Lecture 2
Kinematic variables

Part I

" NOW ALL WE NEED IS NOMENCLATURE . "

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Notations and Conventions

Units:$c = \hbar = k_B = 1$

<u>examples:</u>

$\hbar c = 1 = 197\,\text{MeV}\,fm \Rightarrow 1\,fm = 1/197\,\text{MeV}^{-1} \sim 1/200\,\text{MeV}^{-1}$
$c = 3\,10^{23} fm\,s^{-1} = 1 \Rightarrow 1\,fm = 1/3\,10^{-23} s \sim 10^{-23} s$
$k_B = 0.86 10^{-10}\,\text{MeV}\,K^{-1} = 1 \Rightarrow 1K = 0.86\,10^{-10}\text{MeV} \sim 10^{-10}\text{MeV}$

Space-time coordinates (contravariant vector)

$x^\mu = (x^0, x^1, x^2, x^3) = (t, x, y, z) = (t, \vec{x})$

4-momentum vector

$p^\mu = (p^0, p^1, p^2, p^3) = (E, p_x, p_y, p_z) = (E, \vec{p}) = (E, \vec{p}_T, p_z)$

Scalar product of 4-vectors:

$a \cdot b = a^0 b^0 - \vec{a} \cdot \vec{b}$

Mandelstam variable:

$s \equiv (p_a + p_b)^2$

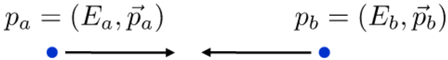
Relativistic energy and momentum:

$E = \gamma m$, $p = \gamma m\, v$ with $\gamma = 1/\sqrt{1 - v^2}$
$E^2 = m^2 + p^2$

$\sqrt{s}$ in center-of-mass system or cms

Consider a collision of two particles.

The cms is defined by $\vec{p}_a = -\vec{p}_b$



$s = (E_a + E_b)^2 - (\vec{p}_a + \vec{p}_b)^2$

$$\boxed{\sqrt{s} \stackrel{cms}{=} (E_a + E_b)}$$

In the cms, $\sqrt{s}$ is the total energy available:

For identical particles: $\vec{p}_a = -\vec{p}_b$, $m_a = m_b$

$\Rightarrow p_a = (E, \vec{p})$, $p_b = (E, -\vec{p})$ and $\sqrt{s} \stackrel{cms}{=} 2E$

Exercise:

At LHC, Pb nuclei can be collided at 2.76 TeV A. What is the energy of each nucleon in a colliding pair?

$\sqrt{s} = 2.76\,A\,\mathrm{TeV}$

$\Rightarrow \sqrt{s}_{NN} = 2.76\,\mathrm{TeV} = 2E \Rightarrow E = 1.38\,TeV$

Exercise:

At LHC, if p+p collisions are performed at $\sqrt{s} = 7\,\mathrm{TeV}$, what is $\sqrt{s}$ for Pb+Pb? (maintaining all collider characteristics unchanged)

In p+p, the proton energy is $E_p = |\vec{p}_p| = \sqrt{s}/2 = 3.5\,\mathrm{TeV}$

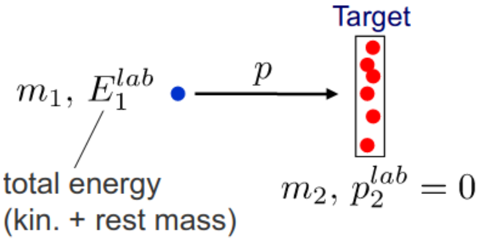
In Pb+Pb, only the protons are accelerated so

$A\,E_{nucleon} = A\,|\vec{p}_{nucleon}| = Z|\vec{p}_p| = Z\,E_p$

$\Rightarrow \sqrt{s}_{NN} = 2\,E_{nucleon} = (82/208)7\,\mathrm{TeV} = 2.76\,\mathrm{TeV}$

and $\sqrt{s} = 2.76\,A\,\mathrm{TeV}$

$\sqrt{s}$ for fixed-target experiment



$p_1 = (E_1^{lab}, \vec{p}_1)$ and $p_2 = (m_2, \vec{0})$
$s = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 p_2 = m_1^2 + m_2^2 + 2E_1^{lab} m_2$

If $E_1^{lab} >> m_1, m_2 \Rightarrow \boxed{\sqrt{s} \sim \sqrt{2E_1^{lab} m_2}}$

Exercise:

At the SPS, precursor of LEP and LHC, Pb nuclei were acelerated in fixed target mode, with beam energy 158 AGeV (among other energies). What is $\sqrt{s_{NN}}$ in the cms of each nucleon pair?

(158 AGeV means that each of the A nucleons of the incident nucleus collide with energy 158 GeV with the target nucleus.)

For a nucleon-nucleon pair:

$E_1^{lab} = 158\,\text{GeV} >> m_1 = m_2 \sim 1\text{GeV}$

$\Rightarrow \sqrt{s_{NN}} \sim \sqrt{2E_1^{lab} m_2} = 17.8\,\text{GeV} \sim 20\,\text{GeV}$

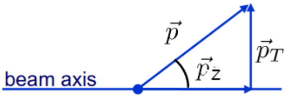
(N.B. everything is expressed in GeV)

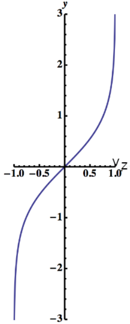
This $\sqrt{s_{NN}}$ at SPS is about 10 times lower than the highest RHIC one.

Rapidity

It is a generalization of longitudinal velocity $v_z = p_z/E$

$$\boxed{y \equiv \text{arctanh}\;\; v_z = \frac{1}{2}\ln\frac{1+v_z}{1-v_z} = \frac{1}{2}\ln\frac{E+p_z}{E-p_z}}$$

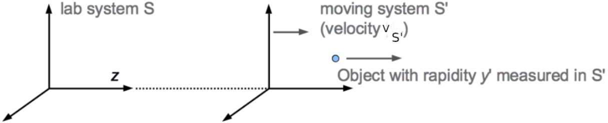




Note: $v_z << 1 \Rightarrow y \sim v_z$ (it is the case at midrapidity $y \sim 0$ in the cms)

A neat property

When going to a new referential, the new rapidities are obtained from the old ones by adding/subtracting a constant:



Lorentz transformation: $E = \gamma(E' + v_{S'} p'_z)$, $p_z = \gamma(p'_z + v_{S'} E')$

$\Rightarrow y = \frac{1}{2} \ln \frac{E+p_z}{E-p_z} = ... = \frac{1}{2} \ln \frac{E'+p'_z}{E'-p'_z} + \frac{1}{2} \ln \frac{1+v_{S'}}{1-v_{S'}} = y' + y_{S'}$

y is not Lorentz invariant but it changes in a simple way: $\boxed{y = y' + y_{S'}}$

$y_{S'}$ is the rapidity of S' measured in S.

Exercise: write E and p_z in term of rapidity and $m_T \equiv \sqrt{m^2 + p_T^2}$

Summing and subtracting

$e^y = \sqrt{\frac{E+p_z}{E-p_z}}$, $e^{-y} = \sqrt{\frac{E-p_z}{E+p_z}}$

$\Rightarrow \boxed{\mathrm{E{=}m}_T \cosh y, p_z = m_T \sinh y}$

$m_T \equiv \sqrt{m^2 + p_T^2}$ is called transverse mass and is invariant under boost along the beam axis.

Exercise:

Consider 2 objects colliding along the z-axis with $p_a = (E_a, 0, 0, p_{za})$ and $p_b = (E_b, 0, 0, p_{zb})$ in some frame. What is their cm rapidity in that frame?

In the cms:

$p'_{za} = \gamma_{cm}(p_{za} - v_{cm}E_a)$ and $p'_{zb} = \gamma_{cm}(p_{zb} - v_{cm}E_b)$ with $p'_{zb} = -p'_{za}$

$\Rightarrow v_{cm} = \frac{p_{za}+p_{zb}}{E_b+E_a}$ and $\boxed{\mathrm{y}_{cm} = \frac{1}{2}\ln\frac{E_a+p_{za}+E_b+p_{zb}}{E_a-p_{za}+E_b-p_{zb}}}$

It can also be written:

$$y_{cm} = \frac{1}{2}\ln\frac{m_{Ta}(\cosh y_a+\sinh y_a)+m_{Tb}(\cosh y_b+\sinh y_b)}{m_{Ta}(\cosh y_a-\sinh y_a)+m_{Tb}(\cosh y_b-\sinh y_b)} = \frac{1}{2}\ln\frac{m_a e^{y_a}+m_b e^{y_b}}{m_a e^{-y_a}+m_b e^{-y_b}}$$

$$\Rightarrow \boxed{\mathrm{y}_{cm} = \frac{1}{2}(y_a+y_b) + \frac{1}{2}\ln\frac{m_a e^{y_a}+m_b e^{y_b}}{m_a e^{y_b}+m_b e^{y_a}}}$$

If $m_a = m_b$, $y_{cm} = \frac{1}{2}(y_a + y_b)$

Exercise:

Consider 2 **equal mass** objects colliding along the z-axis with $p_a = (E_a, 0, 0, p_{za})$ and $p_b = (E_b, 0, 0, p_{zb})$ in some frame. Compute the rapidity of each object in the cms.

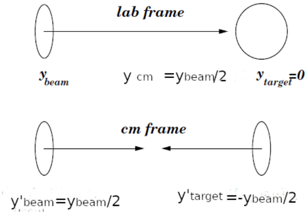
In the original frame: $y_a = \frac{1}{2} \ln \frac{E_a + p_{za}}{E_a - p_{za}}$, $y_b = \frac{1}{2} \ln \frac{E_b + p_{zb}}{E_b - p_{zb}}$ and $y_{cm} = (y_a + y_b)/2$

From the additivity property, in the cms:$y'_a = y_a - y_{cm} = (y_a - y_b)/2$ and $y'_b = y_b - y_{cm} = (y_b - y_a)/2$

Applications:

- Original frame is a fixed target one:

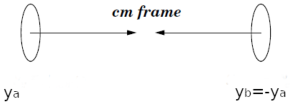
$y_{cm} = (y_{target} + y_{beam})/2 = y_{beam}/2$, $y'_{target} = -y_{beam}/2$, $y'_{beam} = +y_{beam}/2$



- Original frame is cms:

$p_{zb} = -p_{za} \Rightarrow y_b = -y_a$

$y_{cm} = (y_a + y_b)/2 = 0$



Challenge

Show that for an ultrarelativistic $A_1 + A_2$ collision, the center-of-momentum frame of nucleon-nucleon collisions has the rapidity

$$y_{cm} = \frac{1}{2} \ln \frac{Z_1 A_2}{Z_2 A_1}$$

(which reduces to 0 for $A + A$ as it should, cf.previous slide).

What value does it have for p+Pb collisions at the LHC?

Homework

1) At SPS, Pb nuclei collided in fixed target mode with momentum 158 GeV (per nucleon).
a) Compute the contraction factor γ and rapidity for the beam.
b) In the cms, compute the total energy for nucleon-nucleon collisions and Pb+Pb collisions as well as the γ's and rapidities.

2) At RHIC, Au nuclei can be collided with $\sqrt{s_{NN}} = 200\,\text{GeV}$.
a) In the cms, compute the total energy for nucleon-nucleon collisions and Au+Au collisions as well as the γ's and rapidities.
b) For fixed target mode, what would be the beam momentum?

Other references on this topic

- W. Florkowski, Phenomenology of Ultra-Relativistic Heavy-Ion Collisions, World Scientific, 2010
- R. Vogt, Ultrarelativistic Heavy-ion Collisions, Elsevier, 2007
- C.Y. Wong, Introduction to High-Energy Heavy-Ion Collisions, World Scientific, 1994
- https://www.physi.uni-heidelberg.de/~reygers/lectures/2019/qgp/qgp2019_02_kinematics.pdf