Ising model

Let us suppose a lattice composed of N spins with magnetic dipole moment moment $\vec{\mu}$ a strength $g\mu_B\sqrt{J(J+1)}$ and degeneracy $(2J+1)$. Each atom interacts with their q nearest neighbors and they are subject to the magnetic field $\vec{H}$.

The analysis of this system reveals that, depending on the dimensionality, a ferromagnetic - paramagnetic can emerge at the critical temperature T_C.

Most of system have a curve $M(0,T)$ that are better fitted for $M \times T/T_C$ $J = 1/2$. This reveals that the ferromagnetism is associated to the spin of electron and not with the orbital contribution. Next, I will present a better discussion about the interaction which is responsible for the ferromagnetism. It is called "exchange interaction". When two electrons have equal spins, they should be "more separated" from each other (due to the Pauli-exclusion principle). Conversely, when their spins are different, they can be closed to each other. Hence, different spatial separations give rise to different electrostatic interactions and they are closely related to the relative ~~sep~~ orientations of their spins.

Now, we are going to present an argument / explanation, based on perturbation theory, about the "exchange interaction".

The wave-function of two electrons should be anti-symmetric. For instance, there are two possibilities: $\underbrace{\phi_S(\vec{r_1}, \vec{r_2})}_{\text{Symmetric Spatial wave-function}} \underbrace{\chi_A(S_1, S_2)}_{\text{anti-symmetric Spin Wave-function}}$

or $\underbrace{\phi_A(\vec{r_1}, \vec{r_2})}_{\text{Anti-Symmetric Spatial}} \underbrace{\chi_S(S_1, S_2)}_{\text{Symmetric Spin}}$

The first one have a spatial wave-function proportional to $\frac{1}{\sqrt{2}}\left[\phi_1(\vec{r_1})\,\phi_2(\vec{r_2}) + \phi_2(\vec{r_1})\,\phi_1(\vec{r_2})\right]$,

Whereas the latter one have a spatial wave function of type $\frac{1}{\sqrt{2}}\left[\phi_1(\vec{r_1})\,\phi_2(\vec{r_2}) - \phi_2(\vec{r_1})\,\phi_1(\vec{r_2})\right]$

Regarding the spin wave-function, the first one is anti-symmetric and has total spin $S=0$ (singlet) and is of type $\frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right]$, whereas

the latter one is symmetric and accounts (2)
for three possibilities: $|\uparrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$ or $\frac{1}{\sqrt{2}}[|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle]$

An estimate of the difference energy ~~about~~ between above states is obtained by considering the quantity $\langle \psi_1 | -\frac{e^2}{r_{12}} | \psi_1 \rangle - \langle \psi_2 | -\frac{e^2}{r_{12}} | \psi_2 \rangle$

where $|\psi_1\rangle \longrightarrow$ first state (Symmetric Spatial wave function)

$|\psi_2\rangle \rightarrow$ second state (Anti-Symmetric Spatial wave function),

Both $|\psi_1\rangle$ and $|\psi_2\rangle$ can be expressed as: $|\psi_{1(2)}\rangle = \frac{1}{\sqrt{2}}[|k_1^1, k_2^2\rangle \pm |k_1^2, k_2^1\rangle]$ ($|\psi_1\rangle$ ... $|\psi_2\rangle$)

$\langle \psi_1 | -\frac{e^2}{r_{12}} | \psi_1 \rangle$ can be written as

$$\int d^3\vec{r}_1 \int d^3\vec{r}_2 \langle \psi_1 | \vec{r}_1, \vec{r}_2 \rangle \langle \vec{r}_1, \vec{r}_2 | -\frac{e^2}{r_{12}} | \psi_1 \rangle =$$

$$= A + B + C + D \quad \text{where}$$

$$A = \int d^3\vec{r}_1 \int d^3\vec{r}_2 \, |\phi_1(\vec{r}_1)|^2 |\phi_2(\vec{r}_2)|^2 \left(-\frac{e^2}{r_1 r_2}\right)$$

$$B = \int d^3\vec{r}_1 \int d^3\vec{r}_2 \, |\phi_2(\vec{r}_1)|^2 |\phi_1(\vec{r}_2)|^2 \left(-\frac{e^2}{r_1 r_2}\right)$$

$$C = \int d^3\vec{r}_1 \int d^3\vec{r}_2 \, \phi_1^*(\vec{r}_1) \phi_2^*(\vec{r}_2) \left(-\frac{e^2}{r_1 r_2}\right) \phi_2(\vec{r}_1) \phi_1(\vec{r}_2)$$

$$D = \int d^3\vec{r}_1 \int d^3\vec{r}_2 \, \phi_1^*(\vec{r}_2) \phi_2^*(\vec{r}_1) \left(-\frac{e^2}{r_1 r_2}\right) \phi_1(\vec{r}_1) \phi_2(\vec{r}_2)$$

The quantity $\langle \psi_2 | -\frac{e^2}{r_{12}} | \psi_2 \rangle$ is analogous and given by

$$(A+B) - (C+D)$$

Hence, the difference between the "singlet" and "triplet" states is given by $2(C+D)$ and depends solely on the "exchange" term. This energy is closely dependent on the coulombian interaction.

Now, we are going to show that a Hamiltonian of type $\mathcal{H} = \tilde{C}\, \vec{S}_1 \cdot \vec{S}_2 + D$ is able to reproduce the "separation" of two states of electrons. More specifically, we are going to evaluate $\underset{\substack{\uparrow \\ \text{singlet}}}{\langle s |} \vec{S}_1 \cdot \vec{S}_2 | s \rangle$ -

$$\underset{\substack{\uparrow \\ \text{triplet}}}{\langle t |} \vec{S}_1 \cdot \vec{S}_2 | t \rangle$$

$$\vec{S}_1 \cdot \vec{S}_2 = S_{1x} S_{2x} + S_{1y} S_{2y} + S_{13} S_{23},$$

where S_x, S_y, S_3 are Pauli matrices given by $S_x = \frac{\hbar}{2}\sigma_x$, $S_y = \frac{\hbar}{2}\sigma_y$, $S_3 = \frac{\hbar}{2}\sigma_3$

and $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ (3)

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

all of them are Hermitian matrices and ~~ph~~ have the same eigenvalues $\lambda = \pm 1$.

Their eingenvectors are given by

$$\sigma_x |\pm\rangle_x = \pm|\pm\rangle_x$$

$$\sigma_y |\pm\rangle_y = \pm|\pm\rangle_y$$

$$\sigma_z |\pm\rangle = \pm|\pm\rangle$$

where $|\pm\rangle_x = \frac{1}{\sqrt{2}}\left[|+\rangle \pm |-\rangle\right]$

$$|\pm\rangle_y = \frac{1}{\sqrt{2}}\left[|+\rangle \pm i|-\rangle\right]$$

It is easy to show that

$$\begin{cases} S_{1x} S_{2x} |+-\rangle = \frac{\hbar^2}{4}|-+\rangle \\ S_{1x} S_{2x} |-+\rangle = \frac{\hbar^2}{4}|+-\rangle \\ S_{1z} S_{2z} |+-\rangle = -\frac{\hbar^2}{4}|+-\rangle \\ S_{1y} S_{2y} |+-\rangle = \frac{\hbar^2}{4}|-+\rangle \\ S_{1y} S_{2y} |-+\rangle = \frac{\hbar^2}{4}|+-\rangle \\ S_{1z} S_{2z} |-+\rangle = -\frac{\hbar^2}{4}|-+\rangle. \end{cases}$$

$$\begin{cases} S_{1x} S_{2x} |++\rangle = 0 \\ S_{1y} S_{2y} |++\rangle = 0 \\ S_{1z} S_{2z} |++\rangle = \frac{\hbar^2}{4}|++\rangle \\ S_{1x} S_{2x} |--\rangle = 0 \\ S_{1y} S_{2y} |--\rangle = 0 \\ S_{1z} S_{2z} |--\rangle = \frac{\hbar^2}{4}|--\rangle \end{cases}$$

Hence, a singlet state $|s\rangle = \frac{1}{\sqrt{2}}\left[|+-\rangle - |-+\rangle\right]$

and a triplet state $|t\rangle = \frac{1}{\sqrt{2}}\left[|+-\rangle + |-+\rangle\right]$

have difference energies given by

$$\langle s|\vec{S}_1 \cdot \vec{S}_2|s\rangle = -\frac{3}{4}\hbar^2\tilde{C}$$

$$\langle t|\vec{S}_1 \cdot \vec{S}_2|t\rangle = \frac{\hbar^2\tilde{C}}{4}$$

$> = -\hbar^2\tilde{C},$

which is qualitatively similar to the previous argument, allowing us to identify the coefficient $\tilde{C}$ with the scalar product between spin vectors and the exchange term. When $\tilde{C} > 0$ the ferromagnetism is favored, whereas $\tilde{C} < 0$ favors the anti-ferromagnetism.

Once presented physical arguments for the Hamiltonian $\mathcal{H} = -\tilde{J}\sum_{\langle ij\rangle}\vec{S}_i \cdot \vec{S}_j + \tilde{H}\sum_i \vec{S}_i$,

we are going to study their main features in the realm of mean-field theory and beyond the mean-field. For simplicity we are going to consider the simplified case $\vec{S}_i \cdot \vec{S}_j = S_{1z}S_{2z}$ where $S_{iz} = \pm\frac{1}{2}$ which is the Ising model and quantum treatment is not required.

Bragg - Williams mean - field - approximation

This first kind of mean - field approximation is called Bragg - Williams and consists of neglecting correlation among spin sites

$$\langle \sigma_i \sigma_j \rangle \simeq \langle \sigma_i \rangle \langle \sigma_j \rangle$$

From this approximation, the ensemble average of above Hamiltonian $\left(\mathcal{H} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i \right)$ reads.

with $\sigma_i = \pm 1$

$$u = \langle \mathcal{H} \rangle$$

$$= -J d N \langle \sigma_i \rangle \langle \sigma_j \rangle - h N \langle \sigma_i \rangle,$$

Where $\underline{\underline{d}}$ is the system dimensionality.

Since we are dealing with the simplest case of $S_i = \pm \frac{1}{2}$ $\left(\text{or } \sigma_i = \pm 1 \right)$ we have that the magnetization per site is given by

number of spins "up"

$$m = \langle \sigma_i \rangle = \frac{N_+ - N_-}{N}$$

number of spins "down"

magnetization per site

and $\underline{\underline{u}}$ then reads

$$u = -J d N m^2 - h N m.$$

The entropy system is expressed in terms of N_+ and N_- and given by

$$S = K_B \ln \frac{N!}{N_+! \, N_-!} \qquad \text{Where} \quad \begin{aligned} N_+ - N_- &= Nm \\ N_+ + N_- &= N \end{aligned}$$

$$= K_B \ln \frac{N!}{\left(\frac{N+M}{2}\right)! \left(\frac{N-M}{2}\right)!}$$

By using the Stirling approximation, S is given by

$$S = -K_B \left[\left(\frac{N+M}{2}\right) \ln \left(\frac{1+m}{2}\right) - \left(\frac{N-M}{2}\right) \ln \left(\frac{1-m}{2}\right) \right]$$

or $$s = \frac{S}{N} = -K_B \left[\left(\frac{1+m}{2}\right) \ln \left(\frac{1+m}{2}\right) + \left(\frac{1-m}{2}\right) \ln \left(\frac{1-m}{2}\right) \right]$$

The functional $g(T, H; m)$ is given by

$$g(T, H; m) = u(T, H; m) - T\, s(m)$$

or $$g(T, H) = \min_m \{ g(T, H; m) \}$$

where $$g(T, H; m) = -J d m^2 - h m + K_B T \left[\left(\frac{1+m}{2}\right) \ln \left(\frac{1+m}{2}\right) + - \left(\frac{1-m}{2}\right) \ln \left(\frac{1-m}{2}\right) \right]$$

The minimization of above functional leads to the following equation:

$$\left(\frac{\partial g}{\partial m}\right) = 0 \iff -2Jdm - h +$$

$$+ K_B T \left[\left(\frac{1+m}{2}\right) \frac{1/2}{\frac{1+m}{2}} + \frac{1}{2} \ln\left(\frac{1+m}{2}\right) + \right.$$

$$\left. + \left(\frac{1-m}{2}\right) \frac{-1/2}{\left(\frac{1-m}{2}\right)} - \frac{1}{2} \ln\left(\frac{1-m}{2}\right) \right]$$

$$\iff h = -2Jdm + \frac{K_B T}{2} \ln \frac{1+m}{1-m} \quad \text{or}$$

even $m = \tanh\left(2\beta J dm + \beta H\right)$,

Which is the previous Curie - Weiss with ($\beta\lambda \equiv 2Jd$).

The critical point is then given by $\frac{2Jd}{K_B T_c} = 1$ or $T_c = \frac{2Jd}{K_B}$.

Therefore, such Mean - field treatment Ising model predicts a ferromagnetic - paramagnetic phase transition, even for $d=1$ (this is actually incorrect), whose equation of State is the Same as before (but proposed under a phenomenological way)

Curie-Weiss Model

Instead of neglecting correlations, as performed previously, one can modify the Ising model allowing it exactly solvable. Such a model modification is given by

$$\mathcal{H} \longrightarrow -\frac{J}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_i \sigma_j - H \sum_i \sigma_i$$

interaction among all spins (instead of only among nearest-neighbor ones), but they decay with the distance as $1/N$, in order to ensure the "thermodynamic limit". The partition function then reads

$$Z = \sum_{\{\sigma\}} e^{\frac{\beta J}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_i \sigma_j + \beta H \sum_i \sigma_i}$$

$$= \sum_{\{\sigma\}} e^{\frac{\beta J}{2N} \left(\sum_{i=1}^{N} \sigma_i \right)^2 + \beta H \sum_i \sigma_i}$$

By using the gaussian identity

$$\int_{-\infty}^{\infty} e^{-x^2 + 2ax}\, dx = \sqrt{\pi}\; a^2 ,$$

the partition function then reads

$$e^{\frac{\beta J}{2N}\left(\sum_{i=1}^{N} \sigma_i\right)^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2 + 2x\sqrt{\frac{\beta J}{2N}} \sum_{i=1}^{N} \sigma_i} \, dx$$

and then we have that

$$Z = \sum_{\{\sigma\}} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2 + 2\sqrt{\frac{\beta J}{2N}}\, x \sum_{i=1}^{N} \sigma_i + \beta H \sum_{i=1}^{N} \sigma_i} \, dx$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} dx \sum_{\{\sigma\}} e^{\underbrace{\left(2\sqrt{\frac{2\beta J}{N}}\, x + \beta H\right)}_{\alpha} \sum_{i=1}^{N} \sigma_i}$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} dx \left(e^{\alpha} + e^{-\alpha}\right)^N$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\left(x^2 - N \ln(2\cosh \alpha)\right)} \, dx$$

where $\alpha = 2\sqrt{\frac{2\beta J}{N}}\, x + \beta H$

By introducing the new variables

$$\sqrt{\frac{2\beta J}{N}}\, x = \beta J m,$$

above partition function becomes

$$Z = \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{\infty} dm \, e^{-\left[\frac{N}{2\beta J} \beta^2 J^2 m^2 - N \ln\left[2\cosh(\beta J m + \beta H)\right]\right]}$$

or even

$$Z = \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{\infty} dm \; e^{-N\beta g(T,H;m)}$$ (7)

where

$$g(T,H;m) = \frac{Jm^2}{2} - \frac{1}{\beta} \ln\left[2\cosh(\beta J m + \beta H)\right].$$

(*)

above integral can be estimated through the Laplace method:

$$\int_a^b e^{Mf(x)}\,dx \approx e^{Mf(x_0)} \int_a^b e^{-M\left[f''(x_0)\frac{(x-x_0)^2}{2}\right]}.$$

Minimizing the above expression for (*) we have that

$$\frac{\partial g}{\partial m} = 0 \iff Jm - \frac{1}{\beta}\,\frac{\sinh(\beta J m + \beta H)\,(\beta J)}{\cosh(\beta J m + \beta H)}$$

We obtain that

$$m = \tanh(\beta J m + \beta H),$$

Which is the same equation of state as before.

Mean-field approximation - ~~[illegible]~~ Bogoliubov inequality

This approach consists of "replacing" the expressions for the system we are interested (in which analytic solutions are generally not available) for an exactly solvable "upper bound"

More specifically, let us consider the Hamiltonian $\mathcal{H}$ (we are not able to solve it exactly" and the $\mathcal{H}_0$ (exactly solvable) and $\mathcal{H}_1$ being the difference between $\mathcal{H}$ and $\mathcal{H}_0$.

We then have that

$$\mathcal{H}(\lambda) = \mathcal{H}_0 + \lambda \mathcal{H}_1$$

$\lambda = 0 \longrightarrow$ one recovers the exactly solvable case $\mathcal{H}_0$

$\lambda = 1 \longrightarrow$ We recover the Hamiltonian $\mathcal{H}$ we want to study.

The Bogoliubov inequality consists of "replacing / approaching" the Helmholtz / Gibbs free energy ~~for~~ the following expression

$$F(\lambda) \leq F(0) + \lambda \langle \mathcal{H}_1 \rangle_0 ,$$

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Where $\Gamma(0)$ is the free energy of the Hamiltonian (non perturbed) $\mathcal{H}_0$ and

$\langle \mathcal{H}_1 \rangle_0$ is the ensemble average of the Hamiltonian $\mathcal{H}_1$ evaluated over the the space of configurations respecting the Hamiltonian $\mathcal{H}_0$.

Let us prove the above relation.

For instance, we have that the free energy (Gibbs) $F(\lambda)$ is given by

$$-\beta F(\lambda) = \ln \underbrace{\Big(\underbrace{\sum_\sigma}_{\text{configurations}} e^{-\beta \mathcal{H}(\lambda)} \Big)}_{\text{partition function } Z(\lambda).}$$

By deriving above expression ~~with~~ respecting to λ we have that

$$-\beta \frac{dF(\lambda)}{d\lambda} = -\beta \frac{\sum_\sigma (-\beta \mathcal{H}_1) e^{-\beta(\mathcal{H}_0 + \lambda \mathcal{H}_1)}}{\sum_\sigma e^{-\beta(\mathcal{H}_0 + \lambda \mathcal{H}_1)}}$$

$$= \langle \mathcal{H}_1 \rangle$$

By proceeding analogously with $\frac{d^2F}{d\lambda^2}$ we have that

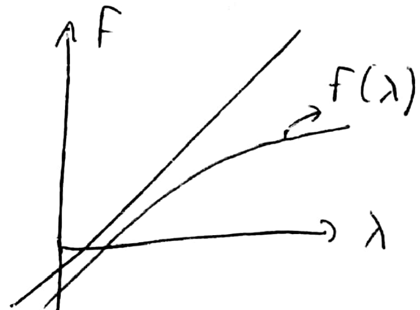
$$\frac{d^2F}{d\lambda^2} = \frac{d}{d\lambda}\left[\frac{\sum_\sigma \mathcal{H}_1 e^{-\beta(\mathcal{H}_0+\lambda\mathcal{H}_1)}}{\sum_\sigma e^{-\beta(\mathcal{H}_0+\lambda\mathcal{H}_1)}}\right]$$

$= -\beta(A-B)$, where

$$A = \frac{\sum_\sigma \mathcal{H}_1^2 e^{-\beta(\mathcal{H}_0+\lambda\mathcal{H}_1)}}{\sum_\sigma e^{-\beta(\mathcal{H}_0+\lambda\mathcal{H}_1)}},$$

$$B = \left(\frac{\sum_\sigma \mathcal{H}_1 e^{-\beta(\mathcal{H}_0+\lambda\mathcal{H}_1)}}{\sum_\sigma e^{-\beta(\mathcal{H}_0+\lambda\mathcal{H}_1)}}\right)^2$$

Since $A-B \geqslant 0$ (always) the derivative $\frac{d^2F}{d\lambda^2} \leq 0$ and hence the Gibbs free energy is concave with respect to λ and then it must to have the following shape



Which justifies the following upper bound

$$F(\lambda) \leq F(0) + \underbrace{\lambda \left(\frac{dF}{d\lambda}\right)\Bigg|_{\lambda=0}}$$

straight line whose linear coefficient taken at $\lambda = 0$.

Since $\frac{dF}{d\lambda} = \langle \mathcal{H}_1 \rangle$ we arrive at the following relation

$$F(\lambda) \leq F(0) + \lambda \langle \mathcal{H}_1 \rangle_0 ,$$

where

$$\langle \mathcal{H}_1 \rangle_0 = \frac{\sum_\sigma \mathcal{H}_1 e^{-\beta \mathcal{H}_0}}{\sum_\sigma e^{-\beta \mathcal{H}_0}}$$ and

$$F(0) = -K_B T \ln Z_0 , \text{ with } Z_0 = \sum_{\{\sigma\}} e^{-\beta \mathcal{H}_0}.$$

Next, we shall exemplify for the

Let $\mathcal{H} = -J \sum_{(ij)} \sigma_i \sigma_j - H \sum_i \sigma_i$ $\rightarrow$ the Hamiltonian

in which we don't know the exact solution and $\mathcal{H}_0 = -\eta \sum_i \sigma_i$ the Hamiltonian which is exactly solvable

$$\mathcal{H}_1 = +\mathcal{H} - \mathcal{H}_0 = -J \sum_{(ij)} \sigma_i \sigma_j - (H-\eta) \sum_i \sigma_i$$

$$\langle \mathcal{H}_1 \rangle = -J \left\langle \sum_{(ij)} \sigma_i \sigma_j \right\rangle_0 - (H-\eta) \left\langle \sum_i \sigma_i \right\rangle_0$$

$$Z_0 = \sum_{\{\sigma\}} e^{+\beta\eta \sum_i \sigma_i} = (2\cosh \beta\eta)^N$$

and then $F(0) = -N k_B T \ln (2\cosh \beta\eta)$

$$\left\langle \sum_{(ij)} \sigma_i \sigma_j \right\rangle_0 = N d \langle \sigma_i \sigma_j \rangle_0$$

$$= N d \langle \sigma_i \rangle_0 \langle \sigma_j \rangle_0$$

the Hamiltonian $\mathcal{H}_0$ does not presenting coupling between sites $\underline{i}$ and $\underline{j}$.

Hence $$\langle \sigma_i \rangle_0 = \frac{\frac{1}{N} \sum_{\{\sigma\}} \left(\sum_i \sigma_i\right) e^{-\beta \mathcal{H}_0}}{\sum_{\{\sigma\}} e^{-\beta \mathcal{H}_0}}$$

$$= \tanh \beta\eta$$

Hence

$$\frac{\langle \mathcal{H}_1 \rangle_0}{N} = - J d (\tanh \beta\eta)^2 - (H - \eta) \tanh \beta\eta$$

and the functional free energy

$$\mathcal{F}(T, H; \eta) = - K_B T \ln (2 \cosh \beta\eta) + \\ - J d (\tanh \beta\eta)^2 - (H-\eta) \tanh \beta\eta$$

Since above expression for $\mathcal{F}(T, H; \eta)$ is an upper bound, by maximizing it with respect to η, we have that

$$\frac{\partial \mathcal{F}}{\partial \eta} = - K_B T (\beta \tanh \beta\eta) - 2 J d \tanh \beta\eta \times \\ \times \left(- \frac{\beta\eta}{\cosh^2 \beta\eta} \right) + \tanh \beta\eta + \\ - (H - \eta) \frac{(-\beta)}{\cosh^2 \beta\eta} = 0 .$$

$$= \frac{-\beta}{\cosh^2 \beta\eta} \left[- 2 J d \tanh \beta\eta - H + \eta \right] = 0.$$

Since $\cosh^2 \beta\eta \neq 0$ we have that

$$H = \eta - 2 J d \tanh \beta\eta$$

By noting that $\langle \sigma_i \rangle_0 = \tanh \beta \eta = m$.

(magnetization per site), one finally arrives at

$$H + 2Jdm = \eta$$

$$H + 2Jdm = \frac{1}{\beta} \tanh^{-1} m \quad \text{or}$$

$$m = \tanh(\beta H + 2\beta J d m),$$

which correspond to the Curie-Weiss equation obtained previously under three different ways.