbol{\beta}$ is the $(k + 1) \times 1$ dimensional vector of the $k + 1$ unknown regression coefficients.

The multiple regression model in Equation (19.1) for the i^{th} observation, written using the vectors $\boldsymbol{\beta}$ and $\boldsymbol{X}_i$, is

$$Y_i = \boldsymbol{X}_i'\boldsymbol{\beta} + u_i, i = 1, \ldots, n. \tag{19.4}$$

In Equation (19.4), the first regressor is the "constant" regressor that always equals 1, and its coefficient is the intercept. Thus the intercept does not appear separately in Equation (19.4); rather, it is the first element of the coefficient vector $\boldsymbol{\beta}$.

Stacking all n observations in Equation (19.4) yields the multiple regression model in matrix form:

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{U}. \tag{19.5}$$

The Extended Least Squares Assumptions

The extended least squares assumptions for the multiple regression model are the four least squares assumptions for causal inference in the multiple regression model in Key Concept 6.4 plus the two additional assumptions of homoskedasticity and normally distributed errors. The assumption of homoskedasticity is used when we study the efficiency of the OLS estimator, and the assumption of normality is used when we study the exact sampling distribution of the OLS estimator and test statistics.

The extended least squares assumptions are summarized in Key Concept 19.1.

Except for notational differences, the first three assumptions in Key Concept 19.1 are identical to the first three assumptions in Key Concept 6.4.

The fourth assumptions in Key Concepts 6.4 and 19.1 might appear different, but, in fact, they are the same: They are simply different ways of saying that there cannot be perfect multicollinearity. Recall that perfect multicollinearity arises when one regressor can be written as a perfect linear combination of the others. In the matrix notation of Equation (19.2), perfect multicollinearity means that one column of $\boldsymbol{X}$ is a perfect linear combination of the other columns of $\boldsymbol{X}$, but if this is true, then $\boldsymbol{X}$ does not have full column rank. Thus saying that $\boldsymbol{X}$ has rank $k + 1$—that is, rank equal to the number of columns of $\boldsymbol{X}$—is just another way to say that the regressors are not perfectly multicollinear.

The fifth least squares assumption in Key Concept 19.1 is that the error term is conditionally homoskedastic, and the sixth assumption is that the conditional distribution of u_i given $\boldsymbol{X}_i$ is normal. These two assumptions are the same as the final two assumptions in Key Concept 18.1 except that they are now stated for multiple regressors.

***Implications for the mean vector and covariance matrix of* U.** The least squares assumptions in Key Concept 19.1 imply simple expressions for the mean vector and covariance matrix of the conditional distribution of $\boldsymbol{U}$ given the matrix of regressors $\boldsymbol{X}$. (The mean vector and covariance matrix of a vector of random variables are defined in Appendix 19.2.) Specifically, the first and second assumptions in Key Concept 19.1 imply that $E(u_i|\boldsymbol{X}) = E(u_i|\boldsymbol{X}_i) = 0$ and that $\text{cov}(u_i, u_j|\boldsymbol{X}) = \boldsymbol{E}(\boldsymbol{u_i u_j}|\boldsymbol{X}) = E(u_i u_j|\boldsymbol{X}_i, \boldsymbol{X}_j) = E(u_i|\boldsymbol{X}_i)E(u_j|\boldsymbol{X}_j) = 0$ for $i \neq j$ (Exercise 18.7). The first, second, and fifth assumptions imply that $E(u_i^2|\boldsymbol{X}) = E(u_i^2|\boldsymbol{X}_i) = \sigma_u^2$. Combining these results, we have that

$$\text{under assumptions 1 and 2, } E(\boldsymbol{U}|\boldsymbol{X}) = \boldsymbol{0}_n, \text{ and} \tag{19.6}$$

$$\text{under assumptions 1, 2, and 5, } E(\boldsymbol{U}\boldsymbol{U}'|\boldsymbol{X}) = \sigma_u^2 \boldsymbol{I}_n, \tag{19.7}$$

where $\boldsymbol{0}_n$ is the n-dimensional vector of zeros and $\boldsymbol{I}_n$ is the $n \times n$ identity matrix.

Similarly, the first, second, fifth, and sixth assumptions in Key Concept 19.1 imply that the conditional distribution of the n-dimensional random vector $\boldsymbol{U}$, conditional on $\boldsymbol{X}$, is the multivariate normal distribution (defined in Appendix 19.2). That is,

$$\text{under assumptions 1, 2, 5, and 6, the conditional distribution of } \boldsymbol{U} \text{ given } \boldsymbol{X} \text{ is } N(\boldsymbol{0}_n, \sigma_u^2 \boldsymbol{I}_n). \tag{19.8}$$

The OLS Estimator

The OLS estimator minimizes the sum of squared prediction mistakes, $\sum_{i=1}^{n}(Y_i - b_0 - b_1 X_{1i} - \cdots - b_k X_{ki})^2$ [Equation (6.8)]. The formula for the OLS estimator is obtained by taking the derivative of the sum of squared prediction

mistakes with respect to each element of the coefficient vector, setting these derivatives to 0, and solving for the estimator $\hat{\boldsymbol{\beta}}$.

The derivative of the sum of squared prediction mistakes with respect to the j^{th} regression coefficient, b_j, is

$$\frac{\partial}{\partial b_j}\sum_{i=1}^{n}(Y_i - b_0 - b_1X_{1i} - \cdots - b_kX_{ki})^2$$

$$= -2\sum_{i=1}^{n}X_{ji}(Y_i - b_0 - b_1X_{1i} - \cdots - b_kX_{ki}) \tag{19.9}$$

for $j = 0, \ldots, k$, where, for $j = 0$, $X_{0i} = 1$ for all i. The derivative on the right-hand side of Equation (19.9) is the j^{th} element of the $k + 1$ dimensional vector, $-2\boldsymbol{X}'(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{b})$, where $\boldsymbol{b}$ is the $k + 1$ dimensional vector consisting of $b_0, \ldots, b_k$. There are $k + 1$ such derivatives, each corresponding to an element of $\boldsymbol{b}$. Combined, these yield the system of $k + 1$ equations that, when set to 0, constitute the first-order conditions for the OLS estimator $\hat{\boldsymbol{\beta}}$. That is, $\hat{\boldsymbol{\beta}}$ solves the system of $k + 1$ equations:

$$\boldsymbol{X}'(\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}) = \boldsymbol{0}_{k+1} \tag{19.10}$$

or, equivalently, $\boldsymbol{X}'\boldsymbol{Y} = \boldsymbol{X}'\boldsymbol{X}\hat{\boldsymbol{\beta}}$.

Solving the system of equations (19.10) yields the OLS estimator $\hat{\boldsymbol{\beta}}$ in matrix form:

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y}, \tag{19.11}$$

where $(\boldsymbol{X}'\boldsymbol{X})^{-1}$ is the inverse of the matrix $\boldsymbol{X}'\boldsymbol{X}$.

The role of "no perfect multicollinearity." The fourth least squares assumption in Key Concept 19.1 states that $\boldsymbol{X}$ has full column rank. In turn, this implies that the matrix $\boldsymbol{X}'\boldsymbol{X}$ has full rank—that is, that $\boldsymbol{X}'\boldsymbol{X}$ is nonsingular. Because $\boldsymbol{X}'\boldsymbol{X}$ is nonsingular, it is invertible. Thus the assumption that there is no perfect multicollinearity ensures that $(\boldsymbol{X}'\boldsymbol{X})^{-1}$ exists, so Equation (19.10) has a unique solution and the formula in Equation (19.11) for the OLS estimator can actually be computed. Said differently, if $\boldsymbol{X}$ does *not* have full column rank, there is not a unique solution to Equation (19.10), and $\boldsymbol{X}'\boldsymbol{X}$ is singular. Therefore, $(\boldsymbol{X}'\boldsymbol{X})^{-1}$ cannot be computed, and thus $\hat{\boldsymbol{\beta}}$ cannot be computed from Equation (19.11).

19.2 Asymptotic Distribution of the OLS Estimator and *t*-Statistic

If the sample size is large and the first four assumptions of Key Concept 19.1 are satisfied, then the OLS estimator has an asymptotic joint normal distribution, the heteroskedasticity-robust estimator of the covariance matrix is consistent, and the

KEY CONCEPT 19.2

The Multivariate Central Limit Theorem

Suppose that $\boldsymbol{W}_1, \ldots, \boldsymbol{W}_n$ are i.i.d. m-dimensional random variables with mean vector $E(\boldsymbol{W}_i) = \boldsymbol{\mu}_{\boldsymbol{W}}$ and covariance matrix $E[(\boldsymbol{W}_i - \boldsymbol{\mu}_{\boldsymbol{W}})(\boldsymbol{W}_i - \boldsymbol{\mu}_{\boldsymbol{W}})'] = \boldsymbol{\Sigma}_{\boldsymbol{W}}$, where $\boldsymbol{\Sigma}_{\boldsymbol{W}}$ is positive definite and finite. Let $\overline{\boldsymbol{W}} = \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{W}_i$. Then $\sqrt{n}(\overline{\boldsymbol{W}} - \boldsymbol{\mu}_{\boldsymbol{W}}) \xrightarrow{d} N(\boldsymbol{0}_m, \boldsymbol{\Sigma}_{\boldsymbol{W}})$.

heteroskedasticity-robust OLS t-statistic has an asymptotic standard normal distribution. These results make use of the multivariate normal distribution (Appendix 19.2) and a multivariate extension of the central limit theorem.

The Multivariate Central Limit Theorem

The central limit theorem of Key Concept 2.7 applies to a one-dimensional random variable. To derive the *joint* asymptotic distribution of the elements of $\hat{\boldsymbol{\beta}}$, we need a multivariate central limit theorem that applies to vector-valued random variables.

The multivariate central limit theorem extends the univariate central limit theorem to averages of observations on a vector-valued random variable, $\boldsymbol{W}$, where $\boldsymbol{W}$ is m-dimensional. The difference between the central limit theorems for a scalar-valued random variable and that for a vector-valued random variable is the conditions on the variances. In the scalar case in Key Concept 2.7, the requirement is that the variance is both nonzero and finite. In the vector case, the requirement is that the covariance matrix is both positive definite and finite. If the vector-valued random variable $\boldsymbol{W}$ has a finite positive definite covariance matrix, then $0 < \text{var}(\boldsymbol{c}'\boldsymbol{W}) < \infty$ for all nonzero m-dimensional vectors $\boldsymbol{c}$ (Exercise 19.3).

The multivariate central limit theorem that we will use is stated in Key Concept 19.2.

Asymptotic Normality of $\hat{\beta}$

In large samples, the OLS estimator has the multivariate normal asymptotic distribution

$$\sqrt{n}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \xrightarrow{d} N(\boldsymbol{0}_{k+1}, \boldsymbol{\Sigma}_{\sqrt{n}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})}), \text{ where } \boldsymbol{\Sigma}_{\sqrt{n}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})} = \boldsymbol{Q}_{\boldsymbol{X}}^{-1}\boldsymbol{\Sigma}_{\boldsymbol{V}}\boldsymbol{Q}_{\boldsymbol{X}}^{-1}, \quad (19.12)$$

where $\boldsymbol{Q}_{\boldsymbol{X}}$ is the $(k + 1) \times (k + 1)$ dimensional matrix of second moments of the regressors—that is, $\boldsymbol{Q}_{\boldsymbol{X}} = E(\boldsymbol{X}_i\boldsymbol{X}_i')$—and $\boldsymbol{\Sigma}_{\boldsymbol{V}}$ is the $(k + 1) \times (k + 1)$ dimensional covariance matrix of $\boldsymbol{V}_i = \boldsymbol{X}_i u_i$—that is, $\boldsymbol{\Sigma}_{\boldsymbol{V}} = E(\boldsymbol{V}_i\boldsymbol{V}_i')$. Note that the second least squares assumption in Key Concept 19.1 implies that $\boldsymbol{V}_i$, $i = 1, \ldots, n$, are i.i.d.

Written in terms of $\hat{\boldsymbol{\beta}}$ rather than $\sqrt{n}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$, the normal approximation in Equation (19.12) is

$$\hat{\boldsymbol{\beta}}, \text{ in large samples, is approximately distributed } N(\boldsymbol{\beta}, \boldsymbol{\Sigma}_{\hat{\boldsymbol{\beta}}}),$$
$$\text{where } \boldsymbol{\Sigma}_{\hat{\boldsymbol{\beta}}} = \boldsymbol{\Sigma}_{\sqrt{n}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})}/n = \boldsymbol{Q}_{\boldsymbol{X}}^{-1}\boldsymbol{\Sigma}_{\boldsymbol{V}}\boldsymbol{Q}_{\boldsymbol{X}}^{-1}/n. \quad (19.13)$$

The covariance matrix $\Sigma_{\hat{\boldsymbol{\beta}}}$ in Equation (19.13) is the covariance matrix of the approximate normal distribution of $\hat{\boldsymbol{\beta}}$, whereas $\Sigma_{\sqrt{n}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})}$ in Equation (19.12) is the covariance matrix of the asymptotic normal distribution of $\sqrt{n}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})$. These two covariance matrices differ by a factor of n, depending on whether the OLS estimator is scaled by $\sqrt{n}$.

Derivation of Equation (19.12). To derive Equation (19.12), first use Equations (19.3) and (19.11) to write $\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'(\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{U})$, so that

$$\hat{\boldsymbol{\beta}} = \boldsymbol{\beta} + (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{U}. \tag{19.14}$$

Thus $\hat{\boldsymbol{\beta}} - \boldsymbol{\beta} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{U}$, so

$$\sqrt{n}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) = \left(\frac{\boldsymbol{X}'\boldsymbol{X}}{n}\right)^{-1}\left(\frac{\boldsymbol{X}'\boldsymbol{U}}{\sqrt{n}}\right). \tag{19.15}$$

The derivation of Equation (19.12) involves arguing first that the "denominator" matrix in Equation (19.15), $\boldsymbol{X}'\boldsymbol{X}/n$, is consistent for $\boldsymbol{Q}_X$ and second that the "numerator" matrix, $\boldsymbol{X}'\boldsymbol{U}/\sqrt{n}$, obeys the multivariate central limit theorem in Key Concept 19.2. The details are given in Appendix 19.3.

Heteroskedasticity-Robust Standard Errors

The heteroskedasticity-robust estimator of $\Sigma_{\sqrt{n}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})}$ is obtained by replacing the population moments in its definition [Equation (19.12)] by sample moments. Accordingly, the heteroskedasticity-robust estimator of the covariance matrix of $\sqrt{n}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$ is

$$\hat{\Sigma}_{\sqrt{n}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})} = \left(\frac{\boldsymbol{X}'\boldsymbol{X}}{n}\right)^{-1}\hat{\Sigma}_{\hat{V}}\left(\frac{\boldsymbol{X}'\boldsymbol{X}}{n}\right)^{-1}, \text{ where } \hat{\Sigma}_{\hat{V}} = \frac{1}{n-k-1}\sum_{i=1}^{n}\boldsymbol{X}_i\boldsymbol{X}_i'\hat{u}_i^2, \tag{19.16}$$

The estimator $\hat{\Sigma}_{\hat{V}}$ incorporates the same degrees-of-freedom adjustment that is in the standard error of the regression (SER) for the multiple regression model (Section 6.4) to adjust for potential downward bias because of estimation of $k + 1$ regression coefficients.

The proof that $\hat{\Sigma}_{\sqrt{n}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})} \xrightarrow{p} \Sigma_{\sqrt{n}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})}$ is conceptually similar to the proof, presented in Section 18.3, of the consistency of heteroskedasticity-robust standard errors for the single-regressor model.

Heteroskedasticity-robust standard errors. The heteroskedasticity-robust estimator of the covariance matrix of $\hat{\boldsymbol{\beta}}$, $\Sigma_{\hat{\boldsymbol{\beta}}}$, is

$$\hat{\Sigma}_{\hat{\boldsymbol{\beta}}} = n^{-1}\hat{\Sigma}_{\sqrt{n}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})}. \tag{19.17}$$

The heteroskedasticity-robust standard error for the j^{th} regression coefficient is the square root of the j^{th} diagonal element of $\hat{\Sigma}_{\hat{\boldsymbol{\beta}}}$. That is, the heteroskedasticity-robust standard error of the j^{th} coefficient is

$$SE(\hat{\boldsymbol{\beta}}_j) = \sqrt{(\hat{\Sigma}_{\hat{\boldsymbol{\beta}}})_{jj}}, \tag{19.18}$$

where $(\hat{\Sigma}_{\hat{\boldsymbol{\beta}}})_{jj}$ is the (j, j) element of $\hat{\Sigma}_{\hat{\boldsymbol{\beta}}}$.

Other heteroskedasticity-robust variance estimators. The variance estimator in Equation (19.16) is called the HC1 variance estimator. The HC1 estimator is the most commonly used in practice, but it is not the only heteroskedasticity-robust variance estimator. Simulation studies have found that, in small samples, the HC1 estimator can be biased down, yielding standard errors that are too small. Long and Ervin (2000) provide simulation evidence that in small samples HC1 can be improved upon by a variant that weights each squared residual by a function of the X's. Imbens and Kolesar (2016) point out that, in addition to this bias, in small samples the sampling variability of the variance estimator makes the normal approximation a poor one, and they suggest using instead a t approximation to the t-statistic, along with a different variance estimator than HC1 or that suggested by Long and Ervin (2000). Angrist and Pischke (2009) suggest, however, that when the sample size exceeds 50, the HC1 estimator leads to negligible size distortions. Consistent with modern econometric practice, this text focuses on large samples, for which the HC1 estimator works well.

Confidence Intervals for Predicted Effects

Section 8.1 describes two methods for computing the standard error of predicted effects that involve changes in two or more regressors. There are compact matrix expressions for these standard errors and thus for confidence intervals for predicted effects.

Consider a change in the value of the regressors for the i^{th} observation from some initial value—say, $\boldsymbol{X}_{i,0}$—to some new value—$\boldsymbol{X}_{i,0} + \boldsymbol{d}$—so that the change in $\boldsymbol{X}_i$ is $\Delta\boldsymbol{X}_i = \boldsymbol{d}$, where $\boldsymbol{d}$ is a $k + 1$ dimensional vector. This change in $\boldsymbol{X}$ can involve multiple regressors (that is, multiple elements of $\boldsymbol{X}_i$). For example, if two of the regressors are the value of an independent variable and its square, then $\boldsymbol{d}$ is the difference between the subsequent and initial values of these two variables.

The expected effect of this change in $\boldsymbol{X}_i$ is $\boldsymbol{d}'\boldsymbol{\beta}$, and the estimator of this effect is $\boldsymbol{d}'\hat{\boldsymbol{\beta}}$. Because linear combinations of normally distributed random variables are themselves normally distributed, $\sqrt{n}(\boldsymbol{d}'\hat{\boldsymbol{\beta}} - \boldsymbol{d}'\boldsymbol{\beta}) = \boldsymbol{d}'\sqrt{n}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \xrightarrow{d} N(0, \boldsymbol{d}'\Sigma_{\sqrt{n}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})}\boldsymbol{d})$. Thus the standard error of this predicted effect is $(\boldsymbol{d}'\hat{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\beta}}}\boldsymbol{d})^{1/2}$. A 95% confidence interval for this predicted effect is

$$\boldsymbol{d}'\hat{\boldsymbol{\beta}} \pm 1.96\sqrt{\boldsymbol{d}'\hat{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\beta}}}\boldsymbol{d}}. \tag{19.19}$$

Asymptotic Distribution of the t-Statistic

The t-statistic testing the null hypothesis that $\beta_j = \beta_{j,0}$, constructed using the heteroskedasticity-robust standard error in Equation (19.18), is given in Key Concept 7.1. The argument that this t-statistic has an asymptotic standard normal distribution parallels the argument given in Section 18.3 for the single-regressor model.

19.3 Tests of Joint Hypotheses

Section 7.2 considers tests of joint hypotheses that involve multiple restrictions, where each restriction involves a single coefficient, and Section 7.3 considers tests of a single restriction involving two or more coefficients. The matrix setup of Section 19.1 permits a unified representation of these two types of hypotheses as linear restrictions on the coefficient vector, where each restriction can involve multiple coefficients. Under the first four least squares assumptions in Key Concept 19.1, the heteroskedasticity-robust OLS F-statistic testing these hypotheses has an $F_{q,\infty}$ asymptotic distribution under the null hypothesis.

Joint Hypotheses in Matrix Notation

Consider a joint hypothesis that is linear in the coefficients and imposes q restrictions, where $q \leq k + 1$. Each of these q restrictions can involve one or more of the regression coefficients. This joint null hypothesis can be written in matrix notation as

$$\boldsymbol{R\beta} = \boldsymbol{r}, \tag{19.20}$$

where $\boldsymbol{R}$ is a $q \times (k + 1)$ nonrandom matrix with full row rank and $\boldsymbol{r}$ is a nonrandom $q \times 1$ vector. The number of rows of $\boldsymbol{R}$ is q, which is the number of restrictions being imposed under the null hypothesis.

The null hypothesis in Equation (19.20) subsumes all the null hypotheses considered in Sections 7.2 and 7.3. For example, a joint hypothesis of the type considered in Section 7.2 is that $\beta_0 = 0, \beta_1 = 0, \ldots, \beta_{q-1} = 0$. To write this joint hypothesis in the form of Equation (19.20), set $\boldsymbol{R} = [\boldsymbol{I}_q \; \boldsymbol{0}_{q\times(k+1-q)}]$ and $\boldsymbol{r} = \boldsymbol{0}_q$.

The formulation in Equation (19.20) also captures the restrictions of Section 7.3 involving multiple regression coefficients. For example, if $k = 2$, then the hypothesis that $\beta_1 + \beta_2 = 1$ can be written in the form of Equation (19.20) by setting $\boldsymbol{R} = [0\ 1\ 1]$, $r = 1$, and $q = 1$.

Asymptotic Distribution of the F-Statistic

The heteroskedasticity-robust F-statistic testing the joint hypothesis in Equation (19.20) is

$$F = (\boldsymbol{R\hat{\beta}} - \boldsymbol{r})'[\boldsymbol{R}\hat{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\beta}}}\boldsymbol{R}']^{-1}(\boldsymbol{R\hat{\beta}} - \boldsymbol{r})/q. \tag{19.21}$$

If the first four assumptions in Key Concept 19.1 hold, then under the null hypothesis

$$F \xrightarrow{d} F_{q,\infty}. \tag{19.22}$$

This result follows by combining the asymptotic normality of $\hat{\boldsymbol{\beta}}$ with the consistency of the heteroskedasticity-robust estimator $\hat{\boldsymbol{\Sigma}}_{\sqrt{n}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})}$ of the covariance matrix. Specifically, first note that Equation (19.12) and Equation (19.74) in

Appendix 19.2 imply that, under the null hypothesis, $\sqrt{n}(\boldsymbol{R}\hat{\boldsymbol{\beta}} - \boldsymbol{r}) = \sqrt{n}\boldsymbol{R}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \xrightarrow{d} N(\boldsymbol{0}, \boldsymbol{R}\boldsymbol{\Sigma}_{\sqrt{n}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})}\boldsymbol{R}')$. It follows from Equation (19.77) that, under the null hypothesis, $(\boldsymbol{R}\hat{\boldsymbol{\beta}} - \boldsymbol{r})'[\boldsymbol{R}\boldsymbol{\Sigma}_{\hat{\boldsymbol{\beta}}}\boldsymbol{R}']^{-1}(\boldsymbol{R}\hat{\boldsymbol{\beta}} - \boldsymbol{r}) = [\sqrt{n}\boldsymbol{R}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})]'[\boldsymbol{R}\boldsymbol{\Sigma}_{\sqrt{n}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})}\boldsymbol{R}']^{-1}[\sqrt{n}\boldsymbol{R}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})] \xrightarrow{d} \chi^2_q$. However, because $\hat{\boldsymbol{\Sigma}}_{\sqrt{n}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})} \xrightarrow{p} \boldsymbol{\Sigma}_{\sqrt{n}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})}$, it follows from Slutsky's theorem that $[\sqrt{n}\boldsymbol{R}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})]'[\boldsymbol{R}\hat{\boldsymbol{\Sigma}}_{\sqrt{n}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})}\boldsymbol{R}']^{-1}[\sqrt{n}\boldsymbol{R}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})] \xrightarrow{d} \chi^2_q$. or, equivalently (because $\hat{\boldsymbol{\Sigma}}_{\boldsymbol{\beta}} = \hat{\boldsymbol{\Sigma}}_{\sqrt{n}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})}/n$), that $F \xrightarrow{d} \chi^2_q/q$, which is in turn distributed $F_{q,\infty}$.

Confidence Sets for Multiple Coefficients

As discussed in Section 7.4, an asymptotically valid confidence set for two or more elements of $\boldsymbol{\beta}$ can be constructed as the set of values that, when taken as the null hypothesis, are not rejected by the F-statistic. In principle, this set could be computed by repeatedly evaluating the F-statistic for many values of $\boldsymbol{\beta}$, but, as is the case with a confidence interval for a single coefficient, it is simpler to manipulate the formula for the test statistic to obtain an explicit formula for the confidence set.

Here is the procedure for constructing a confidence set for two or more of the elements of $\boldsymbol{\beta}$. Let $\boldsymbol{\delta}$ denote the q-dimensional vector consisting of the coefficients for which we wish to construct a confidence set. For example, if we are constructing a confidence set for the regression coefficients β_1 and β_2, then $q = 2$ and $\boldsymbol{\delta} = (\beta_1\, \beta_2)'$. In general, we can write $\boldsymbol{\delta} = \boldsymbol{R}\boldsymbol{\beta}$, where the matrix $\boldsymbol{R}$ consists of 0's and 1's [as discussed following Equation (19.20)]. The F-statistic testing the hypothesis that $\boldsymbol{\delta} = \boldsymbol{\delta}_0$ is $F = (\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}_0)'[\boldsymbol{R}\hat{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\beta}}}\boldsymbol{R}']^{-1}(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta}_0)/q$, where $\hat{\boldsymbol{\delta}} = \boldsymbol{R}\hat{\boldsymbol{\beta}}$. A 95% confidence set for $\boldsymbol{\delta}$ is the set of values $\boldsymbol{\delta}_0$ that are not rejected by the F-statistic. That is, when $\boldsymbol{\delta} = \boldsymbol{R}\boldsymbol{\beta}$, a 95% confidence set for $\boldsymbol{\delta}$ is

$$\{\boldsymbol{\delta}: (\hat{\boldsymbol{\delta}} - \boldsymbol{\delta})'[\boldsymbol{R}\hat{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\beta}}}\boldsymbol{R}']^{-1}(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta})/q \leq c\}, \tag{19.23}$$

where c is the 95$^{\text{th}}$ percentile (the 5% critical value) of the $F_{q,\infty}$ distribution.

The set in Equation (19.23) consists of all the points contained inside the ellipse determined when the inequality in Equation (19.23) is an equality (this is an ellipsoid when $q > 2$). Thus the confidence set for δ can be computed by solving Equation (19.23) for the boundary ellipse.

19.4 Distribution of Regression Statistics with Normal Errors

The distributions presented in Sections 19.2 and 19.3, which were justified by appealing to the law of large numbers and the central limit theorem, apply when the sample size is large. If, however, the errors are homoskedastic and normally distributed, conditional on $\boldsymbol{X}$, then the OLS estimator has a multivariate normal distribution in a finite sample, conditional on $\boldsymbol{X}$. In addition, the finite sample distribution of the

square of the standard error of the regression is proportional to the chi-squared distribution with $n - k - 1$ degrees of freedom, the homoskedasticity-only OLS t-statistic has a Student t distribution with $n - k - 1$ degrees of freedom, and the homoskedasticity-only F-statistic has an $F_{q,n-k-1}$ distribution. The arguments in this section employ some specialized matrix formulas for OLS regression statistics, which are presented first.

Matrix Representations of OLS Regression Statistics

The OLS predicted values, residuals, and sum of squared residuals have compact matrix representations. These representations make use of two matrices, $\boldsymbol{P_X}$ and $\boldsymbol{M_X}$.

The matrices P_X and M_X. The algebra of OLS in the multivariate model relies on the two symmetric $n \times n$ matrices, $\boldsymbol{P_X}$ and M_X:

$$\boldsymbol{P_X} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}' \text{ and} \tag{19.24}$$

$$\boldsymbol{M_X} = \boldsymbol{I_n} - \boldsymbol{P_X}. \tag{19.25}$$

A matrix $\boldsymbol{C}$ is idempotent if $\boldsymbol{C}$ is square and $\boldsymbol{CC} = \boldsymbol{C}$ (see Appendix 19.1). Because $\boldsymbol{P_X} = \boldsymbol{P_X}\boldsymbol{P_X}$ and $\boldsymbol{M_X} = \boldsymbol{M_X}\boldsymbol{M_X}$ (Exercise 19.5) and because $\boldsymbol{P_X}$ and $\boldsymbol{M_X}$ are symmetric, $\boldsymbol{P_X}$ and $\boldsymbol{M_X}$ are symmetric idempotent matrices.

The matrices $\boldsymbol{P_X}$ and $\boldsymbol{M_X}$ have some additional useful properties (Exercise 19.5), which follow directly from the definitions in Equations (19.24) and (19.25):

$$\begin{gathered}\boldsymbol{P_X}\boldsymbol{X} = \boldsymbol{X} \text{ and } \boldsymbol{M_X}\boldsymbol{X} = \boldsymbol{0}_{n\times(k+1)}; \\ \text{rank}(\boldsymbol{P_X}) = k + 1 \text{ and } \text{rank}(\boldsymbol{M_X}) = n - k - 1,\end{gathered} \tag{19.26}$$

where $\text{rank}(\boldsymbol{P_X})$ is the rank of $\boldsymbol{P_X}$.

The matrices $\boldsymbol{P_X}$ and $\boldsymbol{M_X}$ can be used to decompose an n-dimensional vector $\boldsymbol{Z}$ into two parts: a part that is spanned by the columns of $\boldsymbol{X}$ and a part that is orthogonal to the columns of $\boldsymbol{X}$. In other words, $\boldsymbol{P_X}\boldsymbol{Z}$ is the projection of $\boldsymbol{Z}$ onto the space spanned by the columns of $\boldsymbol{X}$, $\boldsymbol{M_X}\boldsymbol{Z}$ is the part of $\boldsymbol{Z}$ orthogonal to the columns of $\boldsymbol{X}$, and $\boldsymbol{Z} = \boldsymbol{P_X}\boldsymbol{Z} + \boldsymbol{M_X}\boldsymbol{Z}$.

OLS predicted values and residuals. The matrices $\boldsymbol{P_X}$ and $\boldsymbol{M_X}$ provide some simple expressions for OLS predicted values and residuals. The OLS predicted values, $\hat{\boldsymbol{Y}} = \boldsymbol{X}\hat{\boldsymbol{\beta}}$, and the OLS residuals, $\hat{\boldsymbol{U}} = \boldsymbol{Y} - \hat{\boldsymbol{Y}}$, can be expressed as follows (Exercise 19.5):

$$\hat{\boldsymbol{Y}} = \boldsymbol{P_X}\boldsymbol{Y} \text{ and} \tag{19.27}$$

$$\hat{\boldsymbol{U}} = \boldsymbol{M_X}\boldsymbol{Y} = \boldsymbol{M_X}\boldsymbol{U}. \tag{19.28}$$

The expressions in Equations (19.27) and (19.28) provide a simple proof that the OLS residuals and predicted values are orthogonal—that is, that Equation (4.35) holds: $\hat{\boldsymbol{Y}}'\hat{\boldsymbol{U}} = \boldsymbol{Y}'\boldsymbol{P_X}'\boldsymbol{M_X}\boldsymbol{Y} = 0$, where the second equality follows from $\boldsymbol{P_X}'\boldsymbol{M_X} = \boldsymbol{0}_{n\times n}$, which in turn follows from $\boldsymbol{M_X}\boldsymbol{X} = \boldsymbol{0}_{n\times(k+1)}$ in Equation (19.26).

The standard error of the regression. The SER, defined in Section 4.3, is $s_{\hat{u}}$, where

$$s_{\hat{u}}^2 = \frac{1}{n-k-1}\sum_{i=1}^{n}\hat{u}_i^2 = \frac{1}{n-k-1}\hat{\boldsymbol{U}}'\hat{\boldsymbol{U}} = \frac{1}{n-k-1}\boldsymbol{U}'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{U}, \quad (19.29)$$

where the final equality follows because $\hat{\boldsymbol{U}}'\hat{\boldsymbol{U}} = (\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{U})'(\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{U}) = \boldsymbol{U}'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{U} = \boldsymbol{U}'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{U}$ (because $\boldsymbol{M}_{\boldsymbol{X}}$ is symmetric and idempotent).

Distribution of $\hat{\beta}$ with Independent Normal Errors

Because $\hat{\boldsymbol{\beta}} = \boldsymbol{\beta} + (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{U}$ [Equation (19.14)] and because the distribution of $\boldsymbol{U}$, conditional on $\boldsymbol{X}$, is, by assumption, $N(\boldsymbol{0}_n, \sigma_u^2\boldsymbol{I}_n)$ [Equation (19.8)], the conditional distribution of $\hat{\boldsymbol{\beta}}$ given $\boldsymbol{X}$ is multivariate normal with mean $\boldsymbol{\beta}$. The covariance matrix of $\hat{\boldsymbol{\beta}}$, conditional on $\boldsymbol{X}$, is $\boldsymbol{\Sigma}_{\hat{\boldsymbol{\beta}}|\boldsymbol{X}} = E[(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'|\boldsymbol{X}] = E[(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{U}\boldsymbol{U}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}|\boldsymbol{X}] = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'(\sigma_u^2\boldsymbol{I}_n)\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1} = \sigma_u^2(\boldsymbol{X}'\boldsymbol{X})^{-1}$. Accordingly, under all six assumptions in Key Concept 19.1, the finite-sample conditional distribution of $\hat{\boldsymbol{\beta}}$ given $\boldsymbol{X}$ is

$$\hat{\boldsymbol{\beta}} \sim N(\boldsymbol{\beta}, \boldsymbol{\Sigma}_{\hat{\boldsymbol{\beta}}|\boldsymbol{X}}), \text{ where } \boldsymbol{\Sigma}_{\hat{\boldsymbol{\beta}}|\boldsymbol{X}} = \sigma_u^2(\boldsymbol{X}'\boldsymbol{X})^{-1}. \quad (19.30)$$

Distribution of $s_{\hat{u}}^2$

If all six assumptions in Key Concept 19.1 hold, then $s_{\hat{u}}^2$ has an exact sampling distribution that is proportional to a chi-squared distribution with $n-k-1$ degrees of freedom:

$$s_{\hat{u}}^2 \sim \frac{\sigma_u^2}{n-k-1} \times \chi_{n-k-1}^2 \quad (19.31)$$

The proof of Equation (19.31) starts with Equation (19.29). Because $\boldsymbol{U}$ is normally distributed, conditional on $\boldsymbol{X}$, and because $\boldsymbol{M}_{\boldsymbol{X}}$ is a symmetric idempotent matrix, the quadratic form $\boldsymbol{U}'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{U}/\sigma_u^2$ has an exact chi-squared distribution with degrees of freedom equal to the rank of $\boldsymbol{M}_{\boldsymbol{X}}$ [Equation (19.78) in Appendix 19.2]. From Equation (19.26), the rank of $\boldsymbol{M}_{\boldsymbol{X}}$ is $n-k-1$. Thus $\boldsymbol{U}'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{U}/\sigma_u^2$ has an exact χ_{n-k-1}^2 distribution, from which Equation (19.31) follows.

The degrees-of-freedom adjustment ensures that $s_{\hat{u}}^2$ is unbiased. The expectation of a random variable with a χ_{n-k-1}^2 distribution is $n-k-1$; thus $E(\boldsymbol{U}'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{U}) = (n-k-1)\sigma_u^2$, so $E(s_{\hat{u}}^2) = \sigma_u^2$.

Homoskedasticity-Only Standard Errors

The homoskedasticity-only estimator $\tilde{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\beta}}}$ of the covariance matrix of $\hat{\boldsymbol{\beta}}$, conditional on $\boldsymbol{X}$, is obtained by substituting the sample variance $s_{\hat{u}}^2$ for the population variance σ_u^2 in the expression for $\boldsymbol{\Sigma}_{\hat{\boldsymbol{\beta}}|\boldsymbol{X}}$ in Equation (19.30). Accordingly,

$$\tilde{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\beta}}} = s_{\hat{u}}^2(\boldsymbol{X}'\boldsymbol{X})^{-1} \quad \text{(homoskedasticity-only)}. \quad (19.32)$$

The estimator of the variance of the normal conditional distribution of $\hat{\beta}_j$ given $\boldsymbol{X}$ is the (j, j) element of $\widetilde{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\beta}}}$. Thus the homoskedasticity-only standard error of $\hat{\beta}_j$ is the square root of the j^{th} diagonal element of $\widetilde{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\beta}}}$. That is, the homoskedasticity-only standard error of $\hat{\beta}_j$ is

$$\widetilde{SE}(\hat{\beta}_j) = \sqrt{(\widetilde{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\beta}}})_{jj}} \quad \text{(homoskedasticity-only)}. \tag{19.33}$$

Distribution of the *t*-Statistic

Let $\widetilde{t}$ be the t-statistic testing the hypothesis $\beta_j = \beta_{j,0}$, constructed using the homoskedasticity-only standard error; that is, let

$$\widetilde{t} = \frac{\hat{\beta}_j - \beta_{j,0}}{\sqrt{(\widetilde{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\beta}}})_{jj}}}. \tag{19.34}$$

Under all six of the extended least squares assumptions in Key Concept 19.1, the exact sampling distribution of $\widetilde{t}$ is the Student t distribution with $n - k - 1$ degrees of freedom; that is,

$$\widetilde{t} \sim t_{n-k-1}. \tag{19.35}$$

The proof of Equation (19.35) is given in Appendix 19.4.

Distribution of the *F*-Statistic

If all six least squares assumptions in Key Concept 19.1 hold, then the F-statistic testing the hypothesis in Equation (19.20), constructed using the homoskedasticity-only estimator of the covariance matrix, has an exact $F_{q,n-k-1}$ distribution under the null hypothesis.

The homoskedasticity-only F-statistic. The homoskedasticity-only F-statistic is similar to the heteroskedasticity-robust F-statistic in Equation (19.21) except that the homoskedasticity-only estimator $\widetilde{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\beta}}}$ is used instead of the heteroskedasticity-robust estimator $\widetilde{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\beta}}}$. Substituting the expression $\widetilde{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\beta}}} = s_{\hat{u}}^2(\boldsymbol{X}'\boldsymbol{X})^{-1}$ into the expression for the F-statistic in Equation (19.21) yields the homoskedasticity-only F-statistic testing the null hypothesis in Equation (19.20):

$$\widetilde{F} = \frac{(\boldsymbol{R}\hat{\boldsymbol{\beta}} - \boldsymbol{r})'[\boldsymbol{R}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{R}']^{-1}(\boldsymbol{R}\hat{\boldsymbol{\beta}} - \boldsymbol{r})/q}{s_{\hat{u}}^2}. \tag{19.36}$$

If all six assumptions in Key Concept 19.1 hold, then under the null hypothesis

$$\widetilde{F} \sim F_{q,n-k-1}. \tag{19.37}$$

The proof of Equation (19.37) is given in Appendix 19.4.

The F-statistic in Equation (19.36) is called the Wald version of the F-statistic (named after the statistician Abraham Wald). Although the formula for the homoskedastic-only F-statistic given in Equation (7.13) appears quite different from the formula for the Wald statistic in Equation (19.36), the homoskedastic-only F-statistic and the Wald F-statistic are two versions of the same statistic. That is, the two expressions are equivalent, a result shown in Exercise 19.13.

19.5 Efficiency of the OLS Estimator with Homoskedastic Errors

Under the Gauss–Markov conditions for multiple regression, the OLS estimator of $\boldsymbol{\beta}$ is efficient among all linear conditionally unbiased estimators; that is, the OLS estimator is the best linear unbiased estimator (BLUE).

The Gauss–Markov Conditions for Multiple Regression

The **Gauss–Markov conditions for multiple regression** are

$$\begin{aligned} &\text{(i) } E(\boldsymbol{U}|\boldsymbol{X}) = \boldsymbol{0}_n, \\ &\text{(ii) } E(\boldsymbol{U}\boldsymbol{U}'|\boldsymbol{X}) = \sigma_u^2 \boldsymbol{I}_n, \text{ and} \\ &\text{(iii) } \boldsymbol{X} \text{ has full column rank.} \end{aligned} \tag{19.38}$$

The Gauss–Markov conditions for multiple regression in turn are implied by the first five assumptions in Key Concept 19.1 [see Equations (19.6) and (19.7)]. The conditions in Equation (19.38) generalize the Gauss–Markov conditions for a single-regressor model to multiple regression. [By using matrix notation, the second and third Gauss–Markov conditions in Equation (5.31) are collected into the single condition (ii) in Equation (19.38).]

Linear Conditionally Unbiased Estimators

We start by describing the class of linear unbiased estimators and by showing that OLS is in that class.

The class of linear conditionally unbiased estimators. An estimator of $\boldsymbol{\beta}$ is said to be linear if it is a linear function of $Y_1, \ldots, Y_n$. Accordingly, the estimator $\widetilde{\boldsymbol{\beta}}$ is linear in $\boldsymbol{Y}$ if it can be written in the form

$$\widetilde{\boldsymbol{\beta}} = \boldsymbol{A}'\boldsymbol{Y}, \tag{19.39}$$

where $\boldsymbol{A}$ is an $n \times (k + 1)$ dimensional matrix of weights that may depend on $\boldsymbol{X}$ and on nonrandom constants but not on $\boldsymbol{Y}$.

Gauss–Markov Theorem for Multiple Regression

KEY CONCEPT
19.3

Suppose that the Gauss–Markov conditions for multiple regression in Equation (19.38) hold. Then the OLS estimator $\hat{\boldsymbol{\beta}}$ is BLUE. That is, let $\tilde{\boldsymbol{\beta}}$ be a linear conditionally unbiased estimator of $\boldsymbol{\beta}$, and let $\boldsymbol{c}$ be a nonrandom $k + 1$ dimensional vector. Then $\text{var}(\boldsymbol{c}'\hat{\boldsymbol{\beta}}|\boldsymbol{X}) \leq \text{var}(\boldsymbol{c}'\tilde{\boldsymbol{\beta}}|\boldsymbol{X})$ for every nonzero vector $\boldsymbol{c}$, where the inequality holds with equality for all $\boldsymbol{c}$ only if $\tilde{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}}$.

An estimator is conditionally unbiased if the mean of its conditional sampling distribution given $\boldsymbol{X}$ is $\boldsymbol{\beta}$. That is, $\tilde{\boldsymbol{\beta}}$ is conditionally unbiased if $E(\tilde{\boldsymbol{\beta}}|\boldsymbol{X}) = \boldsymbol{\beta}$.

The OLS estimator is linear and conditionally unbiased. Comparison of Equations (19.11) and (19.39) shows that the OLS estimator is linear in $\boldsymbol{Y}$; specifically, $\hat{\boldsymbol{\beta}} = \hat{\boldsymbol{A}}'\boldsymbol{Y}$, where $\hat{\boldsymbol{A}} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}$. To show that $\hat{\boldsymbol{\beta}}$ is conditionally unbiased, recall from Equation (19.14) that $\hat{\boldsymbol{\beta}} = \boldsymbol{\beta} + (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{U}$. Taking the conditional expectation of both sides of this expression yields $E(\hat{\boldsymbol{\beta}}|\boldsymbol{X}) = \boldsymbol{\beta} + E[(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{U}|\boldsymbol{X}] = \boldsymbol{\beta} + (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{E}(\boldsymbol{U}|\boldsymbol{X}) = \boldsymbol{\beta}$, where the final equality follows because $E(\boldsymbol{U}|\boldsymbol{X}) = 0$ by the first Gauss–Markov condition.

The Gauss–Markov Theorem for Multiple Regression

The **Gauss–Markov theorem for multiple regression** provides conditions under which the OLS estimator is efficient among the class of linear conditionally unbiased estimators. A subtle point arises, however, because $\hat{\boldsymbol{\beta}}$ is a vector and its "variance" is a covariance matrix. When the variance of an estimator is a matrix, just what does it mean to say that one estimator has a smaller variance than another?

The Gauss–Markov theorem handles this problem by comparing the variance of a candidate estimator of a *linear combination* of the elements of $\boldsymbol{\beta}$ to the variance of the corresponding linear combination of $\hat{\boldsymbol{\beta}}$. Specifically, let $\boldsymbol{c}$ be a $k + 1$ dimensional vector, and consider the problem of estimating the linear combination $\boldsymbol{c}'\boldsymbol{\beta}$ using the candidate estimator $\boldsymbol{c}'\tilde{\boldsymbol{\beta}}$ (where $\tilde{\boldsymbol{\beta}}$ is a linear conditionally unbiased estimator) on the one hand and $\boldsymbol{c}'\hat{\boldsymbol{\beta}}$ on the other hand. Because $\boldsymbol{c}'\tilde{\boldsymbol{\beta}}$ and $\boldsymbol{c}'\hat{\boldsymbol{\beta}}$ are both scalars and are both linear conditionally unbiased estimators of $\boldsymbol{c}'\boldsymbol{\beta}$, it now makes sense to compare their variances.

The Gauss–Markov theorem for multiple regression says that the OLS estimator of $\boldsymbol{c}'\boldsymbol{\beta}$ is efficient; that is, the OLS estimator $\boldsymbol{c}'\hat{\boldsymbol{\beta}}$ has the smallest conditional variance of all linear conditionally unbiased estimators. Remarkably, this is true no matter what the linear combination is. It is in this sense that the OLS estimator is BLUE in multiple regression.

The Gauss–Markov theorem is stated in Key Concept 19.3 and proven in Appendix 19.5.

19.6 Generalized Least Squares[1]

The assumption of i.i.d. sampling fits many applications. For example, suppose that Y_i and $\boldsymbol{X}_i$ correspond to information about individuals, such as their earnings, education, and personal characteristics, where the individuals are selected from a population by simple random sampling. In this case, because of the simple random sampling scheme, $(\boldsymbol{X}_i, Y_i)$ are necessarily i.i.d. Because $(\boldsymbol{X}_i, Y_i)$ and $(\boldsymbol{X}_j, Y_j)$ are independently distributed for $i \neq j$, u_i and u_j are independently distributed for $i \neq j$. This in turn implies that u_i and u_j are uncorrelated for $i \neq j$. In the context of the Gauss–Markov assumptions, the assumption that $E(\boldsymbol{U}\boldsymbol{U}' \mid \boldsymbol{X})$ is diagonal therefore is appropriate if the data are collected in a way that makes the observations independently distributed.

Some sampling schemes encountered in econometrics do not, however, result in independent observations and instead can lead to error terms u_i that are correlated from one observation to the next. The leading example is when the data are sampled over time for the same entity—that is, when the data are time series data. As discussed in Section 16.3, in regressions involving time series data, many omitted factors are correlated from one period to the next, and this can result in regression error terms (which represent those omitted factors) that are correlated from one period of observation to the next. In other words, the error term in one period will not, in general, be distributed independently of the error term in the next period. Instead, the error term in one period could be correlated with the error term in the next period.

The presence of correlated error terms creates two problems for inference based on OLS. First, *neither* the heteroskedasticity-robust nor the homoskedasticity-only standard errors produced by OLS provide a valid basis for inference. The solution to this problem is to use standard errors that are robust to both heteroskedasticity and correlation of the error terms across observations. This topic—heteroskedasticity- and autocorrelation-consistent (HAC) covariance matrix estimation—is the subject of Section 16.4 and we do not pursue it further here.

Second, if the error term is correlated across observations, then $E(\boldsymbol{U}\boldsymbol{U}' \mid \boldsymbol{X})$ is not diagonal, the second Gauss–Markov condition in Equation (19.38) does not hold, and OLS is not BLUE. In this section, we study an estimator, **generalized least squares (GLS)**, that is BLUE (at least asymptotically) when the conditional covariance matrix of the errors is no longer proportional to the identity matrix. A special case of GLS is weighted least squares, discussed in Section 18.5, in which the conditional covariance matrix is diagonal and the i^{th} diagonal element is a function of $\boldsymbol{X}_i$. Like WLS, GLS transforms the regression model so that the errors of the transformed model satisfy the Gauss–Markov conditions. The GLS estimator is the OLS estimator of the coefficients in the transformed model.

[1]The GLS estimator was introduced in Section 16.5 in the context of distributed lag time series regression. The presentation here is a self-contained mathematical treatment of GLS that can be read independently of Section 16.5, but reading that section first will help to make these ideas more concrete.

The GLS Assumptions

KEY CONCEPT 19.4

In the linear regression model $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{U}$, the GLS assumptions are

1. $E(\boldsymbol{U}|\boldsymbol{X}) = \boldsymbol{0}_n$;
2. $E(\boldsymbol{UU}'|\boldsymbol{X}) = \boldsymbol{\Omega}(\boldsymbol{X})$, where $\boldsymbol{\Omega}(\boldsymbol{X})$ is an $n \times n$ positive definite matrix that can depend on $\boldsymbol{X}$;
3. $\boldsymbol{X}_i$ and u_i satisfy suitable moment conditions; and
4. $\boldsymbol{X}$ has full column rank (there is no perfect multicollinearity).

The GLS Assumptions

There are four assumptions under which GLS is valid. The first GLS assumption is that u_i has a mean of 0, conditional on $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$; that is,

$$E(\boldsymbol{U}|\boldsymbol{X}) = \boldsymbol{0}_n. \tag{19.40}$$

This assumption is implied by the first two least squares assumptions in Key Concept 19.1; that is, if $E(u_i|\boldsymbol{X}_i) = 0$ and $(\boldsymbol{X}_i, Y_i), i = 1, \ldots, n$, are i.i.d., then $E(\boldsymbol{U}|\boldsymbol{X}) = \boldsymbol{0}_n$. In GLS, however, we will not want to maintain the i.i.d. assumption; after all, one purpose of GLS is to handle errors that are correlated across observations. We discuss the significance of the assumption in Equation (19.40) after introducing the GLS estimator.

The second GLS assumption is that the conditional covariance matrix of $\boldsymbol{U}$ given $\boldsymbol{X}$ is some function of $\boldsymbol{X}$:

$$E(\boldsymbol{UU}'|\boldsymbol{X}) = \boldsymbol{\Omega}(\boldsymbol{X}), \tag{19.41}$$

where $\boldsymbol{\Omega}(\boldsymbol{X})$ is an $n \times n$ positive definite matrix-valued function of $\boldsymbol{X}$.

There are two main applications of GLS that are covered by this assumption. The first is independent sampling with heteroskedastic errors, in which case $\boldsymbol{\Omega}(\boldsymbol{X})$ is a diagonal matrix with diagonal element $\lambda h(\boldsymbol{X}_i)$, where λ is a constant and h is a function. In this case, discussed in Section 18.5, GLS is WLS.

The second application is to homoskedastic errors that are serially correlated. In practice, in this case a model is developed for the serial correlation. For example, one model is that the error term is correlated with only its neighbor, so $\text{corr}(u_i, u_{i-1}) = \rho \neq 0$ but $\text{corr}(u_i, u_j) = 0$ if $|i - j| \geq 2$. In this case, $\boldsymbol{\Omega}(\boldsymbol{X})$ has σ_u^2 as its diagonal element, $\rho\sigma_u^2$ in the first off-diagonal, and zeros elsewhere. Thus $\boldsymbol{\Omega}(\boldsymbol{X})$ does not depend on $\boldsymbol{X}$, $\boldsymbol{\Omega}_{ii} = \sigma_u^2$, $\boldsymbol{\Omega}_{ij} = \rho\sigma_u^2$ for $|i - j| = 1$, and $\boldsymbol{\Omega}_{ij} = 0$ for $|i - j| > 1$. Other models for serial correlation, including the first-order autoregressive model, are discussed further in the context of GLS in Section 16.5 (also see Exercise 19.8).

One assumption that has appeared on all previous lists of least squares assumptions for cross-sectional data is that $\boldsymbol{X}_i$ and u_i have nonzero finite fourth moments. In the case of GLS, the specific moment assumptions needed to prove asymptotic results depend on the nature of the function $\boldsymbol{\Omega}(\boldsymbol{X})$, whether $\boldsymbol{\Omega}(\boldsymbol{X})$ is known or estimated, and the statistic under consideration (the GLS estimator, t-statistic, etc.). Because the assumptions are case- and model-specific, we do not present specific moment assumptions here, and the discussion of the large-sample properties of GLS assumes that such moment conditions apply for the relevant case at hand. For completeness, as the third GLS assumption, $\boldsymbol{X}_i$ and u_i are simply assumed to satisfy suitable moment conditions.

The fourth GLS assumption is that $\boldsymbol{X}$ has full column rank; that is, the regressors are not perfectly multicollinear.

The GLS assumptions are summarized in Key Concept 19.4.

We consider GLS estimation in two cases. In the first case, $\boldsymbol{\Omega}(\boldsymbol{X})$ is known. In the second case, the functional form of $\boldsymbol{\Omega}(\boldsymbol{X})$ is known up to some parameters that can be estimated. To simplify notation, we refer to the function $\boldsymbol{\Omega}(\boldsymbol{X})$ as the matrix $\boldsymbol{\Omega}$, so the dependence of $\boldsymbol{\Omega}$ on $\boldsymbol{X}$ is implicit.

GLS When $\boldsymbol{\Omega}$ Is Known

When $\boldsymbol{\Omega}$ is known, the GLS estimator uses $\boldsymbol{\Omega}$ to transform the regression model to one with errors that satisfy the Gauss–Markov conditions. Specifically, let $\boldsymbol{F}$ be a matrix square root of $\boldsymbol{\Omega}^{-1}$; that is, let $\boldsymbol{F}$ be a matrix that satisfies $\boldsymbol{F}'\boldsymbol{F} = \boldsymbol{\Omega}^{-1}$ (see Appendix 19.1). A property of $\boldsymbol{F}$ is that $\boldsymbol{F}\boldsymbol{\Omega}\boldsymbol{F}' = \boldsymbol{I}_n$. Now premultiply both sides of Equation (19.3) by $\boldsymbol{F}$ to obtain

$$\tilde{\boldsymbol{Y}} = \tilde{\boldsymbol{X}}\boldsymbol{\beta} + \tilde{\boldsymbol{U}}, \tag{19.42}$$

where $\tilde{\boldsymbol{Y}} = \boldsymbol{F}\boldsymbol{Y}$, $\tilde{\boldsymbol{X}} = \boldsymbol{F}\boldsymbol{X}$, and $\tilde{\boldsymbol{U}} = \boldsymbol{F}\boldsymbol{U}$.

The key insight of GLS is that, under the four GLS assumptions, the Gauss–Markov assumptions hold for the transformed regression in Equation (19.42). That is, by transforming all the variables by the matrix square root of the inverse of $\boldsymbol{\Omega}$, the regression errors in the transformed regression have a conditional mean of 0 and a covariance matrix that equals the identity matrix. To show this mathematically, first note that $E(\tilde{\boldsymbol{U}}|\tilde{\boldsymbol{X}}) = E(\boldsymbol{F}\boldsymbol{U}|\boldsymbol{F}\boldsymbol{X}) = \boldsymbol{F}E(\boldsymbol{U}|\boldsymbol{F}\boldsymbol{X}) = \boldsymbol{0}_n$ by the first GLS assumption [Equation (19.40)]. In addition, $E(\tilde{\boldsymbol{U}}\tilde{\boldsymbol{U}}'|\tilde{\boldsymbol{X}}) = E[(\boldsymbol{F}\boldsymbol{U})(\boldsymbol{F}\boldsymbol{U})'|\boldsymbol{F}\boldsymbol{X}] = \boldsymbol{F}E(\boldsymbol{U}\boldsymbol{U}'|\boldsymbol{F}\boldsymbol{X})\boldsymbol{F}' = \boldsymbol{F}\boldsymbol{\Omega}\boldsymbol{F}' = \boldsymbol{I}_n$, where the second equality follows because $(\boldsymbol{F}\boldsymbol{U})' = \boldsymbol{U}'\boldsymbol{F}'$ and the final equality follows from the definition of $\boldsymbol{F}$. It follows that the transformed regression model in Equation (19.42) satisfies the Gauss–Markov conditions in Key Concept 19.3.

The GLS estimator, $\widetilde{\boldsymbol{\beta}}^{GLS}$, is the OLS estimator of $\boldsymbol{\beta}$ in Equation (19.42); that is, $\widetilde{\boldsymbol{\beta}}^{GLS} = (\tilde{\boldsymbol{X}}'\tilde{\boldsymbol{X}})^{-1}(\tilde{\boldsymbol{X}}'\tilde{\boldsymbol{Y}})$. Because the transformed regression model satisfies the Gauss–Markov conditions, the GLS estimator is the best conditionally unbiased

estimator that is linear in $\tilde{\boldsymbol{Y}}$. But because $\tilde{\boldsymbol{Y}} = \boldsymbol{FY}$ and $\boldsymbol{F}$ is (here) assumed to be known and because $\boldsymbol{F}$ is invertible (because $\boldsymbol{\Omega}$ is positive definite), the class of estimators that are linear in $\tilde{\boldsymbol{Y}}$ is the same as the class of estimators that are linear in $\boldsymbol{Y}$. Thus the OLS estimator of $\boldsymbol{\beta}$ in Equation (19.42) is also the best conditionally unbiased estimator among estimators that are linear in $\boldsymbol{Y}$. In other words, under the GLS assumptions, the GLS estimator is BLUE.

The GLS estimator can be expressed directly in terms of $\boldsymbol{\Omega}$, so in principle there is no need to compute the square root matrix $\boldsymbol{F}$. Because $\tilde{\boldsymbol{X}} = \boldsymbol{FX}$ and $\tilde{\boldsymbol{Y}} = \boldsymbol{FY}$, $\tilde{\boldsymbol{\beta}}^{GLS} = (\boldsymbol{X}'\boldsymbol{F}'\boldsymbol{FX})^{-1}(\boldsymbol{X}'\boldsymbol{F}'\boldsymbol{FY})$. But $\boldsymbol{F}'\boldsymbol{F} = \boldsymbol{\Omega}^{-1}$, so

$$\tilde{\boldsymbol{\beta}}^{GLS} = (\boldsymbol{X}'\boldsymbol{\Omega}^{-1}\boldsymbol{X})^{-1}(\boldsymbol{X}'\boldsymbol{\Omega}^{-1}\boldsymbol{Y}). \tag{19.43}$$

In practice, $\boldsymbol{\Omega}$ is typically unknown, so the GLS estimator in Equation (19.43) typically cannot be computed and thus is sometimes called the **infeasible GLS** estimator. If, however, $\boldsymbol{\Omega}$ has a known functional form but the parameters of that function are unknown, then $\boldsymbol{\Omega}$ can be estimated, and a feasible version of the GLS estimator can be computed.

GLS When $\boldsymbol{\Omega}$ Contains Unknown Parameters

If $\boldsymbol{\Omega}$ is a known function of some parameters that in turn can be estimated, then these estimated parameters can be used to calculate an estimator of the covariance matrix $\boldsymbol{\Omega}$. For example, consider the time series application discussed following Equation (19.41), in which $\boldsymbol{\Omega}(\boldsymbol{X})$ does not depend on $\boldsymbol{X}$, $\boldsymbol{\Omega}_{ii} = \sigma_u^2$, $\boldsymbol{\Omega}_{ij} = \rho\sigma_u^2$ for $|i-j| = 1$, and $\boldsymbol{\Omega}_{ij} = 0$ for $|i-j| > 1$. Then $\boldsymbol{\Omega}$ has two unknown parameters, σ_u^2 and ρ. These parameters can be estimated using the residuals from a preliminary OLS regression; specifically, σ_u^2 can be estimated by $s_{\hat{u}}^2$, and ρ can be estimated by the sample correlation between all neighboring pairs of OLS residuals. These estimated parameters can in turn be used to compute an estimator of $\boldsymbol{\Omega}$, $\hat{\boldsymbol{\Omega}}$.

In general, suppose that you have an estimator $\hat{\boldsymbol{\Omega}}$ of $\boldsymbol{\Omega}$. Then the GLS estimator based on $\hat{\boldsymbol{\Omega}}$ is

$$\hat{\boldsymbol{\beta}}^{GLS} = (\boldsymbol{X}'\hat{\boldsymbol{\Omega}}^{-1}\boldsymbol{X})^{-1}(\boldsymbol{X}'\hat{\boldsymbol{\Omega}}^{-1}\boldsymbol{Y}). \tag{19.44}$$

The GLS estimator in Equation (19.44) is sometimes called the **feasible GLS** estimator because it can be computed if the covariance matrix contains some unknown parameters that can be estimated.

The Conditional Mean Zero Assumption and GLS

For the OLS estimator to be consistent, the first least squares assumption must hold; that is, $E(u_i|\boldsymbol{X}_i)$ must be 0. In contrast, the first GLS assumption is that $E(u_i|\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n) = 0$. In other words, the first OLS assumption is that the error for the i^{th} observation has a conditional mean of 0 given the values of the regressors for

that observation, whereas the first GLS assumption is that u_i has a conditional mean of 0 given the values of the regressors for *all* observations.

As discussed in Section 19.1, the assumptions that $E(u_i|\boldsymbol{X}_i) = 0$ and that sampling is i.i.d. together imply that $E(u_i|\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n) = 0$. Thus, when sampling is i.i.d., so that GLS is WLS, the first GLS assumption is implied by the first least squares assumption in Key Concept 19.1.

When sampling is not i.i.d., however, the first GLS assumption is not implied by the assumption that $E(u_i|\boldsymbol{X}_i) = 0$; that is, the first GLS assumption is stronger. Although the distinction between these two conditions might seem slight, it can be very important in applications to time series data. This distinction is discussed in Section 16.5 in the context of whether the regressor is "past and present" exogenous or "strictly" exogenous; the assumption that $E(u_i|\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n) = 0$ corresponds to strict exogeneity. Here, we discuss this distinction at a more general level using matrix notation. To do so, we focus on the case that $\boldsymbol{U}$ is homoskedastic, $\boldsymbol{\Omega}$ is known, and $\boldsymbol{\Omega}$ has nonzero off-diagonal elements.

The role of the first GLS assumption. To see the source of the difference between these assumptions, it is useful to contrast the consistency arguments for GLS and OLS.

We first sketch the argument for the consistency of the GLS estimator in Equation (19.43). Substituting Equation (19.3) into Equation (19.43), we have $\widetilde{\boldsymbol{\beta}}^{GLS} = \boldsymbol{\beta} + (\boldsymbol{X}'\boldsymbol{\Omega}^{-1}\boldsymbol{X}/n)^{-1}(\boldsymbol{X}'\boldsymbol{\Omega}^{-1}\boldsymbol{U}/n)$. Under the first GLS assumption, $E(\boldsymbol{X}'\boldsymbol{\Omega}^{-1}\boldsymbol{U}) = E[\boldsymbol{X}'\boldsymbol{\Omega}^{-1}E(\boldsymbol{U}|\boldsymbol{X})] = \boldsymbol{0}_n$. If in addition the variance of $\boldsymbol{X}'\boldsymbol{\Omega}^{-1}\boldsymbol{U}/n$ tends to 0 and $\boldsymbol{X}'\boldsymbol{\Omega}^{-1}\boldsymbol{X}/n \xrightarrow{p} \widetilde{\boldsymbol{Q}}$, where $\widetilde{\boldsymbol{Q}}$ is some invertible matrix, then $\widetilde{\boldsymbol{\beta}}^{GLS} \xrightarrow{p} \boldsymbol{\beta}$. Critically, when $\boldsymbol{\Omega}$ has off-diagonal elements, the term $\boldsymbol{X}'\boldsymbol{\Omega}^{-1}\boldsymbol{U} = \sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{X}_i(\boldsymbol{\Omega}^{-1})_{ij}u_j$ involves products of $\boldsymbol{X}_i$ and u_j for different i, j pairs, where $(\boldsymbol{\Omega}^{-1})_{ij}$ denotes the (i, j) element of $\boldsymbol{\Omega}^{-1}$. Thus, for $\boldsymbol{X}'\boldsymbol{\Omega}^{-1}\boldsymbol{U}$ to have a mean of 0, it is not enough that $E(u_i|\boldsymbol{X}_i) = 0$; rather, $E(u_i|\boldsymbol{X}_j)$ must equal 0 for all i, j pairs corresponding to nonzero values of $(\boldsymbol{\Omega}^{-1})_{ij}$. Depending on the covariance structure of the errors, only some of or all the elements of $(\boldsymbol{\Omega}^{-1})_{ij}$ might be nonzero. For example, if u_i follows a first-order autoregression (as discussed in Section 16.5), the only nonzero elements $(\boldsymbol{\Omega}^{-1})_{ij}$ are those for which $|i - j| \leq 1$. In general, however, all the elements of $\boldsymbol{\Omega}^{-1}$ can be nonzero, so, in general, for $\boldsymbol{X}'\boldsymbol{\Omega}^{-1}\boldsymbol{U}/n \xrightarrow{p} \boldsymbol{0}_{(k+1)\times 1}$ (and thus for $\widetilde{\boldsymbol{\beta}}^{GLS}$ to be consistent), we need that $E(\boldsymbol{U}|\boldsymbol{X}) = \boldsymbol{0}_n$; that is, the first GLS assumption must hold.

In contrast, recall the argument that the OLS estimator is consistent. Rewrite Equation (19.14) as $\hat{\boldsymbol{\beta}} = \boldsymbol{\beta} + (\boldsymbol{X}'\boldsymbol{X}/n)^{-1}\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{X}_i u_i$. If $E(u_i|\boldsymbol{X}_i) = 0$, then the term $\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{X}_i u_i$ has mean 0, and if this term has a variance that tends to 0, it converges in probability to 0. If in addition $\boldsymbol{X}'\boldsymbol{X}/n \xrightarrow{p} \boldsymbol{Q}_X$, then $\hat{\boldsymbol{\beta}} \xrightarrow{p} \boldsymbol{\beta}$.

Is the first GLS assumption restrictive? The first GLS assumption requires that the errors for the i^{th} observation be uncorrelated with the regressors for all other observations. This assumption is dubious in some time series applications. This issue is discussed in Section 16.6 in the context of an empirical example, the relationship

between the change in the price of a contract for future delivery of frozen orange concentrate and the weather in Florida. As explained there, the error term in the regression of price changes on the weather is plausibly uncorrelated with current and past values of the weather, so the first OLS assumption holds. However, this error term is plausibly correlated with future values of the weather, so the first GLS assumption does *not* hold.

This example illustrates a general phenomenon in economic time series data that arises when the value of a variable today is set in part based on expectations of the future: Those future expectations typically imply that the error term today depends on a forecast of the regressor tomorrow, which in turn is correlated with the actual value of the regressor tomorrow. For this reason, the first GLS assumption is, in fact, much stronger than the first OLS assumption. Accordingly, in some applications with economic time series data, the GLS estimator is not consistent even though the OLS estimator is.

19.7 Instrumental Variables and Generalized Method of Moments Estimation

This section provides an introduction to the theory of instrumental variables (IV) estimation and the asymptotic distribution of IV estimators. It is assumed throughout that the IV regression assumptions in Key Concepts 12.3 and 12.4 hold and, moreover, that the instruments are strong. These assumptions apply to cross-sectional data with i.i.d. observations. Under certain conditions, the results derived in this section are applicable to time series data as well, and the extension to time series data is briefly discussed at the end of this section. All asymptotic results in this section are developed under the assumption of strong instruments.

This section begins by presenting the IV regression model and the two stage least squares (TSLS) estimator and its asymptotic distribution in the general case of heteroskedasticity, all in matrix form. It is next shown that, in the special case of homoskedasticity, the TSLS estimator is asymptotically efficient among the class of IV estimators in which the instruments are linear combinations of the exogenous variables. Moreover, the J-statistic has an asymptotic chi-squared distribution in which the degrees of freedom equals the number of overidentifying restrictions. This section concludes with a discussion of efficient IV estimation and the test of overidentifying restrictions when the errors are heteroskedastic—a situation in which the efficient IV estimator is known as the efficient generalized method of moments (GMM) estimator [Hansen (1983)].

The IV Estimator in Matrix Form

In this section, we let $\boldsymbol{X}$ denote the $n \times (k + r + 1)$ matrix of the regressors in the equation of interest, so $\boldsymbol{X}$ contains the included endogenous regressors (the X's in Key Concept 12.1) *and* the included exogenous regressors (the W's in Key Concept 12.1).

That is, in the notation of Key Concept 12.1, the i^{th} row of $\boldsymbol{X}$ is $\boldsymbol{X}_i' = (1 \quad X_{1i} \quad X_{2i} \quad \ldots \quad X_{ki} \quad W_{1i} \quad W_{2i} \quad \ldots \quad W_{ri})$. Also, let $\boldsymbol{Z}$ denote the $n \times (m + r + 1)$ matrix of all the exogenous regressors, both those included in the equation of interest (the W's) *and* those excluded from the equation of interest (the instruments). That is, in the notation of Key Concept 12.1, the i^{th} row of $\boldsymbol{Z}$ is $\boldsymbol{Z}_i' = (1 \quad Z_{1i} \quad Z_{2i} \quad \ldots \quad Z_{mi} \quad W_{1i} \quad W_{2i} \quad \ldots \quad W_{ri})$.

With this notation, the IV regression model of Key Concept 12.1, written in matrix form, is

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{U}, \tag{19.45}$$

where $\boldsymbol{U}$ is the $n \times 1$ vector of errors in the equation of interest, with i^{th} element u_i.

The matrix $\boldsymbol{Z}$ consists of all the exogenous regressors, so under the IV regression assumptions in Key Concept 12.4,

$$E(\boldsymbol{Z}_i u_i) = \boldsymbol{0} \quad \text{(instrument exogeneity)}. \tag{19.46}$$

Because there are k included endogenous regressors, the first stage regression consists of k equations.

The TSLS estimator. The TSLS estimator is the instrumental variables estimator in which the instruments are the predicted values of $\boldsymbol{X}$ based on OLS estimation of the first-stage regression. Let $\hat{\boldsymbol{X}}$ denote this matrix of predicted values, so that the i^{th} row of $\hat{\boldsymbol{X}}$ is $(\hat{X}_{1i} \quad \hat{X}_{2i} \quad \ldots \quad \hat{X}_{ki} \quad W_{1i} \quad W_{2i} \quad \ldots \quad W_{ri})$, where $\hat{X}_{1i}$ is the predicted value from the regression of X_{1i} on $\boldsymbol{Z}$ and so forth. Because the W's are contained in $\boldsymbol{Z}$, the predicted value from a regression of W_{1i} on $\boldsymbol{Z}$ is just W_{1i} and so forth, so $\hat{\boldsymbol{X}} = \boldsymbol{P}_{\boldsymbol{Z}}\boldsymbol{X}$, where $\boldsymbol{P}_{\boldsymbol{Z}} = \boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'$ [see Equation (19.27)]. Accordingly, the TSLS estimator is

$$\hat{\boldsymbol{\beta}}^{TSLS} = (\hat{\boldsymbol{X}}'\hat{\boldsymbol{X}})^{-1}\hat{\boldsymbol{X}}'\boldsymbol{Y}. \tag{19.47}$$

Because $\hat{\boldsymbol{X}} = \boldsymbol{P}_{\boldsymbol{Z}}\boldsymbol{X}$, $\hat{\boldsymbol{X}}'\hat{\boldsymbol{X}} = \boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{Z}}\boldsymbol{X}$, and $\hat{\boldsymbol{X}}'\boldsymbol{Y} = \boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{Z}}\boldsymbol{Y}$, the TSLS estimator can be rewritten as

$$\hat{\boldsymbol{\beta}}^{TSLS} = (\boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{Z}}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{Z}}\boldsymbol{Y}. \tag{19.48}$$

Asymptotic Distribution of the TSLS Estimator

Substituting Equation (19.45) into Equation (19.48), rearranging, and multiplying by $\sqrt{n}$ yields the expression for the centered and scaled TSLS estimator:

$$\begin{aligned}\sqrt{n}(\hat{\boldsymbol{\beta}}^{TSLS} - \boldsymbol{\beta}) &= \left(\frac{\boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{Z}}\boldsymbol{X}}{n}\right)^{-1}\frac{\boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{Z}}\boldsymbol{U}}{\sqrt{n}} \\ &= \left[\frac{\boldsymbol{X}'\boldsymbol{Z}}{n}\left(\frac{\boldsymbol{Z}'\boldsymbol{Z}}{n}\right)^{-1}\frac{\boldsymbol{Z}'\boldsymbol{X}}{n}\right]^{-1}\left[\frac{\boldsymbol{X}'\boldsymbol{Z}}{n}\left(\frac{\boldsymbol{Z}'\boldsymbol{Z}}{n}\right)^{-1}\frac{\boldsymbol{Z}'\boldsymbol{U}}{\sqrt{n}}\right],\end{aligned} \tag{19.49}$$

where the second equality uses the definition of $\boldsymbol{P}_{\boldsymbol{Z}}$. Under the IV regression assumptions, $\boldsymbol{X}'\boldsymbol{Z}/n \xrightarrow{p} \boldsymbol{Q}_{\boldsymbol{XZ}}$ and $\boldsymbol{Z}'\boldsymbol{Z}/n \xrightarrow{p} \boldsymbol{Q}_{\boldsymbol{ZZ}}$, where $\boldsymbol{Q}_{\boldsymbol{XZ}} = E(\boldsymbol{X}_i\boldsymbol{Z}_i')$ and $\boldsymbol{Q}_{\boldsymbol{ZZ}} = E(\boldsymbol{Z}_i\boldsymbol{Z}_i')$. In addition, under the IV regression assumptions, $\boldsymbol{Z}_i u_i$ is i.i.d. with

mean 0 [Equation (19.46)] and a positive definite covariance matrix, so its sum, divided by $\sqrt{n}$, satisfies the conditions of the multivariate central limit theorem (Key Concept 19.2) and

$$\boldsymbol{Z}'\boldsymbol{U}/\sqrt{n} \xrightarrow{d} \boldsymbol{\Psi}_{ZU}, \text{ where } \boldsymbol{\Psi}_{\boldsymbol{ZU}} \sim N(\boldsymbol{0}, \boldsymbol{H}), \boldsymbol{H} = E(\boldsymbol{Z}_i\boldsymbol{Z}_i' u_i^2) \tag{19.50}$$

and $\boldsymbol{\Psi}_{\boldsymbol{ZU}}$ is $(m + r + 1) \times 1$.

Application of Equation (19.50) and of the limits $\boldsymbol{X}'\boldsymbol{Z}/n \xrightarrow{p} \boldsymbol{Q}_{\boldsymbol{XZ}}$ and $\boldsymbol{Z}'\boldsymbol{Z}/n \xrightarrow{p} \boldsymbol{Q}_{\boldsymbol{ZZ}}$ to Equation (19.49) yields the result that, under the IV regression assumptions, the TSLS estimator is asymptotically normally distributed:

$$\sqrt{n}\,(\hat{\boldsymbol{\beta}}^{TSLS} - \boldsymbol{\beta}) \xrightarrow{d} (\boldsymbol{Q}_{\boldsymbol{XZ}}\boldsymbol{Q}_{\boldsymbol{ZZ}}^{-1}\boldsymbol{Q}_{\boldsymbol{ZX}})^{-1}\boldsymbol{Q}_{\boldsymbol{XZ}}\boldsymbol{Q}_{\boldsymbol{ZZ}}^{-1}\boldsymbol{\Psi}_{\boldsymbol{ZU}} \sim N(\boldsymbol{0}, \boldsymbol{\Sigma}^{TSLS}), \tag{19.51}$$

where

$$\boldsymbol{\Sigma}^{TSLS} = (\boldsymbol{Q}_{\boldsymbol{XZ}}\boldsymbol{Q}_{\boldsymbol{ZZ}}^{-1}\boldsymbol{Q}_{\boldsymbol{ZX}})^{-1}\boldsymbol{Q}_{\boldsymbol{XZ}}\boldsymbol{Q}_{\boldsymbol{ZZ}}^{-1}\boldsymbol{H}\boldsymbol{Q}_{\boldsymbol{ZZ}}^{-1}\boldsymbol{Q}_{\boldsymbol{ZX}}(\boldsymbol{Q}_{\boldsymbol{XZ}}\boldsymbol{Q}_{\boldsymbol{ZZ}}^{-1}\boldsymbol{Q}_{\boldsymbol{ZX}})^{-1}, \tag{19.52}$$

where $\boldsymbol{H}$ is defined in Equation (19.50).

Standard errors for TSLS. The formula in Equation (19.52) is daunting. Nevertheless, it provides a way to estimate $\boldsymbol{\Sigma}^{TSLS}$ by substituting sample moments for the population moments. The resulting variance estimator is

$$\hat{\boldsymbol{\Sigma}}^{TSLS} = (\hat{\boldsymbol{Q}}_{\boldsymbol{XZ}}\hat{\boldsymbol{Q}}_{\boldsymbol{ZZ}}^{-1}\hat{\boldsymbol{Q}}_{\boldsymbol{ZX}})^{-1}\hat{\boldsymbol{Q}}_{\boldsymbol{XZ}}\hat{\boldsymbol{Q}}_{\boldsymbol{ZZ}}^{-1}\hat{\boldsymbol{H}}\hat{\boldsymbol{Q}}_{\boldsymbol{ZZ}}^{-1}\hat{\boldsymbol{Q}}_{\boldsymbol{ZX}}(\hat{\boldsymbol{Q}}_{\boldsymbol{XZ}}\hat{\boldsymbol{Q}}_{\boldsymbol{ZZ}}^{-1}\hat{\boldsymbol{Q}}_{\boldsymbol{ZX}})^{-1}, \tag{19.53}$$

where $\hat{\boldsymbol{Q}}_{\boldsymbol{XZ}} = \boldsymbol{X}'\boldsymbol{Z}/n, \hat{\boldsymbol{Q}}_{\boldsymbol{ZZ}} = \boldsymbol{Z}'\boldsymbol{Z}/n, \hat{\boldsymbol{Q}}_{\boldsymbol{ZX}} = \boldsymbol{Z}'\boldsymbol{X}/n$, and

$$\hat{\boldsymbol{H}} = \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{Z}_i\boldsymbol{Z}_i\hat{u}_i^2, \text{ where } \hat{\boldsymbol{U}} = \boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}^{TSLS}, \tag{19.54}$$

so that $\hat{\boldsymbol{U}}$ is the vector of TSLS residuals, and where $\hat{u}_i$ is the i^{th} element of that vector (the TSLS residual for the i^{th} observation).

The TSLS standard errors are the square roots of the diagonal elements of $\hat{\boldsymbol{\Sigma}}^{TSLS}/n$.

Properties of TSLS When the Errors Are Homoskedastic

If the errors are homoskedastic, then the TSLS estimator is asymptotically efficient among the class of IV estimators in which the instruments are linear combinations of the rows of $\boldsymbol{Z}$. This result is the IV counterpart to the Gauss–Markov theorem and constitutes an important justification for using TSLS.

The TSLS distribution under homoskedasticity. If the errors are homoskedastic—that is, if $E(u_i^2|\boldsymbol{Z}_i) = \sigma_u^2$—then $\boldsymbol{H} = E(\boldsymbol{Z}_i\boldsymbol{Z}_i' u_i^2) = E[E(\boldsymbol{Z}_i\boldsymbol{Z}_i' u_i^2|\boldsymbol{Z}_i)] = E[\boldsymbol{Z}_i\boldsymbol{Z}_i' E(u_i^2|\boldsymbol{Z}_i)] = \boldsymbol{Q}_{\boldsymbol{ZZ}}\sigma_u^2$. In this case, the variance of the asymptotic distribution of the TSLS estimator in Equation (19.52) simplifies to

$$\boldsymbol{\Sigma}^{TSLS} = (\boldsymbol{Q}_{\boldsymbol{XZ}}\boldsymbol{Q}_{\boldsymbol{ZZ}}^{-1}\boldsymbol{Q}_{\boldsymbol{ZX}})^{-1}\sigma_u^2 \quad \text{(homoskedasticity only)}. \tag{19.55}$$

The homoskedasticity-only estimator of the TSLS variance matrix is

$$\widetilde{\boldsymbol{\Sigma}}^{TSLS} = (\hat{\boldsymbol{Q}}_{\boldsymbol{XZ}}\hat{\boldsymbol{Q}}_{\boldsymbol{ZZ}}^{-1}\hat{\boldsymbol{Q}}_{\boldsymbol{ZX}})^{-1}\hat{\sigma}_u^2, \text{ where } \hat{\sigma}_u^2 = \frac{\hat{\boldsymbol{U}}'\hat{\boldsymbol{U}}}{n-k-r-1} \quad \text{(homoskedasticity only)}, \tag{19.56}$$

and the homoskedasticity-only TSLS standard errors are the square roots of the diagonal elements of $\widetilde{\boldsymbol{\Sigma}}^{TSLS}/n$.

The class of IV estimators that use linear combinations of Z. The class of IV estimators that use linear combinations of $\boldsymbol{Z}$ as instruments can be generated in two equivalent ways. Both start with the same moment equation: Under the assumption of instrument exogeneity, the errors $\boldsymbol{U} = \boldsymbol{Y} - \boldsymbol{X\beta}$ are uncorrelated with the exogenous regressors; that is, at the true value of $\boldsymbol{\beta}$, Equation (19.46) implies that

$$E[(\boldsymbol{Y} - \boldsymbol{X\beta})'\boldsymbol{Z}] = 0. \tag{19.57}$$

Equation (19.57) constitutes a system of $m + r + 1$ equations involving the $k + r + 1$ unknown elements of $\boldsymbol{\beta}$. When $m > k$, these equations are redundant in the sense that all are satisfied at the true value of $\boldsymbol{\beta}$. When these population moments are replaced by their sample moments, the system of equations $(\boldsymbol{Y} - \boldsymbol{Xb})'\boldsymbol{Z} = 0$ can be solved for $\boldsymbol{b}$ when there is exact identification ($m = k$). This value of $\boldsymbol{b}$ is the IV estimator of $\boldsymbol{\beta}$. However, when there is overidentification ($m > k$), the equations in the system cannot be simultaneously satisfied by the same value of $\boldsymbol{b}$ because of sampling variation—there are more equations than unknowns—and, in general, this system does not have a solution.

The first approach to the problem of estimating $\boldsymbol{\beta}$ when there is overidentification is to trade off the desire to satisfy each equation by minimizing a quadratic form involving all the equations. Specifically, let $\boldsymbol{A}$ be an $(m + r + 1) \times (m + r + 1)$ symmetric positive semidefinite weight matrix, and let $\hat{\boldsymbol{\beta}}_{\boldsymbol{A}}^{IV}$ denote the estimator that minimizes

$$\min_b (\boldsymbol{Y} - \boldsymbol{Xb})'\boldsymbol{ZAZ}'(\boldsymbol{Y} - \boldsymbol{Xb}). \tag{19.58}$$

The solution to this minimization problem is found by taking the derivative of the objective function with respect to $\boldsymbol{b}$, setting the result equal to 0, and rearranging. Doing so yields $\hat{\boldsymbol{\beta}}_{\boldsymbol{A}}^{IV}$, the IV estimator based on the weight matrix $\boldsymbol{A}$:

$$\hat{\boldsymbol{\beta}}_{\boldsymbol{A}}^{IV} = (\boldsymbol{X}'\boldsymbol{ZAZ}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{ZAZ}'\boldsymbol{Y}. \tag{19.59}$$

Comparison of Equations (19.59) and (19.48) shows that the TSLS estimator is the IV estimator with $\boldsymbol{A} = (\boldsymbol{Z}'\boldsymbol{Z})^{-1}$. That is, TSLS is the solution of the minimization problem in Equation (19.58) with $\boldsymbol{A} = (\boldsymbol{Z}'\boldsymbol{Z})^{-1}$.

The calculations leading to Equations (19.51) and (19.52), applied to $\hat{\boldsymbol{\beta}}_A^{IV}$, show that

$$\sqrt{n}(\hat{\boldsymbol{\beta}}_A^{IV} - \boldsymbol{\beta}) \xrightarrow{d} N(\mathbf{0}, \boldsymbol{\Sigma}_A^{IV}), \text{ where}$$
$$\boldsymbol{\Sigma}_A^{IV} = (\boldsymbol{Q}_{XZ}\boldsymbol{A}\boldsymbol{Q}_{ZX})^{-1}\boldsymbol{Q}_{XZ}\boldsymbol{A}\boldsymbol{H}\boldsymbol{A}\boldsymbol{Q}_{ZX}(\boldsymbol{Q}_{XZ}\boldsymbol{A}\boldsymbol{Q}_{ZX})^{-1}. \tag{19.60}$$

The second way to generate the class of IV estimators that use linear combinations of $\boldsymbol{Z}$ is to consider IV estimators in which the instruments are $\boldsymbol{ZB}$, where $\boldsymbol{B}$ is an $(m + r + 1) \times (k + r + 1)$ matrix with full column rank. Then the system of $(k + r + 1)$ equations, $(\boldsymbol{Y} - \boldsymbol{Xb})'\boldsymbol{ZB} = 0$, can be solved uniquely for the $(k + r + 1)$ unknown elements of $\boldsymbol{b}$. Solving these equations for $\boldsymbol{b}$ yields $\hat{\boldsymbol{\beta}}^{IV} = (\boldsymbol{B}'\boldsymbol{Z}'\boldsymbol{X})^{-1}(\boldsymbol{B}'\boldsymbol{Z}'\boldsymbol{Y})$, and substitution of $\boldsymbol{B} = \boldsymbol{AZ}'\boldsymbol{X}$ into this expression yields Equation (19.59).

Thus the two approaches to defining IV estimators that are linear combinations of the instruments yield the same family of IV estimators. It is conventional to work with the first approach, in which the IV estimator solves the quadratic minimization problem in Equation (19.58), and that is the approach taken here.

Asymptotic efficiency of TSLS under homoskedasticity. If the errors are homoskedastic, then $\boldsymbol{H} = \boldsymbol{Q}_{ZZ}\sigma_u^2$, and the expression for $\boldsymbol{\Sigma}_A^{IV}$ in Equation (19.60) becomes

$$\boldsymbol{\Sigma}_A^{IV} = (\boldsymbol{Q}_{XZ}\boldsymbol{A}\boldsymbol{Q}_{ZX})^{-1}\boldsymbol{Q}_{XZ}\boldsymbol{A}\boldsymbol{Q}_{ZZ}\boldsymbol{A}\boldsymbol{Q}_{ZX}(\boldsymbol{Q}_{XZ}\boldsymbol{A}\boldsymbol{Q}_{ZX})^{-1}\sigma_u^2. \tag{19.61}$$

To show that TSLS is asymptotically efficient among the class of estimators that are linear combinations of $\boldsymbol{Z}$ when the errors are homoskedastic, we need to show that, under homoskedasticity,

$$\boldsymbol{c}'\boldsymbol{\Sigma}_A^{IV}\boldsymbol{c} \geq \boldsymbol{c}'\boldsymbol{\Sigma}^{TSLS}\boldsymbol{c} \tag{19.62}$$

for all positive semidefinite matrices $\boldsymbol{A}$ and all $(k + r + 1) \times 1$ vectors $\boldsymbol{c}$, where $\boldsymbol{\Sigma}^{TSLS} = (\boldsymbol{Q}_{XZ}\boldsymbol{Q}_{ZZ}^{-1}\boldsymbol{Q}_{ZX})^{-1}\sigma_u^2$ [Equation (19.55)]. The inequality (19.62), which is proven in Appendix 19.6, is the same efficiency criterion as is used in the multivariate Gauss–Markov theorem in Key Concept 19.3. Consequently, TSLS is the efficient IV estimator under homoskedasticity among the class of estimators in which the instruments are linear combinations of $\boldsymbol{Z}$.

The J-statistic under homoskedasticity. The J-statistic (Key Concept 12.6) tests the null hypothesis that all the overidentifying restrictions hold against the alternative that some or all of them do not hold.

The idea of the J-statistic is that, if the overidentifying restrictions hold, u_i will be uncorrelated with the instruments, and thus a regression of $\boldsymbol{U}$ on $\boldsymbol{Z}$ will have population regression coefficients that all equal 0. In practice, $\boldsymbol{U}$ is not observed, but it can be estimated by the TSLS residuals $\hat{\boldsymbol{U}}$, so a regression of $\hat{\boldsymbol{U}}$ on $\boldsymbol{Z}$ should yield statistically insignificant coefficients. Accordingly, the TSLS J-statistic is the homoskedasticity-only F-statistic testing the hypothesis that the coefficients on $\boldsymbol{Z}$ are all 0, in the regression of $\hat{\boldsymbol{U}}$ on $\boldsymbol{Z}$, multiplied by $(m + r + 1)$ so that the F-statistic is in its asymptotic chi-squared form.

An explicit formula for the J-statistic can be obtained using Equation (7.13) for the homoskedasticity-only F-statistic. The unrestricted regression is the regression of $\hat{\boldsymbol{U}}$ on the $m + r + 1$ regressors $\boldsymbol{Z}$, and the restricted regression has no regressors. Thus, in the notation of Equation (7.13), $SSR_{unrestricted} = \hat{\boldsymbol{U}}'\boldsymbol{M}_{\boldsymbol{Z}}\hat{\boldsymbol{U}}$, and $SSR_{restricted} = \hat{\boldsymbol{U}}'\hat{\boldsymbol{U}}$, so $SSR_{restricted} - SSR_{unrestricted} = \hat{\boldsymbol{U}}'\hat{\boldsymbol{U}} - \hat{\boldsymbol{U}}'\boldsymbol{M}_{\boldsymbol{Z}}\hat{\boldsymbol{U}} = \hat{\boldsymbol{U}}'\boldsymbol{P}_{\boldsymbol{Z}}\hat{\boldsymbol{U}}$ and the J-statistic is

$$J = \frac{\hat{\boldsymbol{U}}'\boldsymbol{P}_{\boldsymbol{Z}}\hat{\boldsymbol{U}}}{\hat{\boldsymbol{U}}'\boldsymbol{M}_{\boldsymbol{Z}}\hat{\boldsymbol{U}}/(n - m - r - 1)}. \tag{19.63}$$

The method for computing the J-statistic described in Key Concept 12.6 entails testing only the hypothesis that the coefficients on the excluded instruments are 0. Although these two methods have different computational steps, they produce identical J-statistics (Exercise 19.14).

It is shown in Appendix 19.6 that, under the null hypothesis that $E(u_i\boldsymbol{Z}_i) = 0$,

$$J \xrightarrow{d} \chi^2_{m-k}. \tag{19.64}$$

Generalized Method of Moments Estimation in Linear Models

If the errors are heteroskedastic, then the TSLS estimator is no longer efficient among the class of IV estimators that use linear combinations of $\boldsymbol{Z}$ as instruments. The efficient estimator in this case is known as the efficient generalized method of moments (GMM) estimator. In addition, if the errors are heteroskedastic, then the J-statistic as defined in Equation (19.63) no longer has a chi-squared distribution. However, an alternative formulation of the J-statistic, constructed using the efficient GMM estimator, does have a chi-squared distribution with $m - k$ degrees of freedom.

These results parallel the results for the estimation of the usual regression model with exogenous regressors and heteroskedastic errors: If the errors are heteroskedastic, then the OLS estimator is not efficient among estimators that are linear in $\boldsymbol{Y}$ (the Gauss–Markov conditions are not satisfied), and the homoskedasticity-only F-statistic no longer has an F distribution, even in large samples. In the regression model with exogenous regressors and heteroskedasticity, the efficient estimator is weighted least squares; in the IV regression model with heteroskedasticity, the efficient estimator uses a different weighting matrix than TSLS, and the resulting estimator is the efficient GMM estimator.

GMM estimation. **Generalized method of moments (GMM)** estimation is a general method for the estimation of the parameters of linear or nonlinear models, in which the parameters are chosen to provide the best fit to multiple equations, each of which sets a sample moment to 0. These equations, which in the context of GMM are called moment conditions, typically cannot all be satisfied simultaneously. The GMM estimator trades off the desire to satisfy each of the equations by minimizing a quadratic objective function.

In the linear IV regression model with exogenous variables $\boldsymbol{Z}$, the class of GMM estimators consists of all the estimators that are solutions to the quadratic minimization problem in Equation (19.58). Thus the class of GMM estimators based on the full set of instruments $\boldsymbol{Z}$ with different-weight matrices $\boldsymbol{A}$ is the same as the class of IV estimators in which the instruments are linear combinations of $\boldsymbol{Z}$. In the linear IV regression model, GMM is just another name for the class of estimators we have been studying—that is, estimators that solve Equation (19.58).

The asymptotically efficient GMM estimator. Among the class of GMM estimators, the **efficient GMM** estimator is the GMM estimator with the smallest asymptotic variance matrix [where the smallest variance matrix is defined as in Equation (19.62)]. Thus the result in Equation (19.62) can be restated as saying that TSLS is the efficient GMM estimator in the linear model when the errors are homoskedastic.

To motivate the expression for the efficient GMM estimator when the errors are heteroskedastic, recall that when the errors are homoskedastic, $\boldsymbol{H}$ [the variance matrix of $\boldsymbol{Z}_i u_i$; see Equation (19.50)] equals $\boldsymbol{Q}_{\boldsymbol{ZZ}}\sigma_u^2$, and the asymptotically efficient weight matrix is obtained by setting $\boldsymbol{A} = (\boldsymbol{Z}'\boldsymbol{Z})^{-1}$, which yields the TSLS estimator. In large samples, using the weight matrix $\boldsymbol{A} = (\boldsymbol{Z}'\boldsymbol{Z})^{-1}$ is equivalent to using $\boldsymbol{A} = (\boldsymbol{Q}_{\boldsymbol{ZZ}}\sigma_u^2)^{-1} = \boldsymbol{H}^{-1}$. This interpretation of the TSLS estimator suggests that, by analogy, the efficient IV estimator under heteroskedasticity can be obtained by setting $\boldsymbol{A} = \boldsymbol{H}^{-1}$ and solving

$$\min_b (\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{b})'\boldsymbol{Z}\boldsymbol{H}^{-1}\boldsymbol{Z}'(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{b}). \tag{19.65}$$

This analogy is correct: The solution to the minimization problem in Equation (19.65) is the efficient GMM estimator. Let $\widetilde{\boldsymbol{\beta}}^{Eff.GMM}$ denote the solution to the minimization problem in Equation (19.65). By Equation (19.59), this estimator is

$$\widetilde{\boldsymbol{\beta}}^{Eff.GMM} = (\boldsymbol{X}'\boldsymbol{Z}\boldsymbol{H}^{-1}\boldsymbol{Z}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Z}\boldsymbol{H}^{-1}\boldsymbol{Z}'\boldsymbol{Y}. \tag{19.66}$$

The asymptotic distribution of $\widetilde{\boldsymbol{\beta}}^{Eff.GMM}$ is obtained by substituting $\boldsymbol{A} = \boldsymbol{H}^{-1}$ into Equation (19.60) and simplifying; thus

$$\begin{aligned} &\sqrt{n}(\widetilde{\boldsymbol{\beta}}^{Eff.GMM} - \boldsymbol{\beta}) \xrightarrow{d} N(\boldsymbol{0}, \boldsymbol{\Sigma}^{Eff.GMM}), \\ &\text{where } \boldsymbol{\Sigma}^{Eff.GMM} = (\boldsymbol{Q}_{\boldsymbol{XZ}}\boldsymbol{H}^{-1}\boldsymbol{Q}_{\boldsymbol{ZX}})^{-1}. \end{aligned} \tag{19.67}$$

The result that $\widetilde{\boldsymbol{\beta}}^{Eff.GMM}$ is the efficient GMM estimator is proven by showing that $\boldsymbol{c}'\boldsymbol{\Sigma}_A^{IV}\boldsymbol{c} \geq \boldsymbol{c}'\boldsymbol{\Sigma}^{Eff.GMM}\boldsymbol{c}$ for all vectors $\boldsymbol{c}$, where $\boldsymbol{\Sigma}_A^{IV}$ is given in Equation (19.60). The proof of this result is given in Appendix 19.6.

Feasible efficient GMM estimation. The GMM estimator defined in Equation (19.66) is not a feasible estimator because it depends on the unknown variance matrix $\boldsymbol{H}$. However, a feasible efficient GMM estimator can be computed by

substituting a consistent estimator of $\boldsymbol{H}$ into the minimization problem of Equation (19.65) or, equivalently, by substituting a consistent estimator of $\boldsymbol{H}$ into the formula for $\hat{\boldsymbol{\beta}}^{Eff.GMM}$ in Equation (19.66).

The efficient GMM estimator can be computed in two steps. In the first step, estimate $\boldsymbol{\beta}$ using any consistent estimator. Use this estimator of $\boldsymbol{\beta}$ to compute the residuals from the equation of interest, and then use these residuals to compute an estimator of $\boldsymbol{H}$. In the second step, use this estimator of $\boldsymbol{H}$ to estimate the optimal weight matrix $\boldsymbol{H}^{-1}$ and to compute the efficient GMM estimator. To be concrete, in the linear IV regression model, it is natural to use the TSLS estimator in the first step and to use the TSLS residuals to estimate $\boldsymbol{H}$. If TSLS is used in the first step, then the feasible efficient GMM estimator computed in the second step is

$$\hat{\boldsymbol{\beta}}^{Eff.GMM} = (\boldsymbol{X}'\boldsymbol{Z}\hat{\boldsymbol{H}}^{-1}\boldsymbol{Z}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Z}\hat{\boldsymbol{H}}^{-1}\boldsymbol{Z}'\boldsymbol{Y}, \tag{19.68}$$

where $\hat{\boldsymbol{H}}$ is given in Equation (19.54).

Because $\hat{\boldsymbol{H}} \xrightarrow{p} \boldsymbol{H}$, $\sqrt{n}(\hat{\boldsymbol{\beta}}^{Eff.GMM} - \tilde{\boldsymbol{\beta}}^{Eff.GMM}) \xrightarrow{p} 0$ (Exercise 19.12), and

$$\sqrt{n}(\hat{\boldsymbol{\beta}}^{Eff.GMM} - \boldsymbol{\beta}) \xrightarrow{d} N(0, \boldsymbol{\Sigma}^{Eff.GMM}), \tag{19.69}$$

where $\boldsymbol{\Sigma}^{Eff.GMM} = (\boldsymbol{Q}_{XZ}\boldsymbol{H}^{-1}\boldsymbol{Q}_{ZX})^{-1}$ [Equation (19.67)]. That is, the feasible two-step estimator $\hat{\boldsymbol{\beta}}^{Eff.GMM}$ in Equation (19.68) is, asymptotically, the efficient GMM estimator.

The heteroskedasticity-robust J-statistic. The **heteroskedasticity-robust *J*-statistic**, also known as the **GMM *J*-statistic**, is the counterpart of the TSLS-based *J*-statistic, computed using the efficient GMM estimator and weight function. That is, the GMM *J*-statistic is given by

$$J^{GMM} = (\boldsymbol{Z}'\hat{\boldsymbol{U}}^{GMM})'\hat{\boldsymbol{H}}^{-1}(\boldsymbol{Z}'\hat{\boldsymbol{U}}^{GMM})/n, \tag{19.70}$$

where $\hat{\boldsymbol{U}}^{GMM} = \boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}^{Eff.GMM}$ are the residuals from the equation of interest, estimated by (feasible) efficient GMM, and $\hat{\boldsymbol{H}}^{-1}$ is the weight matrix used to compute $\hat{\boldsymbol{\beta}}^{Eff.GMM}$.

Under the null hypothesis $E(\boldsymbol{Z}_i u_i) = \boldsymbol{0}$, $J^{GMM} \xrightarrow{d} \chi^2_{m-k}$ (see Appendix 19.6).

GMM with time series data. The results in this section were derived under the IV regression assumptions for cross–sectional data. In many applications, however, these results extend to time series applications of IV regression and GMM. Although a formal mathematical treatment of GMM with time series data is beyond the scope of this book (for such a treatment, see Hayashi, 2000, Chapter 6), we nevertheless will summarize the key ideas of GMM estimation with time series data. This summary assumes familiarity with the material in Chapters 14 and 16. For this discussion, it is assumed that the variables are stationary.

It is useful to distinguish between two types of applications: applications in which the error term u_t is serially correlated and applications in which u_t is serially uncorrelated. If the error term u_t is serially correlated, then the asymptotic distribution of the GMM estimator continues to be normally distributed, but the formula for $\boldsymbol{H}$ in Equation (19.50) is no longer correct. Instead, the correct expression for $\boldsymbol{H}$ depends on the autocovariances of $\boldsymbol{Z}_t u_t$ and is analogous to the formula given in Equation (16.14) for the variance of the OLS estimator when the error term is serially correlated. The efficient GMM estimator is still constructed using a consistent estimator of $\boldsymbol{H}$; however, that consistent estimator must be computed using the HAC methods discussed in Chapter 16.

If $\boldsymbol{Z}_t u_t$ is not serially correlated, then HAC estimation of $\boldsymbol{H}$ is unnecessary, and the formulas presented in this section all extend to time series GMM applications. In modern applications to finance and macroeconometrics, it is common to encounter models in which the error term represents an unexpected or unforecastable disturbance, in which case the model typically implies that $\boldsymbol{Z}_t u_t$ is serially uncorrelated. For example, consider a model with a single included endogenous variable and no included exogenous variables so that the equation of interest is $Y_t = \beta_0 + \beta_1 X_t + u_t$. Suppose that an economic theory implies that u_t is unpredictable given past information. Then the theory implies the moment condition

$$E(u_t | Y_{t-1}, X_{t-1}, Z_{t-1}, Y_{t-2}, X_{t-2}, Z_{t-2}, \ldots) = 0, \tag{19.71}$$

where Z_{t-1} is the lagged value of some other variable. The moment condition in Equation (19.71) implies that all the lagged variables $Y_{t-1}, X_{t-1}, Z_{t-1}, Y_{t-2}, X_{t-2}, Z_{t-2}, \ldots$ are candidates for being valid instruments (they satisfy the exogeneity condition). Moreover, because $u_{t-1} = Y_{t-1} - \beta_0 - \beta_1 X_{t-1}$, the moment condition in Equation (19.71) is equivalent to $E(u_t | u_{t-1}, X_{t-1}, Z_{t-1}, u_{t-2}, X_{t-2}, Z_{t-2}, \ldots) = 0$. Because u_t is serially uncorrelated, HAC estimation of $\boldsymbol{H}$ is unnecessary. The theory of GMM presented in this section, including efficient GMM estimation and the GMM J-statistic, therefore applies directly to time series applications with moment conditions of the form in Equation (19.71), under the hypothesis that the moment condition in Equation (19.71) is, in fact, correct.

Summary

1. The linear multiple regression model in matrix form is $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{U}$, where $\boldsymbol{Y}$ is the $n \times 1$ vector of observations on the dependent variable, $\boldsymbol{X}$ is the $n \times (k + 1)$ matrix of n observations on the $k + 1$ regressors (including a constant), $\boldsymbol{\beta}$ is the $k + 1$ vector of unknown parameters, and $\boldsymbol{U}$ is the $n \times 1$ vector of error terms.
2. The OLS estimator is $\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y}$. Under the first four least squares assumptions in Key Concept 19.1, $\hat{\boldsymbol{\beta}}$ is consistent and asymptotically normally distributed. If in addition the errors are homoskedastic, then the conditional variance of $\hat{\boldsymbol{\beta}}$ is $\text{var}(\hat{\boldsymbol{\beta}} | \boldsymbol{X}) = \sigma_u^2(\boldsymbol{X}'\boldsymbol{X})^{-1}$.

3. General linear restrictions on $\boldsymbol{\beta}$ can be written as the q equations $\boldsymbol{R\beta} = \boldsymbol{r}$, and this formulation can be used to test joint hypotheses involving multiple coefficients or to construct confidence sets for elements of $\boldsymbol{\beta}$.
4. When the regression errors are i.i.d. and normally distributed, conditional on $\boldsymbol{X}$, $\hat{\boldsymbol{\beta}}$ has an exact normal distribution, and the homoskedasticity-only t- and F-statistics have exact t_{n-k-1} and $F_{q,n-k-1}$ distributions, respectively.
5. The Gauss–Markov theorem says that, if the errors are homoskedastic and conditionally uncorrelated across observations and if $E(u_i|\boldsymbol{X}) = 0$, the OLS estimator is efficient among linear conditionally unbiased estimators (that is, OLS is BLUE).
6. If the error covariance matrix $\boldsymbol{\Omega}$ is not proportional to the identity matrix and if $\boldsymbol{\Omega}$ is known or can be estimated, then the GLS estimator is asymptotically more efficient than OLS. However, GLS requires that, in general, u_i be uncorrelated with *all* observations on the regressors, not just with $\boldsymbol{X}_i$, as is required by OLS, an assumption that must be evaluated carefully in applications.
7. The TSLS estimator is a member of the class of GMM estimators of the linear model. In GMM, the coefficients are estimated by making the sample covariance between the regression error and the exogenous variables as small as possible—specifically, by solving $\min_b [(\boldsymbol{Y} - \boldsymbol{Xb})'\boldsymbol{Z}]\boldsymbol{A}[\boldsymbol{Z}'(\boldsymbol{Y} - \boldsymbol{Xb})]$, where $\boldsymbol{A}$ is a non-random positive definite matrix. The asymptotically efficient GMM estimator sets $\boldsymbol{A} = [E(\boldsymbol{Z}_i\boldsymbol{Z}_i' u_i^2)]^{-1}$. When the errors are homoskedastic, the asymptotically efficient GMM estimator in the linear IV regression model is TSLS.

Key Terms

Gauss–Markov conditions for multiple regression (726)
Gauss–Markov theorem for multiple regression (727)
generalized least squares (GLS) (728)
infeasible GLS (731)
feasible GLS (731)
generalized method of moments (GMM) (738)
efficient GMM (739)
heteroskedasticity-robust J-statistic (740)
GMM J-statistic (740)
mean vector (752)
covariance matrix (752)

Review the Concepts

19.1 A researcher studying the relationship between earnings and workers' sex specifies the regression model $Y_i = \beta_0 + X_{1i}\beta_1 + X_{2i}\beta_2 + u_i$, where X_{1i} is a binary variable that equals 1 if the i^{th} person is a female and X_{2i} is a binary variable that equals 1 if the i^{th} person is a male. Write the model in the matrix form of Equation (19.2) for a hypothetical set of $n = 5$ observations. Show that the columns of $\boldsymbol{X}$ are linearly dependent, so that $\boldsymbol{X}$ does not have full rank. Explain how you would respecify the model to eliminate the perfect multicollinearity.

19.2 You are analyzing a linear regression model with 500 observations and one regressor. Explain how you would construct a confidence interval for β_1 if

a. Assumptions 1 through 4 in Key Concept 19.1 are true but you think assumption 5 or 6 might not be true.

b. Assumptions 1 through 5 are true but you think assumption 6 might not be true. (Give two ways to construct the confidence interval.)

c. Assumptions 1 through 6 are true.

19.3 Suppose that assumptions 1 through 5 in Key Concept 19.1 are true but that assumption 6 is not. Does the result in Equation (19.31) hold? Explain.

19.4 When is the GLS estimator more efficient than the OLS estimator within the class of linear conditionally unbiased estimators?

19.5 Construct an example of a regression model that satisfies the assumption $E(u_i \mid \boldsymbol{X}_i) = 0$ but for which $E(\boldsymbol{U} \mid \boldsymbol{X}) \neq \boldsymbol{0}_n$.

Exercises

19.1 Consider the population regression of test scores against income and the square of income in Equation (8.1).

a. Write the regression in Equation (8.1) in the matrix form of Equation (19.5). Define $\boldsymbol{Y}$, $\boldsymbol{X}$, $\boldsymbol{U}$, and $\boldsymbol{\beta}$.

b. Explain how to test the null hypothesis that the relationship between test scores and income is linear against the alternative that it is quadratic. Write the null hypothesis in the form of Equation (19.20). What are $\boldsymbol{R}$, $\boldsymbol{r}$, and q?

19.2 Suppose that a sample of $n = 20$ households has the sample means and sample covariances below for a dependent variable and two regressors:

		Sample Covariances		
	Sample Means	Y	X_1	X_2
$\boldsymbol{Y}$	6.39	0.26	0.22	0.32
$\boldsymbol{X_1}$	7.24		0.80	0.28
$\boldsymbol{X_2}$	4.00			2.40

a. Calculate the OLS estimates of β_0, β_1, and β_2. Calculate $s_{\hat{u}}^2$. Calculate the R^2 of the regression.

b. Suppose that all six assumptions in Key Concept 19.1 hold. Test the hypothesis that $\beta_1 = 0$ at the 5% significance level.

19.3 Let $\boldsymbol{W}$ be an $m \times 1$ vector with covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{W}}$, where $\boldsymbol{\Sigma}_{\boldsymbol{W}}$ is finite and positive definite. Let $\boldsymbol{c}$ be a nonrandom $m \times 1$ vector, and let $Q = \boldsymbol{c}'\boldsymbol{W}$.

a. Show that $\text{var}(Q) = \boldsymbol{c}'\boldsymbol{\Sigma}_{\boldsymbol{W}}\boldsymbol{c}$.

b. Suppose that $\boldsymbol{c} \neq \boldsymbol{0}_m$. Show that $0 < \text{var}(Q) < \infty$.

19.4 Consider the regression model $Y_i = \beta_0 + \beta_1 X_i + u_i$ from Chapter 4, and assume that the least squares assumptions in Key Concept 4.3 hold.

a. Write the model in the matrix form given in Equations (19.2) and (19.3).

b. Show that assumptions 1 through 4 in Key Concept 19.1 are satisfied.

c. Use the general formula for $\hat{\boldsymbol{\beta}}$ in Equation (19.11) to derive the expressions for $\hat{\beta}_0$ and $\hat{\beta}_1$ given in Key Concept 4.2.

d. Show that the $(1, 1)$ element of $\boldsymbol{\Sigma}_{\hat{\boldsymbol{\beta}}}$ in Equation (19.13) is equal to the expression for $\sigma_{\hat{\beta}_0}^2$ given in Key Concept 4.4.

19.5 Let $\boldsymbol{P}_{\boldsymbol{X}}$ and $\boldsymbol{M}_{\boldsymbol{X}}$ be as defined in Equations (19.24) and (19.25).

a. Prove that $\boldsymbol{P}_{\boldsymbol{X}}\boldsymbol{M}_{\boldsymbol{X}} = \boldsymbol{0}_{n \times n}$ and that $\boldsymbol{P}_{\boldsymbol{X}}$ and $\boldsymbol{M}_{\boldsymbol{X}}$ are idempotent.

b. Derive Equations (19.27) and (19.28).

c. Show that $\text{rank}(\boldsymbol{P}_{\boldsymbol{X}}) = k + 1$ and $\text{rank}(\boldsymbol{M}_{\boldsymbol{X}}) = n - k - 1$. [*Hint:* First solve Exercise 19.10, and then use the fact that $\text{trace}(\boldsymbol{AB}) = \text{trace}(\boldsymbol{BA})$ for conformable matrices $\boldsymbol{A}$ and $\boldsymbol{B}$.]

19.6 Consider the regression model in matrix form, $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{W\gamma} + \boldsymbol{U}$, where $\boldsymbol{X}$ is an $n \times k_1$ matrix of regressors and $\boldsymbol{W}$ is an $n \times k_2$ matrix of regressors. Then, as shown in Exercise 19.17, the OLS estimator $\hat{\boldsymbol{\beta}}$ can be expressed

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{M}_{\boldsymbol{W}}\boldsymbol{X})^{-1}(\boldsymbol{X}'\boldsymbol{M}_{\boldsymbol{W}}\boldsymbol{Y}).$$

Now let $\hat{\beta}_1^{BV}$ be the "binary variable" fixed effects estimator computed by estimating Equation (10.11) by OLS, and let $\hat{\beta}_1^{DM}$ be the "demeaning" fixed effects estimator computed by estimating Equation (10.14) by OLS, in which the entity-specific sample means have been subtracted from X and Y. Use the expression for $\hat{\boldsymbol{\beta}}$ given above to prove that $\hat{\beta}_1^{BV} = \hat{\beta}_1^{DM}$. [*Hint*: Write Equation (10.11) using a full set of fixed effects, $D1_i, D2_i, \ldots, D_{ni}$ and no constant term. Include all of the fixed effects in $\boldsymbol{W}$. Write out the matrix $\boldsymbol{M}_{\boldsymbol{W}}\boldsymbol{X}$.]

19.7 Consider the regression model $Y_i = \beta_1 X_i + \beta_2 W_i + u_i$, where for simplicity the intercept is omitted and all variables are assumed to have a mean of 0. Suppose that X_i is distributed independently of (W_i, u_i) but W_i and u_i might be correlated, and let $\hat{\beta}_1$ and $\hat{\beta}_2$ be the OLS estimators for this model.

a. Show that whether or not W_i and u_i are correlated, $\hat{\beta}_1 \xrightarrow{p} \beta_1$.

b. Show that if W_i and u_i are correlated, then $\hat{\beta}_2$ is inconsistent.

c. Let $\hat{\beta}_1^r$ be the OLS estimator from the regression of Y on X (the restricted regression that excludes W). Will $\hat{\beta}_1$ have a smaller asymptotic variance than $\hat{\beta}_1^r$, allowing for the possibility that W_i and u_i are correlated? Explain.

19.8 Consider the regression model $Y_i = \beta_0 + \beta_1 X_i + u_i$, where $u_1 = \tilde{u}_1$ and $u_i = 0.5u_{i-1} + \tilde{u}_i$ for $i = 2, 3, \ldots, n$. Suppose that $\tilde{u}_i$ are i.i.d. with mean 0 and variance 1 and are distributed independently of X_j for all i and j.

a. Derive an expression for $E(\boldsymbol{U}\boldsymbol{U}') = \boldsymbol{\Omega}$.

b. Explain how to estimate the model by GLS without explicitly inverting the matrix $\boldsymbol{\Omega}$. (*Hint:* Transform the model so that the regression errors are $\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_n$.)

19.9 This exercise shows that the OLS estimator of a subset of the regression coefficients is consistent under the conditional mean independence assumption stated in Key Concept 6.6. Consider the multiple regression model in matrix form $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{W}\boldsymbol{\gamma} + \boldsymbol{U}$, where $\boldsymbol{X}$ and $\boldsymbol{W}$ are, respectively, $n \times k_1$ and $n \times k_2$ matrices of regressors. Let X_i' and W_i' denote the i^{th} rows of $\boldsymbol{X}$ and $\boldsymbol{W}$ [as in Equation (19.4)]. Assume that (i) $E(u_i|\boldsymbol{X}_i, \boldsymbol{W}_i) = \boldsymbol{W}_i'\boldsymbol{\delta}$, where $\boldsymbol{\delta}$ is a $k_2 \times 1$ vector of unknown parameters; (ii) $(\boldsymbol{X}_i, \boldsymbol{W}_i, Y_i)$ are i.i.d.; (iii) $(\boldsymbol{X}_i, \boldsymbol{W}_i, u_i)$ have four finite nonzero moments; and (iv) there is no perfect multicollinearity. These are assumptions 1 through 4 of Key Concept 19.1, with the conditional mean independence assumption (i) replacing the usual conditional mean 0 assumption.

a. Use the expression for $\hat{\boldsymbol{\beta}}$ given in Exercise 19.6 to write $\hat{\boldsymbol{\beta}} - \boldsymbol{\beta} = (n^{-1}\boldsymbol{X}'\boldsymbol{M}_{\boldsymbol{W}}\boldsymbol{X})^{-1}(n^{-1}\boldsymbol{X}'\boldsymbol{M}_{\boldsymbol{W}}\boldsymbol{U})$.

b. Show that $n^{-1}\boldsymbol{X}'\boldsymbol{M}_{\boldsymbol{W}}\boldsymbol{X} \xrightarrow{p} \boldsymbol{\Sigma}_{\boldsymbol{XX}} - \boldsymbol{\Sigma}_{\boldsymbol{XW}}\boldsymbol{\Sigma}_{\boldsymbol{WW}}^{-1}\boldsymbol{\Sigma}_{\boldsymbol{WX}}$, where $\boldsymbol{\Sigma}_{\boldsymbol{XX}} = E(\boldsymbol{X}_i\boldsymbol{X}_i')$, $\boldsymbol{\Sigma}_{\boldsymbol{XW}} = E(\boldsymbol{X}_i\boldsymbol{W}_i')$, and so forth. [The matrix $\boldsymbol{A}_n \xrightarrow{p} \boldsymbol{A}$ if $\boldsymbol{A}_{n,ij} \xrightarrow{p} \boldsymbol{A}_{ij}$ for all i, j pairs, where $\boldsymbol{A}_{n,ij}$ and $\boldsymbol{A}_{ij}$ are the (i, j) elements of $\boldsymbol{A}_n$ and $\boldsymbol{A}$.]

c. Show that assumptions (i) and (ii) imply that $E(\boldsymbol{U}|\boldsymbol{X}, \boldsymbol{W}) = \boldsymbol{W}\boldsymbol{\delta}$.

d. Use (c) and the law of iterated expectations to show that $n^{-1}\boldsymbol{X}'\boldsymbol{M}_{\boldsymbol{W}}\boldsymbol{U} \xrightarrow{p} \boldsymbol{0}_{k_1 \times 1}$.

e. Use (a) through (d) to conclude that, under assumptions (i) through (iv), $\hat{\boldsymbol{\beta}} \xrightarrow{p} \boldsymbol{\beta}$.

19.10 Let $\boldsymbol{C}$ be a symmetric idempotent matrix.

a. Show that the eigenvalues of $\boldsymbol{C}$ are either 0 or 1. (*Hint:* Note that $\boldsymbol{C}\boldsymbol{q} = \gamma\boldsymbol{q}$ implies $0 = \boldsymbol{C}\boldsymbol{q} - \gamma\boldsymbol{q} = \boldsymbol{C}\boldsymbol{C}\boldsymbol{q} - \gamma\boldsymbol{q} = \gamma\boldsymbol{C}\boldsymbol{q} - \gamma\boldsymbol{q} = \gamma^2\boldsymbol{q} - \gamma\boldsymbol{q}$, and solve for γ.)

b. Show that $\text{trace}(\boldsymbol{C}) = \text{rank}(\boldsymbol{C})$.

c. Let $\boldsymbol{d}$ be an $n \times 1$ vector. Show that $\boldsymbol{d}'\boldsymbol{C}\boldsymbol{d} \geq 0$.

19.11 Suppose that $\boldsymbol{C}$ is an $n \times n$ symmetric idempotent matrix with rank r, and let $\boldsymbol{V} \sim \boldsymbol{N}(\boldsymbol{0}_n, \boldsymbol{I}_n)$.

a. Show that $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{A}'$, where $\boldsymbol{A}$ is $n \times r$ with $\boldsymbol{A}'\boldsymbol{A} = \boldsymbol{I}_r$. (*Hint:* $\boldsymbol{C}$ is positive semidefinite and can be written as $\boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}'$, as explained in Appendix 19.1.)

b. Show that $\boldsymbol{A}'\boldsymbol{V} \sim \boldsymbol{N}(\boldsymbol{0}_r, \boldsymbol{I}_r)$.

c. Show that $\boldsymbol{V}'\boldsymbol{C}\boldsymbol{V} \sim \chi^2_r$.

19.12 **a.** Show that $\widetilde{\boldsymbol{\beta}}^{Eff.GMM}$ is the efficient GMM estimator—that is, that $\widetilde{\boldsymbol{\beta}}^{Eff.GMM}$ in Equation (19.66) is the solution to Equation (19.65).

b. Show that $\sqrt{n}(\hat{\boldsymbol{\beta}}^{Eff.GMM} - \widetilde{\boldsymbol{\beta}}^{Eff.GMM}) \xrightarrow{p} 0$.

c. Show that $J^{GMM} \xrightarrow{d} \chi^2_{m-k}$.

19.13 Consider the problem of minimizing the sum of squared residuals, subject to the constraint that $\boldsymbol{R}\boldsymbol{b} = \boldsymbol{r}$, where $\boldsymbol{R}$ is $q \times (k+1)$ with rank q. Let $\widetilde{\boldsymbol{\beta}}$ be the value of $\boldsymbol{b}$ that solves the constrained minimization problem.

a. Show that the Lagrangian for the minimization problem is $L(\boldsymbol{b}, \gamma) = (\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{b})'(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{b}) + \gamma'(\boldsymbol{R}\boldsymbol{b} - \boldsymbol{r})$, where γ is a $q \times 1$ vector of Lagrange multipliers.

b. Show that $\widetilde{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}} - (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{R}'[\boldsymbol{R}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{R}']^{-1}(\boldsymbol{R}\hat{\boldsymbol{\beta}} - \boldsymbol{r})$.

c. Show that $(\boldsymbol{Y} - \boldsymbol{X}\widetilde{\boldsymbol{\beta}})'(\boldsymbol{Y} - \boldsymbol{X}\widetilde{\boldsymbol{\beta}}) - (\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})(\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}) = (\boldsymbol{R}\hat{\boldsymbol{\beta}} - \boldsymbol{r})'[\boldsymbol{R}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{R}']^{-1}(\boldsymbol{R}\hat{\boldsymbol{\beta}} - \boldsymbol{r})$.

d. Show that $\widetilde{F}$ in Equation (19.36) is equivalent to the homoskedasticity-only F-statistic in Equation (7.13).

19.14 Consider the regression model $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{U}$. Partition $\boldsymbol{X}$ as $[\boldsymbol{X}_1\ \boldsymbol{X}_2]$ and $\boldsymbol{\beta}$ as $[\boldsymbol{\beta}_1'\ \ \boldsymbol{\beta}_2']'$, where $\boldsymbol{X}_1$ has k_1 columns and $\boldsymbol{X}_2$ has k_2 columns. Suppose that $\boldsymbol{X}_2'\boldsymbol{Y} = \boldsymbol{0}_{k_2 \times 1}$. Let $\boldsymbol{R} = [\boldsymbol{I}_{k_1}\ \boldsymbol{0}_{k_1 \times k_2}]$.

a. Show that $\hat{\boldsymbol{\beta}}'(\boldsymbol{X}'\boldsymbol{X})\hat{\boldsymbol{\beta}} = (\boldsymbol{R}\hat{\boldsymbol{\beta}})'[\boldsymbol{R}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{R}]^{-1}(\boldsymbol{R}\hat{\boldsymbol{\beta}})$.

b. Consider the regression described in Equation (12.17). Let $\boldsymbol{W} = [\boldsymbol{1}\ \ \boldsymbol{W}_1\ \ \boldsymbol{W}_2\ \ \ldots\ \ \boldsymbol{W}_r]$, where $\boldsymbol{1}$ is an $n \times 1$ vector of 1's, $\boldsymbol{W}_1$ is the $n \times 1$ vector with i^{th} element $\boldsymbol{W}_{1i}$, and so forth. Let $\hat{\boldsymbol{U}}^{TSLS}$ denote the vector of two stage least squares residuals.

i. Show that $\boldsymbol{W}'\hat{\boldsymbol{U}}^{TSLS} = 0$.

ii. Show that the method for computing the J-statistic described in Key Concept 12.6 (using a homoskedasticity-only F-statistic) and that using the formula in Equation (19.63) produce the same value for the J-statistic. [*Hint:* Use the results in (a), (b.i), and Exercise 19.13.]

19.15 (Consistency of clustered standard errors.) Consider the panel data model $Y_{it} = \beta X_{it} + \alpha_i + u_{it}$, where all variables are scalars. Assume that assumptions

1, 2, and 4 in Key Concept 10.3 hold and strengthen assumption 3, so that X_{it} and u_{it} have eight nonzero finite moments. Let $\boldsymbol{M} = \boldsymbol{I}_T - T^{-1}\boldsymbol{\iota}\boldsymbol{\iota}'$, where $\boldsymbol{\iota}$ is a $T \times 1$ vector of 1's. Also let $\boldsymbol{Y}_i = (Y_{i1} \quad Y_{i2} \quad \cdots \quad Y_{iT})'$, $\boldsymbol{X}_i = (X_{i1} \quad X_{i2} \quad \cdots \quad X_{iT})'$, $\boldsymbol{u}_i = (u_{i1} \quad u_{i2} \quad \cdots \quad u_{iT})'$, $\widetilde{\boldsymbol{Y}}_i = \boldsymbol{M}\boldsymbol{Y}_i$, $\widetilde{\boldsymbol{X}}_i = \boldsymbol{M}\boldsymbol{X}_i$, and $\widetilde{\boldsymbol{u}}_i = \boldsymbol{M}\boldsymbol{u}_i$. For the asymptotic calculations in this problem, suppose that T is fixed and $n \longrightarrow \infty$.

a. Show that the fixed effects estimator of β from Section 10.3 can be written as $\hat{\beta} = (\sum_{i=1}^n \widetilde{\boldsymbol{X}}_i'\widetilde{\boldsymbol{X}}_i)^{-1}\sum_{i=1}^n \widetilde{\boldsymbol{X}}_i'\widetilde{\boldsymbol{Y}}_i$.

b. Show that $\hat{\beta} - \beta = (\sum_{i=1}^n \widetilde{\boldsymbol{X}}_i'\widetilde{\boldsymbol{X}}_i)^{-1}\sum_{i=1}^n \widetilde{\boldsymbol{X}}_i'\boldsymbol{u}_i$. (*Hint:* $\boldsymbol{M}$ is idempotent.)

c. Let $Q_{\widetilde{X}} = T^{-1}E(\widetilde{\boldsymbol{X}}_i'\widetilde{\boldsymbol{X}}_i)$ and $\hat{Q}_{\widetilde{X}} = \frac{1}{nT}\sum_{i=1}^n\sum_{t=1}^T \widetilde{X}_{it}^2$. Show that $\hat{Q}_{\widetilde{X}} \xrightarrow{p} Q_{\widetilde{X}}$.

d. Let $\eta_i = \widetilde{\boldsymbol{X}}_i'\boldsymbol{u}_i/\sqrt{T}$ and $\sigma_\eta^2 = \text{var}(\eta_i)$. Show that $\sqrt{\frac{1}{n}}\sum_{i=1}^n \eta_i \xrightarrow{d} N(0, \sigma_\eta^2)$.

e. Use your answers to (b) through (d) to prove Equation (10.25); that is, show that $\sqrt{nT}(\hat{\beta} - \beta) \xrightarrow{d} N(0, \sigma_\eta^2/Q_{\widetilde{X}}^2)$.

f. Let $\widetilde{\sigma}_{\eta,clustered}^2$ be the infeasible clustered variance estimator, computed using the true errors instead of the residuals so that $\widetilde{\sigma}_{\eta,clustered}^2 = \frac{1}{nT}\sum_{i=1}^n(\widetilde{\boldsymbol{X}}_i'\boldsymbol{u}_i)^2$. Show that $\widetilde{\sigma}_{\eta,clustered}^2 \xrightarrow{p} \sigma_\eta^2$.

g. Let $\hat{\widetilde{\boldsymbol{u}}}_1 = \widetilde{\boldsymbol{Y}}_i - \hat{\beta}\widetilde{\boldsymbol{X}}_i$ and $\hat{\sigma}_{\eta,\,clustered}^2 = \frac{n}{n-1}\frac{1}{nT}\sum_{i=1}^n(\widetilde{\boldsymbol{X}}_i'\hat{\widetilde{\boldsymbol{u}}}_i)^2$ [this is Equation (10.27) in matrix form]. Show that $\hat{\sigma}_{\eta,\,clustered}^2 \xrightarrow{p} \sigma_\eta^2$. [*Hint:* Use an argument like that used in Equation (18.16) to show that $\hat{\sigma}_{\eta,\,clustered}^2 - \hat{\sigma}_{\eta,\,clustered}^2 \xrightarrow{p} 0$, and then use your answer to (f).]

19.16 This exercise takes up the problem of missing data discussed in Section 9.2. Consider the regression model $Y_i = X_i\beta + u_i$, $i = 1, \ldots, n$, where all variables are scalars and the constant term/intercept is omitted for convenience.

a. Suppose that the least squares assumptions in Key Concept 4.3 are satisfied. Show that the least squares estimator of β is unbiased and consistent.

b. Now suppose that some of the observations are missing. Let I_i denote a binary random variable that indicates the nonmissing observations; that is, $I_i = 1$ if observation i is not missing, and $I_i = 0$ if observation i is missing. Assume that $\{I_i, X_i, u_i\}$ are i.i.d.

i. Show that the OLS estimator can be written as

$$\hat{\beta} = \left(\sum_{i=1}^n I_i X_i X_i'\right)^{-1}\left(\sum_{i=1}^n I_i X_i Y_i\right) = \beta + \left(\sum_{i=1}^n I_i X_i X_i'\right)^{-1}\left(\sum_{i=1}^n I_i X_i u_i\right).$$

ii. Suppose that data are missing "completely at random" in the sense that $\Pr(I_i = 1 \,|\, X_i, u_i) = p$, where p is a constant. Show that $\hat{\beta}$ is unbiased and consistent.

iii. Suppose that the probability that the i^{th} observation is missing depends of X_i but not on u_i; that is, $\Pr(I_i = 1 \,|\, X_i, u_i) = p(X_i)$. Show that $\hat{\beta}$ is unbiased and consistent.

iv. Suppose that the probability that the i^{th} observation is missing depends on both X_i and u_i; that is, $\Pr(I_i = 1 \mid X_i, u_i) = p(X_i, u_i)$. Is $\hat{\beta}$ unbiased? Is $\hat{\beta}$ consistent? Explain.

c. Suppose that $\beta = 1$ and that X_i and u_i are mutually independent standard normal random variables [so that both X_i and u_i are distributed $N(0, 1)$]. Suppose that $I_i = 1$ when $Y_i \geq 0$ but that $I_i = 0$ when $Y_i < 0$. Is $\hat{\beta}$ unbiased? Is $\hat{\beta}$ consistent? Explain.

19.17 Consider the regression model in matrix form $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{W\gamma} + \boldsymbol{U}$, where $\boldsymbol{X}$ and $\boldsymbol{W}$ are matrices of regressors and $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are vectors of unknown regression coefficients. Let $\widetilde{\boldsymbol{X}} = \boldsymbol{M_W X}$ and $\widetilde{\boldsymbol{Y}} = \boldsymbol{M_W Y}$, where $\boldsymbol{M_W} = \boldsymbol{I} - \boldsymbol{W}(\boldsymbol{W}'\boldsymbol{W})^{-1}\boldsymbol{W}$.

a. Show that the OLS estimators of $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ can be written as

$$\begin{bmatrix} \hat{\boldsymbol{\beta}} \\ \hat{\boldsymbol{\gamma}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{W} \\ \boldsymbol{W}'\boldsymbol{X} & \boldsymbol{W}'\boldsymbol{W} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{X}'\boldsymbol{Y} \\ \boldsymbol{W}'\boldsymbol{Y} \end{bmatrix}$$

b. Show that

$$\begin{bmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{W} \\ \boldsymbol{W}'\boldsymbol{X} & \boldsymbol{W}'\boldsymbol{W} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} (\boldsymbol{X}'\boldsymbol{M_W}\boldsymbol{X})^{-1} & -(\boldsymbol{X}'\boldsymbol{M_W}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{W}(\boldsymbol{W}'\boldsymbol{W})^{-1} \\ -(\boldsymbol{W}'\boldsymbol{W})^{-1}\boldsymbol{W}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{M_W}\boldsymbol{X})^{-1} & (\boldsymbol{W}'\boldsymbol{W})^{-1} + (\boldsymbol{W}'\boldsymbol{W})^{-1}\boldsymbol{W}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{M_W}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{W}(\boldsymbol{W}'\boldsymbol{W})^{-1} \end{bmatrix}.$$

(*Hint:* Show that the product of the two matrices is equal to the identity matrix.)

c. Show that $\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{M_W}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{M_W}\boldsymbol{Y}$.

d. The Frisch–Waugh theorem (Appendix 6.2) says that $\hat{\boldsymbol{\beta}} = (\widetilde{\boldsymbol{X}}'\widetilde{\boldsymbol{X}})^{-1}\widetilde{\boldsymbol{X}}'\widetilde{\boldsymbol{Y}}$. Use the result in (c) to prove the Frisch–Waugh theorem.

19.18 Consider the homoskedastic linear regression model with two regressors, and let $\rho_{X_1, X_2} = \text{corr}(X_1, X_2)$. Show that $\text{corr}(\hat{\beta}_1, \hat{\beta}_2) \to -\rho_{X_1, X_2}$ [Equation (6.21)] as n increases.

APPENDIX

19.1 Summary of Matrix Algebra

This appendix summarizes vectors, matrices, and the elements of matrix algebra used in Chapter 19. The purpose of this appendix is to review some concepts and definitions from a course in linear algebra, not to replace such a course.

Definitions of Vectors and Matrices

A **vector** is a collection of n numbers or elements, collected either in a column (a **column vector**) or in a row (a **row vector**). The n-dimensional column vector $\boldsymbol{b}$ and the n-dimensional row vector $\boldsymbol{c}$ are

$$\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \text{ and } \boldsymbol{c} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix},$$

where b_1 is the first element of $\boldsymbol{b}$ and, in general, b_i is the i^{th} element of $\boldsymbol{b}$.

Throughout, a boldface denotes a vector or matrix.

A **matrix** is a collection, or an array, of numbers or elements, in which the elements are laid out in columns and rows. The dimension of a matrix is $n \times m$, where n is the number of rows and m is the number of columns. The $n \times m$ matrix $\boldsymbol{A}$ is

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix},$$

where a_{ij} is the (i, j) element of $\boldsymbol{A}$; that is, a_{ij} is the element that appears in the i^{th} row and j^{th} column. An $n \times m$ matrix consists of n row vectors or, alternatively, of m column vectors.

To distinguish one-dimensional numbers from vectors and matrices, a one-dimensional number is called a **scalar**.

Types of Matrices

Square, symmetric, and diagonal matrices. A matrix is said to be **square** if the number of rows equals the number of columns. A square matrix is said to be **symmetric** if its (i, j) element equals its (j, i) element. A **diagonal** matrix is a square matrix in which all the off-diagonal elements equal 0; that is, if the square matrix $\boldsymbol{A}$ is diagonal, then $a_{ij} = 0$ for $i \neq j$.

Special matrices. An important matrix is the **identity matrix,** $\boldsymbol{I}_n$, which is an $n \times n$ diagonal matrix with 1's on the diagonal. The **null matrix,** $\boldsymbol{0}_{n \times m}$, is the $n \times m$ matrix with all elements equal to 0.

The transpose. The **transpose** of a matrix switches the rows and the columns. That is, the transpose of a matrix turns the $n \times m$ matrix $\boldsymbol{A}$ into the $m \times n$ matrix, which is denoted by $\boldsymbol{A}'$, where the (i, j) element of $\boldsymbol{A}$ becomes the (j, i) element of $\boldsymbol{A}'$; said differently, the transpose of the matrix $\boldsymbol{A}$ turns the rows of $\boldsymbol{A}$ into the columns of $\boldsymbol{A}'$. If a_{ij} is the (i, j) element of $\boldsymbol{A}$, then $\boldsymbol{A}'$ (the transpose of $\boldsymbol{A}$) is

$$\boldsymbol{A}' = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{bmatrix}.$$

The transpose of a vector is a special case of the transpose of a matrix. Thus the transpose of a vector turns a column vector into a row vector; that is, if $\boldsymbol{b}$ is an $n \times 1$ column vector, then its transpose is the $1 \times n$ row vector:

$$\boldsymbol{b}' = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}.$$

The transpose of a row vector is a column vector.

Elements of Matrix Algebra: Addition and Multiplication

Matrix addition. Two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ that have the same dimensions (for example, that are both $n \times m$) can be added together. The sum of two matrices is the sum of their elements; that is, if $\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$, then $c_{ij} = a_{ij} + b_{ij}$. A special case of matrix addition is vector addition: If $\boldsymbol{a}$ and $\boldsymbol{b}$ are both $n \times 1$ column vectors, then their sum, $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$, is the element-wise sum; that is, $c_i = a_i + b_i$.

Vector and matrix multiplication. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two $n \times 1$ column vectors. Then the product of the transpose of $\boldsymbol{a}$ (which is itself a row vector) and $\boldsymbol{b}$ is $\boldsymbol{a}'\boldsymbol{b} = \sum_{i=1}^{n} a_i b_i$. Applying this definition with $\boldsymbol{b} = \boldsymbol{a}$ yields $\boldsymbol{a}'\boldsymbol{a} = \sum_{i=1}^{n} a_i^2$.

Similarly, the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ can be multiplied together if they are conformable—that is, if the number of columns of $\boldsymbol{A}$ equals the number of rows of $\boldsymbol{B}$. Specifically, suppose that $\boldsymbol{A}$ has dimension $n \times m$ and $\boldsymbol{B}$ has dimension $m \times r$. Then the product of $\boldsymbol{A}$ and $\boldsymbol{B}$ is an $n \times r$ matrix, $\boldsymbol{C}$; that is, $\boldsymbol{C} = \boldsymbol{AB}$, where the (i, j) element of $\boldsymbol{C}$ is $c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$. Said differently, the (i, j) element of $\boldsymbol{AB}$ is the product of multiplying the row vector that is the i^{th} row of $\boldsymbol{A}$ by the column vector that is the j^{th} column of $\boldsymbol{B}$.

The product of a scalar d with the matrix $\boldsymbol{A}$ has the (i, j) element da_{ij}; that is, each element of $\boldsymbol{A}$ is multiplied by the scalar d.

Some useful properties of matrix addition and multiplication. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be matrices. Then

a. $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$;

b. $(\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C} = \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C})$;

c. $(\boldsymbol{A} + \boldsymbol{B})' = \boldsymbol{A}' + \boldsymbol{B}'$;

d. If $\boldsymbol{A}$ is $n \times m$, then $\boldsymbol{A}\boldsymbol{I}_m = \boldsymbol{A}$ and $\boldsymbol{I}_n\boldsymbol{A} = \boldsymbol{A}$;

e. $\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C}$;

f. $(\boldsymbol{A} + \boldsymbol{B})\boldsymbol{C} = \boldsymbol{AC} + \boldsymbol{BC}$; and

g. $(\boldsymbol{AB})' = \boldsymbol{B}'\boldsymbol{A}'$.

In general, matrix multiplication does not commute; that is, in general $\boldsymbol{AB} \neq \boldsymbol{BA}$, although there are some special cases in which matrix multiplication commutes; for example, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are both $n \times n$ diagonal matrices, then $\boldsymbol{AB} = \boldsymbol{BA}$.

Matrix Inverse, Matrix Square Roots, and Related Topics

The matrix inverse. Let $\boldsymbol{A}$ be a square matrix. Assuming that it exists, the **inverse** of the matrix $\boldsymbol{A}$ is defined as the matrix for which $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}_n$. If, in, fact the inverse matrix $\boldsymbol{A}^{-1}$ exists, then $\boldsymbol{A}$ is said to be **invertible** or **nonsingular**. If both $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible, then $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.

Positive definite and positive semidefinite matrices. Let $\boldsymbol{V}$ be an $n \times n$ square matrix. Then $\boldsymbol{V}$ is **positive definite** if $\boldsymbol{c}'\boldsymbol{V}\boldsymbol{c} > 0$ for all nonzero $n \times 1$ vectors $\boldsymbol{c}$. Similarly, $\boldsymbol{V}$ is **positive semidefinite** if $\boldsymbol{c}'\boldsymbol{V}\boldsymbol{c} \geq 0$ for all nonzero $n \times 1$ vectors $\boldsymbol{c}$. If $\boldsymbol{V}$ is positive definite, then it is invertible.

Linear independence. The $n \times 1$ vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are **linearly independent** if there do not exist nonzero scalars c_1 and c_2 such that $c_1\boldsymbol{a}_1 + c_2\boldsymbol{a}_2 = \mathbf{0}_{n\times 1}$. More generally, the set of k vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_k$ is linearly independent if there do not exist nonzero scalars $c_1, c_2, \ldots, c_k$ such that $c_1\boldsymbol{a}_1 + c_2\boldsymbol{a}_2 + \cdots + c_k\boldsymbol{a}_k = \mathbf{0}_{n\times 1}$.

The rank of a matrix. The **rank** of the $n \times m$ matrix $\boldsymbol{A}$ is the number of linearly independent columns of $\boldsymbol{A}$. The rank of $\boldsymbol{A}$ is denoted rank($\boldsymbol{A}$). If the rank of $\boldsymbol{A}$ equals the number of columns of $\boldsymbol{A}$, then $\boldsymbol{A}$ is said to have full column rank. If the $n \times m$ matrix $\boldsymbol{A}$ has full column rank, then there does not exist a nonzero $m \times 1$ vector $\boldsymbol{c}$ such that $\boldsymbol{A}\boldsymbol{c} = \mathbf{0}_{n\times 1}$. If $\boldsymbol{A}$ is $n \times n$ with rank($\boldsymbol{A}$) $= n$, then $\boldsymbol{A}$ is nonsingular. If the $n \times m$ matrix $\boldsymbol{A}$ has full column rank, then $\boldsymbol{A}'\boldsymbol{A}$ is nonsingular.

The trace of a matrix. The **trace** of the $n \times n$ (square) matrix $\boldsymbol{A}$ is the sum of the diagonal elements; that is, trace($\boldsymbol{A}$) $= \sum_{i=1}^{n} a_{ii}$. For $n \times n$ matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ and $n \times 1$ vector $\boldsymbol{c}$, the trace satisfies these properties: trace($\boldsymbol{A}$) = trace($\boldsymbol{A}'$), trace($\boldsymbol{A} + \boldsymbol{B}$) = trace($\boldsymbol{A}$) + trace($\boldsymbol{B}$), trace($\boldsymbol{A}\boldsymbol{B}$) = trace($\boldsymbol{B}\boldsymbol{A}$), trace($\boldsymbol{B}\boldsymbol{A}\boldsymbol{B}^{-1}$) = trace($\boldsymbol{A}$), and $\boldsymbol{c}'\boldsymbol{B}\boldsymbol{c}$ =trace($\boldsymbol{B}\boldsymbol{c}\boldsymbol{c}'$).

The matrix square root. Let $\boldsymbol{V}$ be an $n \times n$ square symmetric positive definite matrix. The matrix square root of $\boldsymbol{V}$ is defined to be an $n \times n$ matrix $\boldsymbol{F}$ such that $\boldsymbol{F}'\boldsymbol{F} = \boldsymbol{V}$. The matrix square root of a positive definite matrix will always exist, but it is not unique. The matrix square root has the property that $\boldsymbol{F}\boldsymbol{V}^{-1}\boldsymbol{F}' = \boldsymbol{I}_n$. In addition, the matrix square root of a positive definite matrix is invertible, so $\boldsymbol{F}'^{-1}\boldsymbol{V}\boldsymbol{F}^{-1} = \boldsymbol{I}_n$.

Eigenvalues and eigenvectors. Let $\boldsymbol{A}$ be an $n \times n$ matrix. If the $n \times 1$ vector $\boldsymbol{q}$ and the scalar λ satisfy $\boldsymbol{A}\boldsymbol{q} = \lambda\boldsymbol{q}$, where $\boldsymbol{q}'\boldsymbol{q} = 1$, then λ is an **eigenvalue** of $\boldsymbol{A}$, and $\boldsymbol{q}$ is the **eigenvector** of $\boldsymbol{A}$ associated with that eigenvalue. An $n \times n$ matrix has n eigenvalues, which need not take on distinct values, and n eigenvectors.

If $\boldsymbol{V}$ is an $n \times n$ symmetric positive definite matrix, then the eigenvalues of $\boldsymbol{V}$ are positive real numbers, and the eigenvectors of $\boldsymbol{V}$ are real. Also, $\boldsymbol{V}$ can be written in terms of its eigenvalues and eigenvectors as $\boldsymbol{V} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}'$, where $\boldsymbol{\Lambda}$ is a diagonal $n \times n$ matrix with diagonal elements that equal the eigenvalues of $\boldsymbol{V}$ and $\boldsymbol{Q}$ is an $n \times n$ matrix consisting of the eigenvectors of $\boldsymbol{V}$, arranged so that the i^{th} column of $\boldsymbol{Q}$ is the eigenvector corresponding to the eigenvalue λ_i, which is the i^{th} diagonal element of $\boldsymbol{\Lambda}$. The eigenvectors are orthonormal, so $\boldsymbol{Q}'\boldsymbol{Q} = \boldsymbol{I}_n$. The trace of $\boldsymbol{V}$ equals the sum of its eigenvectors: trace($\boldsymbol{V}$) = trace($\boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}'$) = trace($\boldsymbol{\Lambda}\boldsymbol{Q}'\boldsymbol{Q}$) = trace($\boldsymbol{\Lambda}$) = $\sum_{i=1}^{n}\lambda_i$.

Idempotent matrices. A matrix $\boldsymbol{C}$ is idempotent if $\boldsymbol{C}$ is square and $\boldsymbol{C}\boldsymbol{C} = \boldsymbol{C}$. If $\boldsymbol{C}$ is an $n \times n$ idempotent matrix that is also symmetric, then $\boldsymbol{C}$ is positive semidefinite, and $\boldsymbol{C}$ has r eigenvalues that equal 1 and $n - r$ eigenvalues that equal 0, where r = rank($\boldsymbol{C}$) (Exercise 19.10).

APPENDIX

19.2 Multivariate Distributions

This appendix collects various definitions and facts about distributions of vectors of random variables. We start by defining the mean and covariance matrix of the n-dimensional random variable $\boldsymbol{V}$. Next we present the multivariate normal distribution. Finally, we summarize some facts about the distributions of linear and quadratic functions of jointly normally distributed random variables.

The Mean Vector and Covariance Matrix

The first and second moments of an $m \times 1$ vector of random variables, $\boldsymbol{V} = (V_1 \quad V_2 \quad \cdots \quad V_m)'$, are summarized by its mean vector and covariance matrix.

Because $\boldsymbol{V}$ is a vector, the vector of its means—that is, its **mean vector**—is $E(\boldsymbol{V}) = \boldsymbol{\mu}_{\boldsymbol{V}}$. The i^{th} element of the mean vector is the mean of the i^{th} element of $\boldsymbol{V}$.

The **covariance matrix** of $\boldsymbol{V}$ is the matrix consisting of the variance $\text{var}(V_i), i = 1, \ldots, m$, along the diagonal and the (i, j) off-diagonal elements $\text{cov}(V_i, V_j)$. In matrix form, the covariance matrix $\boldsymbol{V}$ is

$$\boldsymbol{\Sigma}_{\boldsymbol{V}} = E[(\boldsymbol{V} - \boldsymbol{\mu}_{\boldsymbol{V}})(\boldsymbol{V} - \boldsymbol{\mu}_{\boldsymbol{V}})'] = \begin{bmatrix} \text{var}(V_1) & \cdots & \text{cov}(V_1, V_m) \\ \vdots & \ddots & \vdots \\ \text{cov}(V_m, V_1) & \cdots & \text{var}(V_m) \end{bmatrix}. \quad (19.72)$$

The Multivariate Normal Distribution

The $m \times 1$ vector random variable $\boldsymbol{V}$ has a multivariate normal distribution with mean vector $\boldsymbol{\mu}_{\boldsymbol{V}}$ and covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{V}}$ if it has the joint probability density function

$$f(\boldsymbol{V}) = \frac{1}{\sqrt{(2\pi)^m \det(\boldsymbol{\Sigma}_{\boldsymbol{V}})}} \exp\left[-\frac{1}{2}(\boldsymbol{V} - \boldsymbol{\mu}_{\boldsymbol{V}})'\boldsymbol{\Sigma}_{\boldsymbol{V}}^{-1}(\boldsymbol{V} - \boldsymbol{\mu}_{\boldsymbol{V}})\right], \quad (19.73)$$

where $\det(\boldsymbol{\Sigma}_{\boldsymbol{V}})$ is the determinant of the matrix $\boldsymbol{\Sigma}_{\boldsymbol{V}}$. The multivariate normal distribution is denoted $N(\boldsymbol{\mu}_{\boldsymbol{V}}, \boldsymbol{\Sigma}_{\boldsymbol{V}})$.

An important fact about the multivariate normal distribution is that if two jointly normally distributed random variables are uncorrelated (or, equivalently, have a block-diagonal covariance matrix), then they are independently distributed. That is, let $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ be jointly normally distributed random variables with respective dimensions $m_1 \times 1$ and $m_2 \times 1$. Then if $\text{cov}(\boldsymbol{V}_1, \boldsymbol{V}_2) = E[(\boldsymbol{V}_1 - \boldsymbol{\mu}_{\boldsymbol{V}_1})(\boldsymbol{V}_2 - \boldsymbol{\mu}_{\boldsymbol{V}_2})'] = \boldsymbol{0}_{m_1 \times m_2}$, $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ are independent.

If $\{V_i\}$ are i.i.d. $N(0, \sigma_v^2)$, then $\boldsymbol{\Sigma}_{\boldsymbol{V}} = \sigma_v^2 \boldsymbol{I}_m$, and the multivariate normal distribution simplifies to the product of m univariate normal densities.

Distributions of Linear Combinations and Quadratic Forms of Normal Random Variables

Linear combinations of multivariate normal random variables are themselves normally distributed, and certain quadratic forms of multivariate normal random variables have a chi-squared

distribution. Let $\boldsymbol{V}$ be an $m \times 1$ random variable distributed $N(\boldsymbol{\mu}_V, \boldsymbol{\Sigma}_V)$, let $\boldsymbol{A}$ and $\boldsymbol{B}$ be nonrandom $a \times m$ and $b \times m$ matrices, and let $\boldsymbol{d}$ be a nonrandom $a \times 1$ vector. Then

$$\boldsymbol{d} + \boldsymbol{AV} \text{ is distributed } N(\boldsymbol{d} + \boldsymbol{A\mu}_V, \boldsymbol{A\Sigma}_V\boldsymbol{A}'), \text{ and} \tag{19.74}$$

$$\text{cov}(\boldsymbol{AV}, \boldsymbol{BV}) = \boldsymbol{A\Sigma}_V\boldsymbol{B}'; \tag{19.75}$$

$$\text{if } \boldsymbol{A\Sigma}_V\boldsymbol{B}' = \boldsymbol{0}_{a\times b}, \text{ then } \boldsymbol{AV} \text{ and } \boldsymbol{BV} \text{ are independently distributed; and} \tag{19.76}$$

$$(\boldsymbol{V} - \boldsymbol{\mu}_V)'\boldsymbol{\Sigma}_V^{-1}(\boldsymbol{V} - \boldsymbol{\mu}_V) \text{ is distributed } \chi^2_m. \tag{19.77}$$

Let $\boldsymbol{U}$ be an m-dimensional multivariate standard normal random variable with distribution $N(\boldsymbol{0}, \boldsymbol{I}_m)$. If $\boldsymbol{C}$ is symmetric and idempotent, then

$$\boldsymbol{U}'\boldsymbol{CU} \text{ has a } \chi^2_r \text{ distribution, where } r = \text{rank}(\boldsymbol{C}). \tag{19.78}$$

Equation (19.78) is proven as Exercise 19.11.

APPENDIX

19.3 Derivation of the Asymptotic Distribution of $\hat{\beta}$

This appendix provides the derivation of the asymptotic normal distribution of $\sqrt{n}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$ given in Equation (19.12). An implication of this result is that $\hat{\boldsymbol{\beta}} \xrightarrow{p} \boldsymbol{\beta}$.

First consider the "denominator" matrix $\boldsymbol{X}'\boldsymbol{X}/n = \frac{1}{n}\sum_{i=1}^n \boldsymbol{X}_i\boldsymbol{X}_i'$ in Equation (19.15). The (j, l) element of this matrix is $\frac{1}{n}\sum_{i=1}^n X_{ji}X_{li}$. By the second assumption in Key Concept 19.1, $\boldsymbol{X}_i$ is i.i.d., so $X_{ji}X_{li}$ is i.i.d. By the third assumption in Key Concept 19.1, each element of $\boldsymbol{X}_i$ has four moments, so, by the Cauchy–Schwarz inequality (Appendix 18.2), $X_{ji}X_{li}$ has two moments. Because $X_{ji}X_{li}$ is i.i.d. with two moments, $\frac{1}{n}\sum_{i=1}^n X_{ji}X_{li}$ obeys the law of large numbers, so $\frac{1}{n}\sum_{i=1}^n X_{ji}X_{li} \xrightarrow{p} E(X_{ji}X_{li})$. This is true for all the elements of $\boldsymbol{X}'\boldsymbol{X}/n$, so $\boldsymbol{X}'\boldsymbol{X}/n \xrightarrow{p} E(\boldsymbol{X}_i\boldsymbol{X}_i') = \boldsymbol{Q}_X$.

Next consider the "numerator" matrix in Equation (19.15), $\boldsymbol{X}'\boldsymbol{U}/\sqrt{n} = \sqrt{\frac{1}{n}}\sum_{i=1}^n \boldsymbol{V}_i$, where $\boldsymbol{V}_i = \boldsymbol{X}_i u_i$. By the first assumption in Key Concept 19.1 and the law of iterated expectations, $E(\boldsymbol{V}_i) = E[\boldsymbol{X}_i E(u_i|\boldsymbol{X}_i)] = \boldsymbol{0}_{k+1}$. By the second least squares assumption, $\boldsymbol{V}_i$ is i.i.d. Let $\boldsymbol{c}$ be a finite $k+1$ dimensional vector. By the Cauchy–Schwarz inequality, $E[(\boldsymbol{c}'\boldsymbol{V}_i)^2] = E[(\boldsymbol{c}'\boldsymbol{X}_i u_i)^2] = E[(\boldsymbol{c}'\boldsymbol{X}_i)^2(u_i)^2] \le \sqrt{E[(\boldsymbol{c}'\boldsymbol{X}_i)^4]E(u_i^4)}$, which is finite by the third least squares assumption. This is true for every such vector $\boldsymbol{c}$, so $E(\boldsymbol{V}_i\boldsymbol{V}_i') = \boldsymbol{\Sigma}_V$ is finite and, we assume, positive definite. Thus the multivariate central limit theorem of Key Concept 19.2 applies to $\sqrt{\frac{1}{n}}\sum_{i=1}^n \boldsymbol{V}_i = \frac{1}{\sqrt{n}}\boldsymbol{X}'\boldsymbol{U}$; that is,

$$\frac{1}{\sqrt{n}}\boldsymbol{X}'\boldsymbol{U} \xrightarrow{d} N(\boldsymbol{0}_{k+1}, \boldsymbol{\Sigma}_V). \tag{19.79}$$

The result in Equation (19.12) follows from Equations (19.15) and (19.79), the consistency of $\boldsymbol{X}'\boldsymbol{X}/n$, the fourth least squares assumption (which ensures that $(\boldsymbol{X}'\boldsymbol{X})^{-1}$ exists), and Slutsky's theorem.

APPENDIX

19.4 Derivations of Exact Distributions of OLS Test Statistics with Normal Errors

This appendix presents the proofs of the distributions under the null hypothesis of the homoskedasticity-only *t*-statistic in Equation (19.35) and the homoskedasticity-only *F*-statistic in Equation (19.37), assuming that all six assumptions in Key Concept 19.1 hold.

Proof of Equation (19.35)

If (i) Z has a standard normal distribution, (ii) W has a χ^2_m distribution, and (iii) Z and W are independently distributed, then the random variable $Z/\sqrt{W/m}$ has the t distribution with m degrees of freedom (Appendix 18.1). To put $\tilde{t}$ in this form, notice that $\hat{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\beta}}} = (s^2_{\hat{u}}/\sigma^2_u)\boldsymbol{\Sigma}_{\hat{\boldsymbol{\beta}}|\boldsymbol{X}}$. Then rewrite Equation (19.34) as

$$\tilde{t} = \frac{(\hat{\beta}_j - \beta_{j,0})/\sqrt{(\boldsymbol{\Sigma}_{\hat{\boldsymbol{\beta}}|\boldsymbol{X}})_{jj}}}{\sqrt{W/(n-k-1)}}, \tag{19.80}$$

where $W = (n-k-1)(s^2_{\hat{u}}/\sigma^2_u)$, and let $Z = (\hat{\beta}_j - \beta_{j,0})/\sqrt{(\boldsymbol{\Sigma}_{\hat{\boldsymbol{\beta}}|\boldsymbol{X}})_{jj}}$ and $m = n-k-1$. With these definitions, $\tilde{t} = Z/\sqrt{W/m}$. Thus, to prove the result in Equation (19.35), we must show (i) through (iii) for these definitions of Z, W, and m.

i. An implication of Equation (19.30) is that, under the null hypothesis, $Z = (\hat{\beta}_j - \beta_{j,0})/\sqrt{(\boldsymbol{\Sigma}_{\hat{\boldsymbol{\beta}}|\boldsymbol{X}})_{jj}}$ has an exact standard normal distribution, which shows (i).
ii. From Equation (19.31), W is distributed as χ^2_{n-k-1}, which shows (ii).
iii. To show (iii), it must be shown that $\hat{\beta}_j$ and $s^2_{\hat{u}}$ are independently distributed.

From Equations (19.14) and (19.29), $\hat{\boldsymbol{\beta}} - \boldsymbol{\beta} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{U}$ and $s^2_{\hat{u}} = (\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{U})'(\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{U})/(n-k-1)$. Thus $\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}$ and $s^2_{\hat{u}}$ are independent if $(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{U}$ and $\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{U}$ are independent. Both $(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{U}$ and $\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{U}$ are linear combinations of $\boldsymbol{U}$, which has an $\boldsymbol{N}(\boldsymbol{0}_{n\times 1}, \sigma^2_u\boldsymbol{I}_n)$ distribution, conditional on $\boldsymbol{X}$. But because $\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1} = \boldsymbol{0}_{n\times(k+1)}$ [Equation (19.26)], it follows that $(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{U}$ and $\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{U}$ are independently distributed [Equation (19.76)]. Consequently, under all six assumptions in Key Concept 19.1,

$$\hat{\boldsymbol{\beta}} \text{ and } s^2_{\hat{u}} \text{ are independently distributed,} \tag{19.81}$$

which shows (iii) and thus proves Equation (19.35).

Proof of Equation (19.37)

The F_{n_1,n_2} distribution is the distribution of $(W_1/n_1)/(W_2/n_2)$, where (i) W_1 is distributed $\chi^2_{n_1}$; (ii) W_2 is distributed $\chi^2_{n_2}$; and (iii) W_1 and W_2 are independently distributed (Appendix 18.1). To express $\tilde{F}$ in this form, let $W_1 = (\boldsymbol{R}\hat{\boldsymbol{\beta}} - \boldsymbol{r})'[\boldsymbol{R}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{R}'\sigma^2_u]^{-1}(\boldsymbol{R}\hat{\boldsymbol{\beta}} - \boldsymbol{r})$ and $W_2 = (n-k-1)s^2_{\hat{u}}/\sigma^2_u$. Substitution of these definitions into Equation (19.36) shows that $\tilde{F} = (W_1/q)/[W_2/(n-k-1)]$. Thus, by the definition of the F distribution, $\tilde{F}$ has an $F_{q,n-k-1}$ distribution if (i) through (iii) hold with $n_1 = q$ and $n_2 = n-k-1$.

i. Under the null hypothesis, $\boldsymbol{R}\hat{\boldsymbol{\beta}} - \boldsymbol{r} = \boldsymbol{R}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$. Because $\hat{\boldsymbol{\beta}}$ has the conditional normal distribution in Equation (19.30) and because $\boldsymbol{R}$ is a nonrandom matrix, $\boldsymbol{R}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$ is distributed $N(\boldsymbol{0}_{q\times 1}, \boldsymbol{R}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{R}'\sigma_u^2)$, conditional on $\boldsymbol{X}$. Thus, by Equation (19.77) in Appendix 19.2, $(\boldsymbol{R}\hat{\boldsymbol{\beta}} - \boldsymbol{r})'[\boldsymbol{R}(\boldsymbol{X}'\boldsymbol{X})\boldsymbol{R}'\sigma_u^2]^{-1}(\boldsymbol{R}\hat{\boldsymbol{\beta}} - \boldsymbol{r})$ is distributed χ_q^2, proving (i).

ii. Requirement (ii) is shown in Equation (19.31).

iii. It has already been shown that $\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}$ and $s_{\hat{u}}^2$ are independently distributed [Equation (19.81)]. It follows that $\boldsymbol{R}\hat{\boldsymbol{\beta}} - \boldsymbol{r}$ and $s_{\hat{u}}^2$ are independently distributed, which in turn implies that W_1 and W_2 are independently distributed, proving (iii) and completing the proof.

APPENDIX

19.5 Proof of the Gauss–Markov Theorem for Multiple Regression

This appendix proves the Gauss–Markov theorem (Key Concept 19.3) for the multiple regression model. Let $\widetilde{\boldsymbol{\beta}}$ be a linear conditionally unbiased estimator of $\boldsymbol{\beta}$ so that $\widetilde{\boldsymbol{\beta}} = \boldsymbol{A}'\boldsymbol{Y}$ and $E(\widetilde{\boldsymbol{\beta}} \,|\, \boldsymbol{X}) = \boldsymbol{\beta}$, where $\boldsymbol{A}$ is an $n \times (k + 1)$ matrix that can depend on $\boldsymbol{X}$ and nonrandom constants. We show that $\text{var}(\boldsymbol{c}'\hat{\boldsymbol{\beta}}) \leq \text{var}(\boldsymbol{c}'\widetilde{\boldsymbol{\beta}})$ for all $k + 1$ dimensional vectors $\boldsymbol{c}$, where the inequality holds with equality only if $\widetilde{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}}$.

Because $\widetilde{\boldsymbol{\beta}}$ is linear, it can be written as $\widetilde{\boldsymbol{\beta}} = \boldsymbol{A}'\boldsymbol{Y} = \boldsymbol{A}'(\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{U}) = (\boldsymbol{A}'\boldsymbol{X})\boldsymbol{\beta} + \boldsymbol{A}'\boldsymbol{U}$. By the first Gauss–Markov condition, $E(\boldsymbol{U}|\boldsymbol{X}) = \boldsymbol{0}_{n\times 1}$, so $E(\widetilde{\boldsymbol{\beta}} \,|\boldsymbol{X}) = (\boldsymbol{A}'\boldsymbol{X})\boldsymbol{\beta}$, but because $\widetilde{\boldsymbol{\beta}}$ is conditionally unbiased, $E(\widetilde{\boldsymbol{\beta}} \,|\, \boldsymbol{X}) = \boldsymbol{\beta} = (\boldsymbol{A}'\boldsymbol{X})\boldsymbol{\beta}$, which implies that $\boldsymbol{A}'\boldsymbol{X} = \boldsymbol{I}_{k+1}$. Thus $\widetilde{\boldsymbol{\beta}} = \boldsymbol{\beta} + \boldsymbol{A}'\boldsymbol{U}$, so $\text{var}(\widetilde{\boldsymbol{\beta}} \,|\boldsymbol{X}) = \text{var}(\boldsymbol{A}'\boldsymbol{U}|\boldsymbol{X}) = E(\boldsymbol{A}'\boldsymbol{U}\boldsymbol{U}'\boldsymbol{A}|\boldsymbol{X}) = \boldsymbol{A}'E(\boldsymbol{U}\boldsymbol{U}'|\boldsymbol{X})\boldsymbol{A} = \boldsymbol{\sigma}_u^2\boldsymbol{A}'\boldsymbol{A}$, where the third equality follows because $\boldsymbol{A}$ can depend on $\boldsymbol{X}$ but not $\boldsymbol{U}$ and the final equality follows from the second Gauss–Markov condition. That is, if $\widetilde{\boldsymbol{\beta}}$ is linear and unbiased, then under the Gauss–Markov conditions,

$$\boldsymbol{A}'\boldsymbol{X} = \boldsymbol{I}_{k+1} \text{ and } \text{var}(\widetilde{\boldsymbol{\beta}}|\boldsymbol{X}) = \sigma_u^2\boldsymbol{A}'\boldsymbol{A}. \tag{19.82}$$

The results in Equation (19.82) also apply to $\hat{\boldsymbol{\beta}}$ with $\boldsymbol{A} = \hat{\boldsymbol{A}} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}$, where $(\boldsymbol{X}'\boldsymbol{X})^{-1}$ exists by the third Gauss–Markov condition.

Now let $\boldsymbol{A} = \hat{\boldsymbol{A}} + \boldsymbol{D}$, so that $\boldsymbol{D}$ is the difference between the matrices $\boldsymbol{A}$ and $\hat{\boldsymbol{A}}$. Note that $\hat{\boldsymbol{A}}'\boldsymbol{A} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{A} = (\boldsymbol{X}'\boldsymbol{X})^{-1}$ [by Equation (19.82)] and $\hat{\boldsymbol{A}}'\hat{\boldsymbol{A}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1} = (\boldsymbol{X}'\boldsymbol{X})^{-1}$, so $\hat{\boldsymbol{A}}'\boldsymbol{D} = \hat{\boldsymbol{A}}'(\boldsymbol{A} - \hat{\boldsymbol{A}}) = \hat{\boldsymbol{A}}'\boldsymbol{A} - \hat{\boldsymbol{A}}'\hat{\boldsymbol{A}} = \boldsymbol{0}_{(k+1)\times(k+1)}$. Substituting $\boldsymbol{A} = \hat{\boldsymbol{A}} + \boldsymbol{D}$ into the formula for the conditional variance in Equation (19.82) yields

$$\begin{aligned}
\text{var}(\widetilde{\boldsymbol{\beta}}|\boldsymbol{X}) &= \sigma_u^2(\hat{\boldsymbol{A}} + \boldsymbol{D})'(\hat{\boldsymbol{A}} + \boldsymbol{D}) \\
&= \sigma_u^2[\hat{\boldsymbol{A}}'\hat{\boldsymbol{A}} + \hat{\boldsymbol{A}}'\boldsymbol{D} + \boldsymbol{D}'\hat{\boldsymbol{A}} + \boldsymbol{D}'\boldsymbol{D}] \\
&= \sigma_u^2(\boldsymbol{X}'\boldsymbol{X})^{-1} + \sigma_u^2\boldsymbol{D}'\boldsymbol{D},
\end{aligned} \tag{19.83}$$

where the final equality uses the facts $\hat{\boldsymbol{A}}'\hat{\boldsymbol{A}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}$ and $\hat{\boldsymbol{A}}'\boldsymbol{D} = \boldsymbol{0}_{(k+1)\times(k+1)}$.

Because $\text{var}(\hat{\boldsymbol{\beta}}\,|\,\boldsymbol{X}) = \sigma_u^2(\boldsymbol{X}'\boldsymbol{X})^{-1}$, Equations (19.82) and (19.83) imply that $\text{var}(\widetilde{\boldsymbol{\beta}}\,|\boldsymbol{X}) - \text{var}(\hat{\boldsymbol{\beta}}|\boldsymbol{X}) = \sigma_u^2\boldsymbol{D}'\boldsymbol{D}$. The difference between the variances of the two estimators of the linear combination $\boldsymbol{c}'\boldsymbol{\beta}$ thus is

$$\text{var}(\boldsymbol{c}'\widetilde{\boldsymbol{\beta}}|\boldsymbol{X}) - \text{var}(\boldsymbol{c}'\hat{\boldsymbol{\beta}}|\boldsymbol{X}) = \sigma_u^2\boldsymbol{c}'\boldsymbol{D}'\boldsymbol{D}\boldsymbol{c} \geq \boldsymbol{0}. \tag{19.84}$$

The inequality in Equation (19.84) holds for all linear combinations $\boldsymbol{c}'\boldsymbol{\beta}$, and the inequality holds with equality for all nonzero $\boldsymbol{c}$ only if $\boldsymbol{D} = \boldsymbol{0}_{n\times(k+1)}$—that is, if $\boldsymbol{A} = \hat{\boldsymbol{A}}$ or, equivalently, $\widetilde{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}}$. Thus $\boldsymbol{c}'\hat{\boldsymbol{\beta}}$ has the smallest variance of all linear conditionally unbiased estimators of $\boldsymbol{c}'\boldsymbol{\beta}$; that is, the OLS estimator is BLUE.

APPENDIX

19.6 Proof of Selected Results for IV and GMM Estimation

The Efficiency of TSLS Under Homoskedasticity [Proof of Equation (19.62)]

When the errors u_i are homoskedastic, the difference between $\boldsymbol{\Sigma}_{\boldsymbol{A}}^{IV}$ [Equation (19.61)] and $\boldsymbol{\Sigma}^{TSLS}$ [Equation (19.55)] is given by

$$\begin{aligned}\boldsymbol{\Sigma}_{\boldsymbol{A}}^{IV} - \boldsymbol{\Sigma}^{TSLS} &= (\boldsymbol{Q}_{\boldsymbol{XZ}}\boldsymbol{A}\boldsymbol{Q}_{\boldsymbol{ZX}})^{-1}\boldsymbol{Q}_{\boldsymbol{XZ}}\boldsymbol{A}\boldsymbol{Q}_{\boldsymbol{ZZ}}\boldsymbol{A}\boldsymbol{Q}_{\boldsymbol{ZX}}(\boldsymbol{Q}_{\boldsymbol{XZ}}\boldsymbol{A}\boldsymbol{Q}_{\boldsymbol{ZX}})^{-1}\sigma_u^2 - (\boldsymbol{Q}_{\boldsymbol{XZ}}\boldsymbol{Q}_{\boldsymbol{ZZ}}^{-1}\boldsymbol{Q}_{\boldsymbol{ZX}})^{-1}\sigma_u^2 \\ &= (\boldsymbol{Q}_{\boldsymbol{XZ}}\boldsymbol{A}\boldsymbol{Q}_{\boldsymbol{ZX}})^{-1}\boldsymbol{Q}_{\boldsymbol{XZ}}\boldsymbol{A}[\boldsymbol{Q}_{\boldsymbol{ZZ}} - \boldsymbol{Q}_{\boldsymbol{ZX}}(\boldsymbol{Q}_{\boldsymbol{XZ}}\boldsymbol{Q}_{\boldsymbol{ZZ}}^{-1}\boldsymbol{Q}_{\boldsymbol{ZX}})^{-1}\boldsymbol{Q}_{\boldsymbol{XZ}}]\boldsymbol{A}\boldsymbol{Q}_{\boldsymbol{ZX}}(\boldsymbol{Q}_{\boldsymbol{XZ}}\boldsymbol{A}\boldsymbol{Q}_{\boldsymbol{ZX}})^{-1}\sigma_u^2, \end{aligned} \tag{19.85}$$

where the second term within the brackets in the second equality follows from $(\boldsymbol{Q}_{\boldsymbol{XZ}}\boldsymbol{A}\boldsymbol{Q}_{\boldsymbol{ZX}})^{-1}\boldsymbol{Q}_{\boldsymbol{XZ}}\boldsymbol{A}\boldsymbol{Q}_{\boldsymbol{ZX}} = \boldsymbol{I}_{(k+r+1)}$. Let $\boldsymbol{F}$ be the matrix square root of $\boldsymbol{Q}_{\boldsymbol{ZZ}}$, so $\boldsymbol{Q}_{\boldsymbol{ZZ}} = \boldsymbol{F}'\boldsymbol{F}$ and $\boldsymbol{Q}_{\boldsymbol{ZZ}}^{-1} = \boldsymbol{F}^{-1}\boldsymbol{F}^{-1'}$. [The latter equality follows from noting that $(\boldsymbol{F}'\boldsymbol{F})^{-1} = \boldsymbol{F}^{-1}\boldsymbol{F}'^{-1}$ and $\boldsymbol{F}'^{-1} = \boldsymbol{F}^{-1'}$.] Then the final expression in Equation (19.85) can be rewritten to yield

$$\begin{aligned}\boldsymbol{\Sigma}_{\boldsymbol{A}}^{IV} - \boldsymbol{\Sigma}^{TSLS} = (\boldsymbol{Q}_{\boldsymbol{XZ}}\boldsymbol{A}\boldsymbol{Q}_{\boldsymbol{ZX}})^{-1}\boldsymbol{Q}_{\boldsymbol{XZ}}\boldsymbol{A}\boldsymbol{F}'[\boldsymbol{I} - \boldsymbol{F}^{-1'}\boldsymbol{Q}_{\boldsymbol{ZX}}(\boldsymbol{Q}_{\boldsymbol{XZ}}\boldsymbol{F}^{-1}\boldsymbol{F}^{-1'}\boldsymbol{Q}_{\boldsymbol{ZX}})^{-1}\boldsymbol{Q}_{\boldsymbol{XZ}}\boldsymbol{F}^{-1}] \\ \times \boldsymbol{F}\boldsymbol{A}\boldsymbol{Q}_{\boldsymbol{ZX}}(\boldsymbol{Q}_{\boldsymbol{XZ}}\boldsymbol{A}\boldsymbol{Q}_{\boldsymbol{ZX}})^{-1}\sigma_u^2,\end{aligned} \tag{19.86}$$

where the second expression within the brackets uses $\boldsymbol{F}'\boldsymbol{F}^{-1'} = \boldsymbol{I}$. Thus

$$\boldsymbol{c}'(\boldsymbol{\Sigma}_{\boldsymbol{A}}^{IV} - \boldsymbol{\Sigma}^{TSLS})\boldsymbol{c} = \boldsymbol{d}'[\boldsymbol{I} - \boldsymbol{D}(\boldsymbol{D}'\boldsymbol{D})^{-1}\boldsymbol{D}']\boldsymbol{d}\sigma_u^2, \tag{19.87}$$

where $\boldsymbol{d} = \boldsymbol{F}\boldsymbol{A}\boldsymbol{Q}_{\boldsymbol{ZX}}(\boldsymbol{Q}_{\boldsymbol{XZ}}\boldsymbol{A}\boldsymbol{Q}_{\boldsymbol{ZX}})^{-1}\boldsymbol{c}$ and $\boldsymbol{D} = \boldsymbol{F}^{-1'}\boldsymbol{Q}_{\boldsymbol{ZX}}$. Now $\boldsymbol{I} - \boldsymbol{D}(\boldsymbol{D}'\boldsymbol{D})^{-1}\boldsymbol{D}'$ is a symmetric idempotent matrix (Exercise 19.5). As a result, $\boldsymbol{I} - \boldsymbol{D}(\boldsymbol{D}'\boldsymbol{D})^{-1}\boldsymbol{D}'$ has eigenvalues that are either 0 or 1, and $\boldsymbol{d}'[\boldsymbol{I} - \boldsymbol{D}(\boldsymbol{D}'\boldsymbol{D})^{-1}\boldsymbol{D}']\boldsymbol{d} \geq 0$ (Exercise 19.10). Thus $\boldsymbol{c}'(\boldsymbol{\Sigma}_{\boldsymbol{A}}^{IV} - \boldsymbol{\Sigma}^{TSLS})\boldsymbol{c} \geq \boldsymbol{0}$, proving that TSLS is efficient under homoskedasticity.

Asymptotic Distribution of the *J*-Statistic Under Homoskedasticity

The *J*-statistic is defined in Equation (19.63). First note that

$$\begin{aligned}\hat{\boldsymbol{U}} &= \boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}^{TSLS} \\ &= \boldsymbol{Y} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{Z}}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{Z}}\boldsymbol{Y} \\ &= (\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{U}) - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{Z}}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{Z}}(\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{U}) \\ &= \boldsymbol{U} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{Z}}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{Z}}\boldsymbol{U} \\ &= [\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{Z}}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{Z}}]\boldsymbol{U}. \end{aligned} \tag{19.88}$$

Thus

$$\begin{aligned}\hat{\boldsymbol{U}}\boldsymbol{P}_{\boldsymbol{Z}}\hat{\boldsymbol{U}} &= \boldsymbol{U}'[\boldsymbol{I} - \boldsymbol{P}_{\boldsymbol{Z}}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{Z}}\boldsymbol{X})^{-1}\boldsymbol{X}']\boldsymbol{P}_{\boldsymbol{Z}}[\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{Z}}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{Z}}]\boldsymbol{U} \\ &= \boldsymbol{U}'[\boldsymbol{P}_{\boldsymbol{Z}} - \boldsymbol{P}_{\boldsymbol{Z}}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{Z}}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{Z}}]\boldsymbol{U}, \end{aligned} \tag{19.89}$$

where the second equality follows by simplifying the preceding expression. Because $\boldsymbol{Z}'\boldsymbol{Z}$ is symmetric and positive definite, it can be written in terms of its matrix square root, $\boldsymbol{Z}'\boldsymbol{Z} = (\boldsymbol{Z}'\boldsymbol{Z})^{1/2'}(\boldsymbol{Z}'\boldsymbol{Z})^{1/2}$, and this matrix square root is invertible, so $(\boldsymbol{Z}'\boldsymbol{Z})^{-1} = (\boldsymbol{Z}'\boldsymbol{Z})^{-1/2}(\boldsymbol{Z}'\boldsymbol{Z})^{-1/2'}$, where $(\boldsymbol{Z}'\boldsymbol{Z})^{-1/2} = [(\boldsymbol{Z}'\boldsymbol{Z})^{1/2}]^{-1}$. Thus $\boldsymbol{P}_{\boldsymbol{Z}}$ can be written as $\boldsymbol{P}_{\boldsymbol{Z}} = \boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}' = \boldsymbol{B}\boldsymbol{B}'$ where $\boldsymbol{B} = \boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1/2}$. Substituting this expression for $\boldsymbol{P}_{\boldsymbol{Z}}$ into the final expression in Equation (19.89) yields

$$\begin{aligned}\hat{\boldsymbol{U}}'\boldsymbol{P}_{\boldsymbol{Z}}\hat{\boldsymbol{U}} &= \boldsymbol{U}'[\boldsymbol{B}\boldsymbol{B}' - \boldsymbol{B}\boldsymbol{B}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{B}\boldsymbol{B}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{B}\boldsymbol{B}']\boldsymbol{U} \\ &= \boldsymbol{U}'\boldsymbol{B}\,[\boldsymbol{I} - \boldsymbol{B}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{B}\boldsymbol{B}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{B}]\boldsymbol{B}'\boldsymbol{U} \\ &= \boldsymbol{U}'\boldsymbol{B}\boldsymbol{M}_{\boldsymbol{B}'\boldsymbol{X}}\boldsymbol{B}'\boldsymbol{U}, \end{aligned} \tag{19.90}$$

where $\boldsymbol{M}_{\boldsymbol{B}'\boldsymbol{X}} = \boldsymbol{I} - \boldsymbol{B}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{B}\boldsymbol{B}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{B}$ is a symmetric idempotent matrix.

The asymptotic null distribution of $\hat{\boldsymbol{U}}'\boldsymbol{P}_{\boldsymbol{Z}}\hat{\boldsymbol{U}}$ is found by computing the limits in probability and in distribution of the various terms in the final expression in Equation (19.90) under the null hypothesis. Under the null hypothesis that $E(\boldsymbol{Z}_i u_i) = 0$, $\boldsymbol{Z}'\boldsymbol{U}/\sqrt{n}$ has mean 0, and the central limit theorem applies, so $\boldsymbol{Z}'\boldsymbol{U}/\sqrt{n} \xrightarrow{d} N(0, \boldsymbol{Q}_{\boldsymbol{Z}\boldsymbol{Z}}\sigma_u^2)$. In addition, $\boldsymbol{Z}'\boldsymbol{Z}/n \xrightarrow{p} \boldsymbol{Q}_{\boldsymbol{Z}\boldsymbol{Z}}$ and $\boldsymbol{X}'\boldsymbol{Z}/n \xrightarrow{p} \boldsymbol{Q}_{\boldsymbol{X}\boldsymbol{Z}}$. Thus $\boldsymbol{B}'\boldsymbol{U} = (\boldsymbol{Z}'\boldsymbol{Z})^{-1/2'}\boldsymbol{Z}'\boldsymbol{U} = (\boldsymbol{Z}'\boldsymbol{Z}/n)^{-1/2'}(\boldsymbol{Z}'\boldsymbol{U}/\sqrt{n}) \xrightarrow{d} \sigma_u z$, where z is distributed $N(\boldsymbol{0}_{m+r+1}, \boldsymbol{I}_{m+r+1})$. In addition, $\boldsymbol{B}'\boldsymbol{X}/\sqrt{n} = (\boldsymbol{Z}'\boldsymbol{Z}/n)^{-1/2'}(\boldsymbol{Z}'\boldsymbol{X}/n) \xrightarrow{p} \boldsymbol{Q}_{\boldsymbol{Z}\boldsymbol{Z}}^{-1/2}\boldsymbol{Q}_{\boldsymbol{Z}\boldsymbol{X}}$, so $\boldsymbol{M}_{\boldsymbol{B}'\boldsymbol{X}} \xrightarrow{p} \boldsymbol{I} - \boldsymbol{Q}_{\boldsymbol{Z}\boldsymbol{Z}}^{-1/2}\boldsymbol{Q}_{\boldsymbol{Z}\boldsymbol{X}}(\boldsymbol{Q}_{\boldsymbol{X}\boldsymbol{Z}}\boldsymbol{Q}_{\boldsymbol{Z}\boldsymbol{Z}}^{-1/2'}\boldsymbol{Q}_{\boldsymbol{Z}\boldsymbol{Z}}^{-1/2}\boldsymbol{Q}_{\boldsymbol{Z}\boldsymbol{X}})^{-1}\boldsymbol{Q}_{\boldsymbol{X}\boldsymbol{Z}}\boldsymbol{Q}_{\boldsymbol{Z}\boldsymbol{Z}}^{-1/2'} = \boldsymbol{M}_{\boldsymbol{Q}_{\boldsymbol{Z}\boldsymbol{Z}}^{-1/2}\boldsymbol{Q}_{\boldsymbol{Z}\boldsymbol{X}}}$. Thus

$$\hat{\boldsymbol{U}}'\boldsymbol{P}_{\boldsymbol{Z}}\hat{\boldsymbol{U}} \xrightarrow{d} (\boldsymbol{z}'\boldsymbol{M}_{\boldsymbol{Q}_{\boldsymbol{X}\boldsymbol{Z}}\boldsymbol{Q}_{\boldsymbol{Z}\boldsymbol{Z}}^{-1/2}}\boldsymbol{z})\sigma_u^2. \tag{19.91}$$

Under the null hypothesis, the TSLS estimator is consistent, and the coefficients in the regression of $\hat{\boldsymbol{U}}$ on $\boldsymbol{Z}$ converge in probability to 0 [an implication of Equation (19.91)], so the denominator in the definition of the *J*-statistic is a consistent estimator of σ_u^2:

$$\hat{\boldsymbol{U}}'\boldsymbol{M}_{\boldsymbol{Z}}\hat{\boldsymbol{U}}/(n - m - r - 1) \xrightarrow{p} \sigma_u^2. \tag{19.92}$$

From the definition of the *J*-statistic and Equations (19.91) and (19.92), it follows that

$$\boldsymbol{J} = \frac{\hat{\boldsymbol{U}}'\boldsymbol{P}_{\boldsymbol{Z}}\hat{\boldsymbol{U}}}{\hat{\boldsymbol{U}}'\boldsymbol{M}_{\boldsymbol{Z}}\hat{\boldsymbol{U}}/(n - m - r - 1)} \xrightarrow{d} \boldsymbol{z}'\boldsymbol{M}_{\boldsymbol{Q}_{\boldsymbol{Z}\boldsymbol{Z}}^{-1/2}\boldsymbol{Q}_{\boldsymbol{X}\boldsymbol{Z}}}\boldsymbol{z}. \tag{19.93}$$

Because $\boldsymbol{z}$ is a standard normal random vector and $\boldsymbol{M}_{\boldsymbol{Q}_{\boldsymbol{ZZ}}^{-1/2}\boldsymbol{Q}_{\boldsymbol{ZX}}}$ is a symmetric idempotent matrix, $\boldsymbol{J}$ is distributed as a chi-squared random variable with degrees of freedom that equals the rank of $\boldsymbol{M}_{\boldsymbol{Q}_{\boldsymbol{ZZ}}^{-1/2}\boldsymbol{Q}_{\boldsymbol{ZX}}}$ [Equation (19.78)]. Because $\boldsymbol{Q}_{\boldsymbol{ZZ}}^{-1/2}\boldsymbol{Q}_{\boldsymbol{ZX}}$ is $(m + r + 1) \times (k + r + 1)$ and $m > k$, the rank of $\boldsymbol{M}_{\boldsymbol{Q}_{\boldsymbol{ZZ}}^{-1/2}\boldsymbol{Q}_{\boldsymbol{ZX}}}$ is $m - k$ [Exercise 19.5]. Thus $\boldsymbol{J} \xrightarrow{d} \chi^2_{m-k}$, which is the result stated in Equation (19.64).

The Efficiency of the Efficient GMM Estimator

The infeasible efficient GMM estimator, $\widetilde{\boldsymbol{\beta}}^{Eff.GMM}$, is defined in Equation (19.66). The proof that $\widetilde{\boldsymbol{\beta}}^{Eff.GMM}$ is efficient entails showing that $\boldsymbol{c}'(\boldsymbol{\Sigma}_{\boldsymbol{A}}^{IV} - \boldsymbol{\Sigma}^{Eff.GMM})\boldsymbol{c} \geq 0$ for all vectors $\boldsymbol{c}$. The proof closely parallels the proof of the efficiency of the TSLS estimator in the first section of this appendix, with the sole modification that $\boldsymbol{H}^{-1}$ replaces $\boldsymbol{Q}_{\boldsymbol{ZZ}}\sigma_u^2$ in Equation (19.85) and subsequently.

Distribution of the GMM *J*-Statistic

The GMM *J*-statistic is given in Equation (19.70). The proof that, under the null hypothesis, $J^{GMM} \xrightarrow{d} \chi^2_{m-k}$ closely parallels the corresponding proof for the TSLS *J*-statistic under homoskedasticity.

APPENDIX

19.7 Regression with Many Predictors: MSPE, Ridge Regression, and Principal Components Analysis

This appendix presents the derivations for various results used in Chapter 14 that rely on matrix calculations.

The MSPE for Linear Regression Estimated by OLS

We first derive Equation (14.4), the mean squared prediction error (MSPE) of the OLS estimator under homoskedasticity.

Let the $k \times 1$ vector $\boldsymbol{X}^{oos}$ denote the values of the X's for the out-of-sample observation ("oos") to be predicted. With this notation, the MSPE in Equation (14.3), written using matrix notation, is

$$\text{MSPE} = \sigma_u^2 + E[(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\boldsymbol{X}^{oos}]^2, \tag{19.94}$$

where $\hat{\boldsymbol{\beta}}$ denotes any estimator of $\boldsymbol{\beta}$, not just the OLS estimator.

Under the least squares assumptions for prediction, the out-of-sample observation is assumed to be an i.i.d. draw from the same population as the estimation sample. Under this assumption, the MSPE in Equation (19.94) can be written

$$\text{MSPE} = \sigma_u^2 + \text{trace}\{E[(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})']\boldsymbol{Q}_{\boldsymbol{X}}\}, \tag{19.95}$$

where $\boldsymbol{Q}_X = E(\boldsymbol{X}'\boldsymbol{X})$. Equation (19.95) follows from Equation (19.94) by writing, $E[(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\boldsymbol{X}^{oos}]^2 = E[\boldsymbol{X}^{oos\prime}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\boldsymbol{X}^{oos}] = \text{trace}E[(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'\boldsymbol{X}^{oos}\boldsymbol{X}^{oos'}] = \text{trace}E[(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})']\boldsymbol{Q}_X$, where the second inequality uses the property of the trace that $\boldsymbol{a}'\boldsymbol{B}\boldsymbol{a} = \text{trace}(\boldsymbol{B}\boldsymbol{a}\boldsymbol{a}')$ for $n \times n$ matrix $\boldsymbol{B}$ and $n \times 1$ vector $\boldsymbol{a}$ and where the final equality uses the assumptions that the out-of-sample observation is independent of the estimation observations and that it is drawn from the same distribution, so that $\boldsymbol{E}(\boldsymbol{X}^{oos}\boldsymbol{X}^{oos'}) = \boldsymbol{Q}_X$.

The MSPE for OLS obtains by substituting the expression for OLS in Equation (19.14) into Equation (19.95) and simplifying. First note that, under the assumption of homoskedasticity, for the OLS estimator,

$$\begin{aligned} E[(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'] &= E[(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{u}\boldsymbol{u}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}] \\ &= E[(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'E(\boldsymbol{u}\boldsymbol{u}'|\boldsymbol{X})\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}] \\ &= E[(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}]\sigma_u^2 = E[(\boldsymbol{X}'\boldsymbol{X})^{-1}]\sigma_u^2, \end{aligned}$$

where the first equality uses Equation (19.14); the second equality uses the law of iterated expectations; the third equality uses the assumption of homoskedasticity, so $E(\boldsymbol{u}\boldsymbol{u}'|\boldsymbol{X}) = \sigma_u^2 I_n$; and the final equality simplifies. Substitution of $E[(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'] = E[(\boldsymbol{X}'\boldsymbol{X})^{-1}]\sigma_u^2$ into Equation (19.95) and multiplying and dividing the second term by $1/n$ yields

$$\text{MSPE}_{\text{OLS}} = \sigma_u^2 + \frac{1}{n}\text{trace}\left\{E\left[\left(\frac{\boldsymbol{X}'\boldsymbol{X}}{n}\right)^{-1}\right]\boldsymbol{Q}_X\right\}\sigma_u^2. \tag{19.96}$$

Equation (19.96) is the MSPE for a prediction made using the OLS estimator under the least squares assumptions for prediction with homoskedastic errors.

Equation (14.4) is an approximation to Equation (19.96) when n is large relative to k. In that case, $\boldsymbol{X}'\boldsymbol{X}/n \cong \boldsymbol{Q}_X$ (specifically, for fixed k, $\boldsymbol{X}'\boldsymbol{X}/n \xrightarrow{p} \boldsymbol{Q}_X$) so $\text{trace}\{E[(\boldsymbol{X}'\boldsymbol{X}/n)^{-1}]\boldsymbol{Q}_X\} \cong \text{trace}\{\boldsymbol{Q}_X^{-1}\boldsymbol{Q}_X\} = \text{trace}\{\boldsymbol{I}_k\} = k$. Substitution of this final expression into Equation (19.96) and collecting terms yields Equation (14.4):

$$\text{MSPE}_{\text{OLS}} \cong \left(1 + \frac{k}{n}\right)\sigma_u^2. \tag{19.97}$$

Connection to the final prediction error (FPE). Equation (19.97) is used in the derivation of the final prediction error (FPE) for time series forecasting given in Equation (15.21) (with a change in notation so that n is replaced by T and k is replaced by $p + 1$). The key difference between the cross-section and time-series cases is the relation of the out-of-sample observation to the in-sample observations. In the deriviation here, the in- and out-of-sample observations are independent. If the values of the predictors in the time series application are independent of the data used to estimate the coefficients, then the derivation here applies directly. Typically this will not be the case, however, because the final observations in the sample (the ones used to make the out-of-sample forecast) are correlated with the in-sample observations. If the sample size is large, however, then the dependence between the estimated regression coefficients and the out-of-sample predictors is small, so Equation (19.97) still holds as an approximation when the sample size is large relative to the number of regressors.

Ridge Regression

Equation (14.8) provides an expression for the ridge regression estimator with a single regressor. This appendix derives an expression for the case of multiple regressors.

The ridge regression estimator minimizes the penalized sum of squared residuals in Equation (14.7), written here using matrix notation:

$$S^{Ridge}(b; \lambda_{Ridge}) = (\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{b})'(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{b}) + \lambda_{Ridge}\boldsymbol{b}'\boldsymbol{b}. \tag{19.98}$$

Taking the derivative of the right-hand side of Equation (19.98) and setting it to 0 yields the system solved by the ridge regression estimator $\hat{\boldsymbol{\beta}}^{Ridge}$, $-2\boldsymbol{X}'(\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}^{Ridge}) + \mathbf{2}\lambda_{Ridge}\hat{\boldsymbol{\beta}}^{Ridge} = \mathbf{0}$ [cf. Equations (19.9) and (19.10) for OLS)]. Solving this system yields the formula for the ridge regression estimator,

$$\hat{\boldsymbol{\beta}}^{Ridge} = (\boldsymbol{X}'\boldsymbol{X} + \lambda_{Ridge}I_k)^{-1}\boldsymbol{X}'\boldsymbol{Y}. \tag{19.99}$$

We note two implications of this formula that are discussed in Sections 14.3 and 14.4, respectively.

First, if the regressors are uncorrelated in the estimation sample, the ridge regression estimator can be written as the OLS estimator, shrunk toward 0 by a factor that depends on the data, that is, $\hat{\beta}_j^{Ridge} = (1 + \lambda^{Ridge}/\sum_{i=1}^n X_{ji}^2)^{-1}\hat{\beta}_j$, which is Equation (14.8). Moreover, if in addition the regressors are standardized using the sample standard deviation, as they are in the empirical work in Chapter 14, that shrinkage factor simplifies to $[1 + \lambda^{Ridge}/(n-1)]^{-1}$. To show these results, note that if the regressors are uncorrelated, then $\boldsymbol{X}'\boldsymbol{X}$ is diagonal, so that $\boldsymbol{X}'\boldsymbol{X} + \lambda_{Ridge}I_k$ is diagonal with j^{th} diagonal element $\sum_{i=1}^n X_{ji}^2 + \lambda^{Ridge}$. Then Equation (19.99) simplifies, so that the ridge estimator of the j^{th} coefficient β_j is $\hat{\beta}_j^{Ridge} = (\sum_{i=1}^n X_{ji}^2 + \lambda^{Ridge})^{-1}\sum_{i=1}^n X_{ji}Y_i = (1 + \lambda^{Ridge}/\sum_{i=1}^n X_{ji}^2)^{-1}(\sum_{i=1}^n X_{ji}^2)^{-1}\sum_{i=1}^n X_{ji}Y_i = (1 + \lambda^{Ridge}/\sum_{i=1}^n X_{ji}^2)^{-1}\hat{\beta}_j$, where $\hat{\beta}_j$ is the OLS estimator for these uncorrelated regressors. Thus, with uncorrelated regressors, the ridge regression estimator shrinks the OLS estimator toward 0 by the factor $(1 + \lambda^{Ridge}/\sum_{i=1}^n X_{ji}^2)^{-1}$. If in addition the regressors are standardized using the sample standard deviation, then $\sum_{i=1}^n X_{ji}^2 = n - 1$, in which case $\hat{\boldsymbol{\beta}}^{Ridge} = [1 + \lambda^{Ridge}/(n-1)]^{-1}\hat{\boldsymbol{\beta}}$.

Second, as is discussed in Section 14.4, predictions made using the ridge regression estimator, in general, change if different linear combinations of the regressors are used as predictors. Specifically, if $\boldsymbol{X}$ denotes the matrix of predictors, then the ridge predictions made using $\boldsymbol{X}$ and using $\boldsymbol{X}\boldsymbol{A}$ differ, where $\boldsymbol{A}$ is a nonsingular $k \times k$ matrix. This is an important difference between ridge and OLS because OLS yields the same predictions whether $\boldsymbol{X}$ or $\boldsymbol{X}\boldsymbol{A}$ is used.

To show this result, consider the ridge regression estimator computed using $\boldsymbol{X}\boldsymbol{A}$, and denote that estimator by $\hat{\boldsymbol{\beta}}_{\boldsymbol{A}}^{Ridge}$. In this notation, the ridge regression estimator computed using $\boldsymbol{X}$ without the linear transformation is $\hat{\boldsymbol{\beta}}_{\boldsymbol{I}}^{Ridge}$. The same linear transformation must be applied to the out-of-sample and in-sample predictors, so the transformed out-of-sample observation is $\boldsymbol{A}'\boldsymbol{X}^{OOS}$. Thus the out-of-sample predicted value using $\hat{\boldsymbol{\beta}}_{\boldsymbol{A}}^{Ridge}$ is $\hat{Y}_{\boldsymbol{A}}^{OOS} = (\boldsymbol{A}'\boldsymbol{X}^{OOS})'\hat{\boldsymbol{\beta}}_{\boldsymbol{A}}^{Ridge} = \boldsymbol{X}^{OOS'}\boldsymbol{A}\hat{\boldsymbol{\beta}}_{\boldsymbol{A}}^{Ridge}$. In this notation, the out-of-sample predicted value using the original regressors $\boldsymbol{X}$ is $\hat{Y}_{\boldsymbol{I}}^{OOS} = \boldsymbol{X}^{OOS'}\hat{\boldsymbol{\beta}}_{\boldsymbol{I}}^{Ridge}$. From Equation (19.99), the ridge estimator is $\hat{\boldsymbol{\beta}}_{\boldsymbol{A}}^{Ridge} = [(\boldsymbol{X}\boldsymbol{A})'(\boldsymbol{X}\boldsymbol{A}) + \lambda_{Ridge}I_k]^{-1}(\boldsymbol{X}\boldsymbol{A})'\boldsymbol{Y} = (\boldsymbol{A}'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{A} + \lambda_{Ridge}I_k)^{-1}\boldsymbol{A}'\boldsymbol{X}\boldsymbol{Y} = [\boldsymbol{A}'(\boldsymbol{X}'\boldsymbol{X} +$

$\lambda_{Ridge}\boldsymbol{A}'^{-1}\boldsymbol{A}^{-1})\boldsymbol{A}]^{-1}\boldsymbol{A}'\boldsymbol{XY} = \boldsymbol{A}^{-1}[\boldsymbol{X}'\boldsymbol{X} + \lambda_{Ridge}(\boldsymbol{AA}')^{-1}]^{-1}\boldsymbol{XY}$, where the equalities follow by collecting terms using the properties of matrix inverses. Thus the ridge prediction for the out-of-sample observation is $\hat{Y}_A^{OOS} = \boldsymbol{X}^{OOS'}\boldsymbol{A}\hat{\boldsymbol{\beta}}_{\boldsymbol{A}}^{Ridge} = \boldsymbol{X}^{OOS'}[\boldsymbol{X}'\boldsymbol{X} + \lambda_{Ridge}(\boldsymbol{AA}')^{-1}]^{-1}\boldsymbol{XY}$, whereas using the X's without the linear rotation yields the prediction $\hat{Y}_I^{OOS} = \boldsymbol{X}^{OOS'}(\boldsymbol{X}'\boldsymbol{X} + \lambda_{Ridge}\boldsymbol{I}_k)^{-1}\boldsymbol{XY}$. The two predictions differ because the matrix $(\boldsymbol{AA}')^{-1}$ appears in the expression for $\hat{Y}_A^{OOS}$ but not in the expression for $\hat{Y}_I^{OOS}$. The only time that a linear transformation $\boldsymbol{A}$ does not change the ridge predicted value is when the linear transformation is orthonormal—that is, when $\boldsymbol{AA}' = \boldsymbol{I}_k$, so that $(\boldsymbol{AA}')^{-1} = \boldsymbol{I}_k$.

To see that OLS produces the same predicted value, regardless of the linear transformation $\boldsymbol{A}$ (as long as $\boldsymbol{A}$ is nonsingular), note that the OLS predicted value is the ridge predicted value when $\lambda_{Ridge} = 0$. The result follows from substituting $\lambda^{Ridge} = 0$ into the expressions for the ridge predictions $\hat{Y}_A^{OOS}$ and $\hat{Y}_I^{OOS}$ in the previous paragraph.

Principal Components Analysis

This section presents formulas for the principal components of X and shows that the sum of the variances of the principal components equals the sum of the variances of the X's [Equation (14.10)]. The section concludes with an expression for the out-of-sample prediction, computed using the first r principal components, as in Section 14.5, expressed in terms of the out-of-sample values of the predictors, $\boldsymbol{X}^{OOS}$.

In Key Concept 14.2, the j^{th} principal component of $\boldsymbol{X}$ is defined to be the linear combination of $\boldsymbol{X}$ such that (a) the squared weights of the linear combinations sum to 1; (b) the j^{th} principal component is uncorrelated with the previous $j - 1$ principal components; and (c) the j^{th} principal component maximizes the variance of the linear combination, subject to (a) and (b). We now state these criteria mathematically and use them to derive explicit formulas for the principal components. In particular, we show that the linear combination weights used to form the first r principal components are the eigenvectors of $\boldsymbol{X}'\boldsymbol{X}$ corresponding to its r largest eigenvalues.

Let $\boldsymbol{PC}_j$ denote the j^{th} principal component, and let $\boldsymbol{W}_j$ denote the $k \times 1$ vector of weights used to construct $\boldsymbol{PC}_j$, so that $\boldsymbol{PC}_j = \boldsymbol{XW}_j$. The sum of squares of $\boldsymbol{PC}_j$ is $\boldsymbol{PC}_j'\boldsymbol{PC}_j = \boldsymbol{W}_j'\boldsymbol{X}'\boldsymbol{XW}_j$, and the sum of squares weights is $\boldsymbol{W}_j'\boldsymbol{W}_j$. Because $\boldsymbol{X}$ has mean 0 (the X's are standardized), $\boldsymbol{PC}_j'\boldsymbol{PC}_j/(n-1)$ is the sample variance of the j^{th} principal component. The weights $\boldsymbol{W}_j$ are chosen to solve

$$\max_{\boldsymbol{W}_j} \boldsymbol{PC}_j'\boldsymbol{PC}_j = \boldsymbol{W}_j'\boldsymbol{X}'\boldsymbol{XW}_j \text{ subject to } \boldsymbol{W}_j'\boldsymbol{W}_j = 1 \text{ and } \boldsymbol{PC}_j'\boldsymbol{PC}_i = 0 \text{ for } i < j. \quad (19.100)$$

For $j = 1$, the constrained maximization problem is to choose $\boldsymbol{W}_1$ to maximize $\boldsymbol{W}_1'\boldsymbol{X}'\boldsymbol{XW}_1$ subject to $\boldsymbol{W}_1'\boldsymbol{W}_1 = 1$. This constrained maximization is done by maximizing the Lagrangian, $\boldsymbol{W}_1'\boldsymbol{X}'\boldsymbol{XW}_1 - \lambda_1(\boldsymbol{W}_1'\boldsymbol{W}_1 - 1)$, where λ_1 is the Lagrange multiplier. Taking the derivative of the Lagrangian with respect to $\boldsymbol{W}_1$ and setting it to 0 yields

$$\boldsymbol{X}'\boldsymbol{XW}_1 = \lambda_1\boldsymbol{W}_1. \quad (19.101)$$

Equation (19.101) shows that $\boldsymbol{W}_1$ is an eigenvector of $\boldsymbol{X}'\boldsymbol{X}$ and λ_1 is its corresponding eigenvalue, where the eigenvector is normalized to have unit length. Moreover, multiplying

both sides of Equation (19.101) by $\boldsymbol{W}_1'$ shows that $\boldsymbol{W}_1'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{W}_1 = \boldsymbol{PC}_1'\boldsymbol{PC}_1 = \lambda_1$, so that maximizing $\boldsymbol{PC}_1'\boldsymbol{PC}_1$ requires that λ_1 be the largest eigenvalue of $\boldsymbol{X}'\boldsymbol{X}$ and that $\boldsymbol{W}_1$ be the eigenvector of $\boldsymbol{X}'\boldsymbol{X}$ corresponding to the largest eigenvalue.

Now consider $\boldsymbol{W}_2$. There are two constraints, $\boldsymbol{W}_2'\boldsymbol{W}_2 = 1$ and $\boldsymbol{PC}_2'\boldsymbol{PC}_1 = \boldsymbol{W}_2'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{W}_1 = 0$, so the Lagrangian is $\boldsymbol{W}_2'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{W}_2 - \lambda_2(\boldsymbol{W}_2'\boldsymbol{W}_2 - 1) - \gamma_{21}\boldsymbol{W}_2'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{W}_1$, where λ_2 and γ_{21} are Lagrange multipliers. Taking the derivative of the Lagrangian with respect to $\boldsymbol{W}_2$ and setting it to 0 yields

$$\boldsymbol{X}'\boldsymbol{X}\boldsymbol{W}_2 = \lambda_2\boldsymbol{W}_2 + \frac{1}{2}\gamma_{21}\boldsymbol{X}'\boldsymbol{X}\boldsymbol{W}_1. \tag{19.102}$$

First note that multiplying both sides of Equation (19.101) by $\boldsymbol{W}_2'$ yields $\boldsymbol{W}_2'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{W}_1 = \lambda_1\boldsymbol{W}_2'\boldsymbol{W}_1$; because $\boldsymbol{W}_2'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{W}_1 = 0$, it follows that $\boldsymbol{W}_2'\boldsymbol{W}_1 = 0$. Now multiplying both sides of Equation (19.102) by $\boldsymbol{W}_1'$ yields $\boldsymbol{W}_1'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{W}_2 = \lambda_2\boldsymbol{W}_1'\boldsymbol{W}_2 + \frac{1}{2}\gamma_{21}\boldsymbol{W}_1'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{W}_1 = \frac{1}{2}\gamma_{21}\boldsymbol{W}_1'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{W}_1$, but because $\boldsymbol{W}_1'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{W}_2 = \boldsymbol{W}_1'\boldsymbol{W}_2 = 0$, it must be that $\gamma_{21} = 0$. Thus Equation (19.102) reduces to $\boldsymbol{X}'\boldsymbol{X}\boldsymbol{W}_2 = \lambda_2\boldsymbol{W}_2$, so that $\boldsymbol{W}_2$ is an eigenvector of $\boldsymbol{X}'\boldsymbol{X}$ and λ_2 is its corresponding eigenvalue. Multiplying both sides of $\boldsymbol{X}'\boldsymbol{X}\boldsymbol{W}_2 = \lambda_2\boldsymbol{W}_2$ by $\boldsymbol{W}_2'$ and imposing the unit normalization yields $\boldsymbol{W}_2'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{W}_2 = \lambda_2$. Thus, the Lagrangian is maximized by choosing $\boldsymbol{W}_2$ to be the eigenvector corresponding to the largest of the remaining eigenvalues—that is, to the second-largest eigenvalue of $\boldsymbol{X}'\boldsymbol{X}$.

Continuing, these calculations shows that $\boldsymbol{W}_j$ is the unit-length eigenvector of $\boldsymbol{X}'\boldsymbol{X}$ associated with λ_j, the j^{th}-largest eigenvalue of $\boldsymbol{X}'\boldsymbol{X}$; that $\boldsymbol{PC}_j'\boldsymbol{PC}_j = \lambda_j$; and that $\boldsymbol{PC}_j'\boldsymbol{PC}_i = 0$ for $i \neq j$. If $k < n$, only the first k eigenvalues of $\boldsymbol{X}'\boldsymbol{X}$ are nonzero, so the total number of principal components is $\min(n, k)$.

Because the trace of a matrix is equal to the sum of its eigenvalues,

$$\text{trace}(\boldsymbol{X}'\boldsymbol{X}) = \sum_{j=1}^{\min(n,k)} \lambda_j = \sum_{j=1}^{\min(n,k)} \boldsymbol{PC}_j'\boldsymbol{PC}_j. \tag{19.103}$$

Dividing the first and last expressions in Equation (19.103) by $n - 1$ yields Equation (14.10).

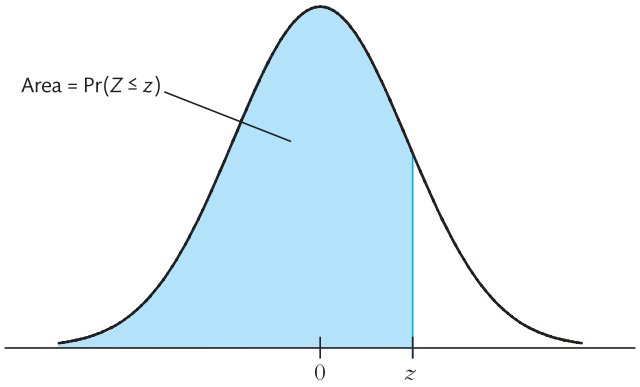
Finally, we provide an expression for the out-of-sample prediction in terms of the out-of-sample value of the predictors, $\boldsymbol{X}^{OOS}$. The first r out-of-sample values of the principal components are $\boldsymbol{PC}_{1:r}^{OOS} = [\boldsymbol{PC}_1^{OOS} \quad \boldsymbol{PC}_2^{OOS} \quad \cdots \quad \boldsymbol{PC}_r^{OOS}] = \boldsymbol{W}_{1:r}' \quad \boldsymbol{X}^{OOS}$, where $\boldsymbol{W}_{1:r} = [\boldsymbol{W}_1 \quad \boldsymbol{W}_2 \quad \cdots \quad \boldsymbol{W}_r]$ are the first r eigenvectors of $\boldsymbol{X}'\boldsymbol{X}$ in the estimation sample. Let $\hat{\gamma}$ denote the $r \times 1$ vector of OLS coefficients in the regression of Y on the first r principal components in the estimation sample. Then the principal components prediction of Y^{OOS} is $\hat{Y}^{OOS} = \hat{\gamma}'\boldsymbol{PC}_{1:r}^{OOS}$. Written in terms of the original regressors, the principal components prediction is

$$\hat{Y}^{OOS} = \hat{\gamma}'\boldsymbol{W}_{1:r}'\boldsymbol{X}^{OOS}. \tag{19.104}$$

This expression was used to compute the entries in Table 14.4 for the principal components prediction.

Appendix

TABLE 1 The Cumulative Standard Normal Distribution Function, $\Phi(z) = \Pr(Z \leq z)$



	Second Decimal Value of z									
z	0	1	2	3	4	5	6	7	8	9
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611

(Table 1 continued)

(Table 1 continued)

	Second Decimal Value of z									
z	0	1	2	3	4	5	6	7	8	9
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

This table can be used to calculate $\Pr(Z \leq z)$ where Z is a standard normal variable. For example, when $z = 1.17$, this probability is 0.8790, which is the table entry for the row labeled 1.1 and the column labeled 7.

TABLE 2 Critical Values for Two-Sided and One-Sided Tests Using the Student *t* Distribution

	Significance Level				
Degrees of Freedom	20% (2-Sided) 10% (1-Sided)	10% (2-Sided) 5% (1-Sided)	5% (2-Sided) 2.5% (1-Sided)	2% (2-Sided) 1% (1-Sided)	1% (2-Sided) 0.5% (1-Sided)
1	3.08	6.31	12.71	31.82	63.66
2	1.89	2.92	4.30	6.96	9.92
3	1.64	2.35	3.18	4.54	5.84
4	1.53	2.13	2.78	3.75	4.60
5	1.48	2.02	2.57	3.36	4.03
6	1.44	1.94	2.45	3.14	3.71
7	1.41	1.89	2.36	3.00	3.50
8	1.40	1.86	2.31	2.90	3.36
9	1.38	1.83	2.26	2.82	3.25
10	1.37	1.81	2.23	2.76	3.17
11	1.36	1.80	2.20	2.72	3.11
12	1.36	1.78	2.18	2.68	3.05
13	1.35	1.77	2.16	2.65	3.01
14	1.35	1.76	2.14	2.62	2.98
15	1.34	1.75	2.13	2.60	2.95
16	1.34	1.75	2.12	2.58	2.92
17	1.33	1.74	2.11	2.57	2.90
18	1.33	1.73	2.10	2.55	2.88
19	1.33	1.73	2.09	2.54	2.86
20	1.33	1.72	2.09	2.53	2.85
21	1.32	1.72	2.08	2.52	2.83
22	1.32	1.72	2.07	2.51	2.82
23	1.32	1.71	2.07	2.50	2.81
24	1.32	1.71	2.06	2.49	2.80
25	1.32	1.71	2.06	2.49	2.79
26	1.32	1.71	2.06	2.48	2.78
27	1.31	1.70	2.05	2.47	2.77
28	1.31	1.70	2.05	2.47	2.76
29	1.31	1.70	2.05	2.46	2.76
30	1.31	1.70	2.04	2.46	2.75
60	1.30	1.67	2.00	2.39	2.66
90	1.29	1.66	1.99	2.37	2.63
120	1.29	1.66	1.98	2.36	2.62
∞	1.28	1.64	1.96	2.33	2.58

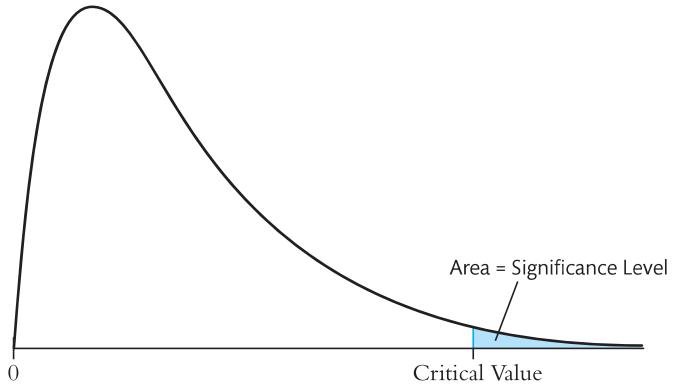
Values are shown for the critical values for two-sided ($\neq$) and one-sided ($>$) alternative hypotheses. The critical value for the one-sided ($<$) test is the negative of the one-sided ($>$) critical value shown in the table. For example, 2.13 is the critical value for a two-sided test with a significance level of 5% using the Student *t* distribution with 15 degrees of freedom.

TABLE 3 Critical Values for the χ^2 Distribution

	Significance Level		
Degrees of Freedom	**10%**	**5%**	**1%**
1	2.71	3.84	6.63
2	4.61	5.99	9.21
3	6.25	7.81	11.34
4	7.78	9.49	13.28
5	9.24	11.07	15.09
6	10.64	12.59	16.81
7	12.02	14.07	18.48
8	13.36	15.51	20.09
9	14.68	16.92	21.67
10	15.99	18.31	23.21
11	17.28	19.68	24.72
12	18.55	21.03	26.22
13	19.81	22.36	27.69
14	21.06	23.68	29.14
15	22.31	25.00	30.58
16	23.54	26.30	32.00
17	24.77	27.59	33.41
18	25.99	28.87	34.81
19	27.20	30.14	36.19
20	28.41	31.41	37.57
21	29.62	32.67	38.93
22	30.81	33.92	40.29
23	32.01	35.17	41.64
24	33.20	36.41	42.98
25	34.38	37.65	44.31
26	35.56	38.89	45.64
27	36.74	40.11	46.96
28	37.92	41.34	48.28
29	39.09	42.56	49.59
30	40.26	43.77	50.89

This table contains the 90^{th}, 95^{th}, and 99^{th} percentiles of the χ^2 distribution. These serve as critical values for tests with significance levels of 10%, 5%, and 1%.

TABLE 4 Critical Values for the $F_{m,\infty}$ Distribution



	Significance Level		
Degrees of Freedom	**10%**	**5%**	**1%**
1	2.71	3.84	6.63
2	2.30	3.00	4.61
3	2.08	2.60	3.78
4	1.94	2.37	3.32
5	1.85	2.21	3.02
6	1.77	2.10	2.80
7	1.72	2.01	2.64
8	1.67	1.94	2.51
9	1.63	1.88	2.41
10	1.60	1.83	2.32
11	1.57	1.79	2.25
12	1.55	1.75	2.18
13	1.52	1.72	2.13
14	1.50	1.69	2.08
15	1.49	1.67	2.04
16	1.47	1.64	2.00
17	1.46	1.62	1.97
18	1.44	1.60	1.93
19	1.43	1.59	1.90
20	1.42	1.57	1.88
21	1.41	1.56	1.85
22	1.40	1.54	1.83
23	1.39	1.53	1.81
24	1.38	1.52	1.79
25	1.38	1.51	1.77
26	1.37	1.50	1.76
27	1.36	1.49	1.74
28	1.35	1.48	1.72
29	1.35	1.47	1.71
30	1.34	1.46	1.70

This table contains the 90^{th}, 95^{th}, and 99^{th} percentiles of the $F_{m,\infty}$ distribution. These serve as critical values for tests with significance levels of 10%, 5%, and 1%.

TABLE 5A Critical Values for the F_{n_1,n_2} Distribution—10% Significance Level

Denominator Degrees of Freedom (n_2)	Numerator Degrees of Freedom (n_1)									
	1	2	3	4	5	6	7	8	9	10
1	39.86	49.50	53.59	55.83	57.24	58.20	58.90	59.44	59.86	60.20
2	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38	9.39
3	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24	5.23
4	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94	3.92
5	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32	3.30
6	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96	2.94
7	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72	2.70
8	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56	2.54
9	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44	2.42
10	3.29	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.35	2.32
11	3.23	2.86	2.66	2.54	2.45	2.39	2.34	2.30	2.27	2.25
12	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.24	2.21	2.19
13	3.14	2.76	2.56	2.43	2.35	2.28	2.23	2.20	2.16	2.14
14	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.12	2.10
15	3.07	2.70	2.49	2.36	2.27	2.21	2.16	2.12	2.09	2.06
16	3.05	2.67	2.46	2.33	2.24	2.18	2.13	2.09	2.06	2.03
17	3.03	2.64	2.44	2.31	2.22	2.15	2.10	2.06	2.03	2.00
18	3.01	2.62	2.42	2.29	2.20	2.13	2.08	2.04	2.00	1.98
19	2.99	2.61	2.40	2.27	2.18	2.11	2.06	2.02	1.98	1.96
20	2.97	2.59	2.38	2.25	2.16	2.09	2.04	2.00	1.96	1.94
21	2.96	2.57	2.36	2.23	2.14	2.08	2.02	1.98	1.95	1.92
22	2.95	2.56	2.35	2.22	2.13	2.06	2.01	1.97	1.93	1.90
23	2.94	2.55	2.34	2.21	2.11	2.05	1.99	1.95	1.92	1.89
24	2.93	2.54	2.33	2.19	2.10	2.04	1.98	1.94	1.91	1.88
25	2.92	2.53	2.32	2.18	2.09	2.02	1.97	1.93	1.89	1.87
26	2.91	2.52	2.31	2.17	2.08	2.01	1.96	1.92	1.88	1.86
27	2.90	2.51	2.30	2.17	2.07	2.00	1.95	1.91	1.87	1.85
28	2.89	2.50	2.29	2.16	2.06	2.00	1.94	1.90	1.87	1.84
29	2.89	2.50	2.28	2.15	2.06	1.99	1.93	1.89	1.86	1.83
30	2.88	2.49	2.28	2.14	2.05	1.98	1.93	1.88	1.85	1.82
60	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74	1.71
90	2.76	2.36	2.15	2.01	1.91	1.84	1.78	1.74	1.70	1.67
120	2.75	2.35	2.13	1.99	1.90	1.82	1.77	1.72	1.68	1.65
∞	**2.71**	**2.30**	**2.08**	**1.94**	**1.85**	**1.77**	**1.72**	**1.67**	**1.63**	**1.60**

This table contains the 90th percentile of the F_{n_1,n_2} distribution, which serves as the critical values for a test with a 10% significance level.

TABLE 5B Critical Values for the F_{n_1, n_2} Distribution—5% Significance Level

Denominator Degrees of Freedom (n_2)	Numerator Degrees of Freedom (n_1)									
	1	2	3	4	5	6	7	8	9	10
1	161.40	199.50	215.70	224.60	230.20	234.00	236.80	238.90	240.50	241.90
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.39	19.40
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99
90	3.95	3.10	2.71	2.47	2.32	2.20	2.11	2.04	1.99	1.94
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91
∞	**3.84**	**3.00**	**2.60**	**2.37**	**2.21**	**2.10**	**2.01**	**1.94**	**1.88**	**1.83**

This table contains the 95th percentile of the distribution F_{n_1,n_2} which serves as the critical values for a test with a 5% significance level.