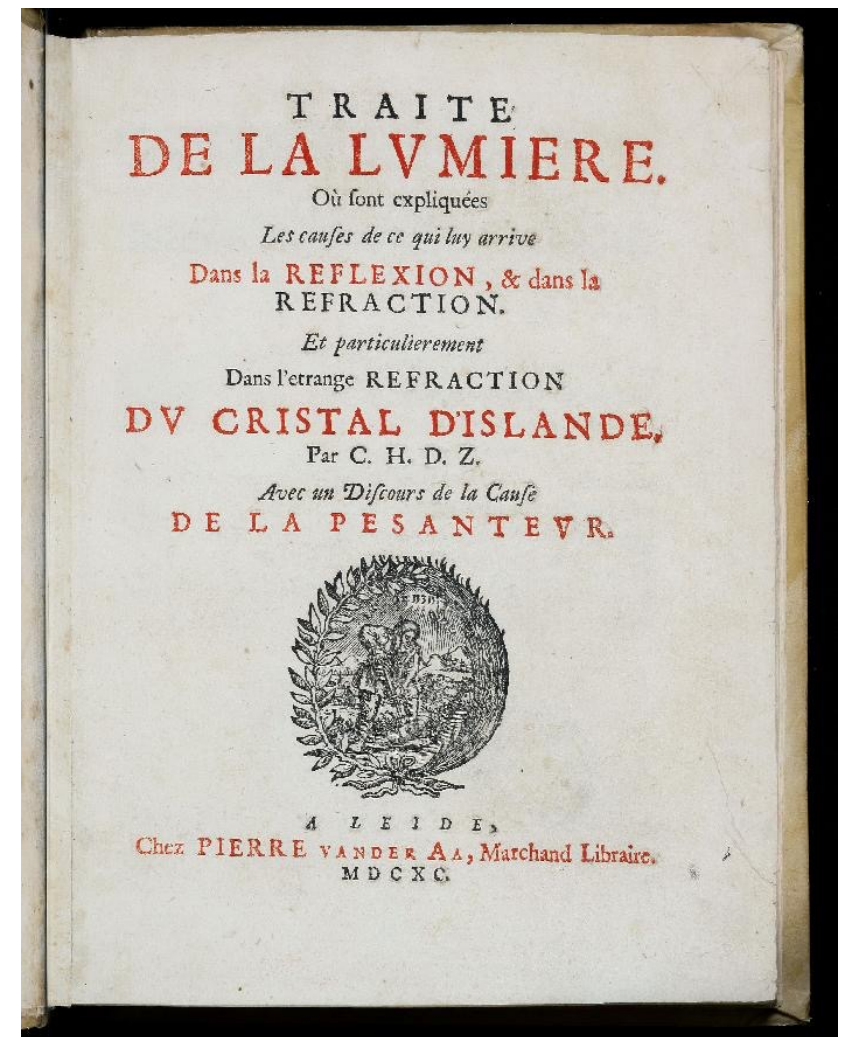
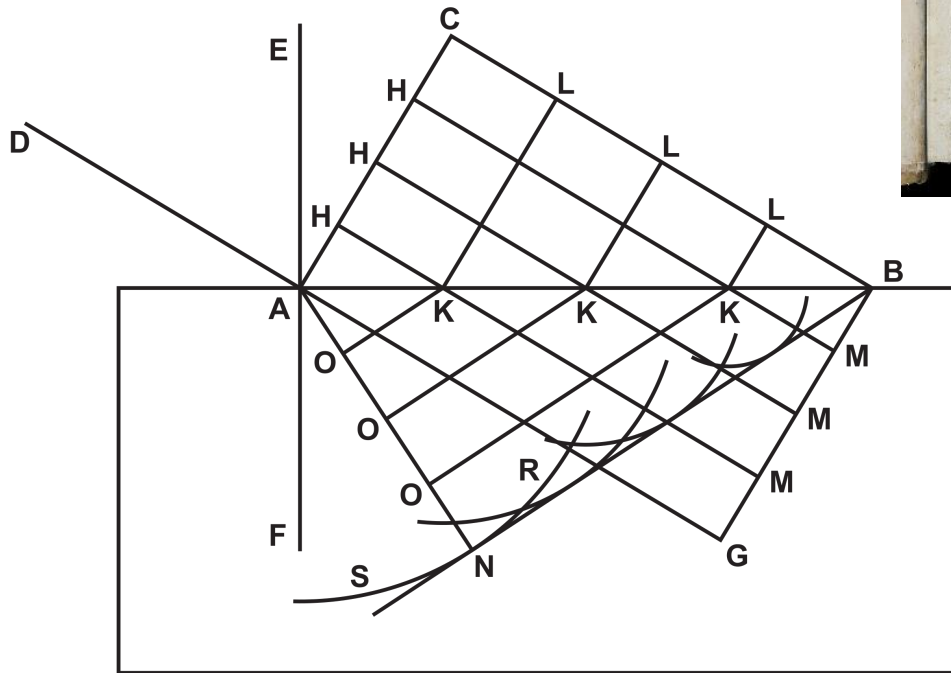
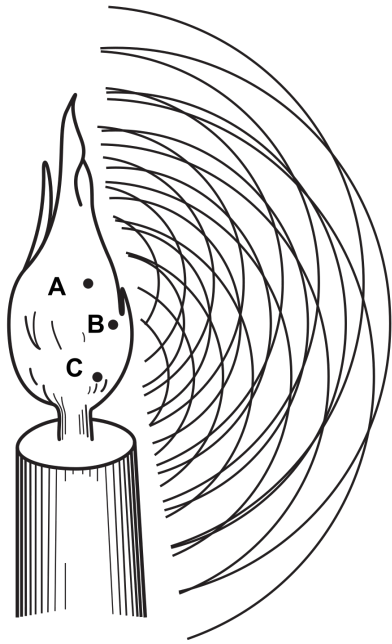
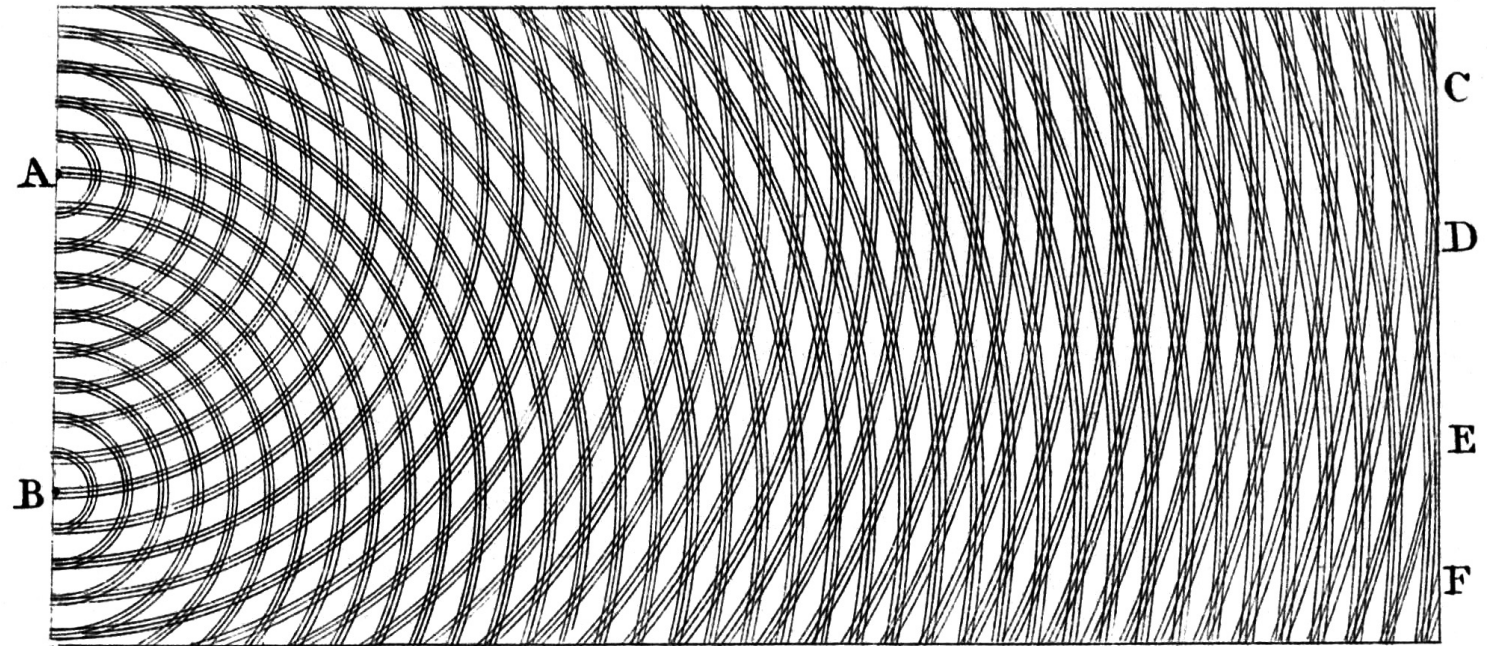
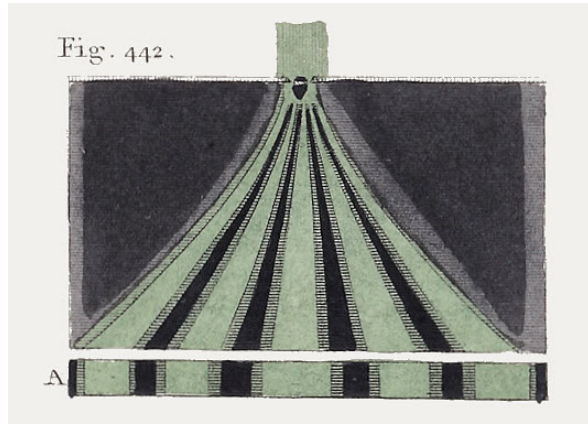


- Até onde podemos lidar com ondas planas?
- Qual a forma de propagação de uma onda limitada espacialmente?
- Como a onda se propaga ao encontrar um obstáculo?

Christian Huygens - 1690



Thomas Young - 1808



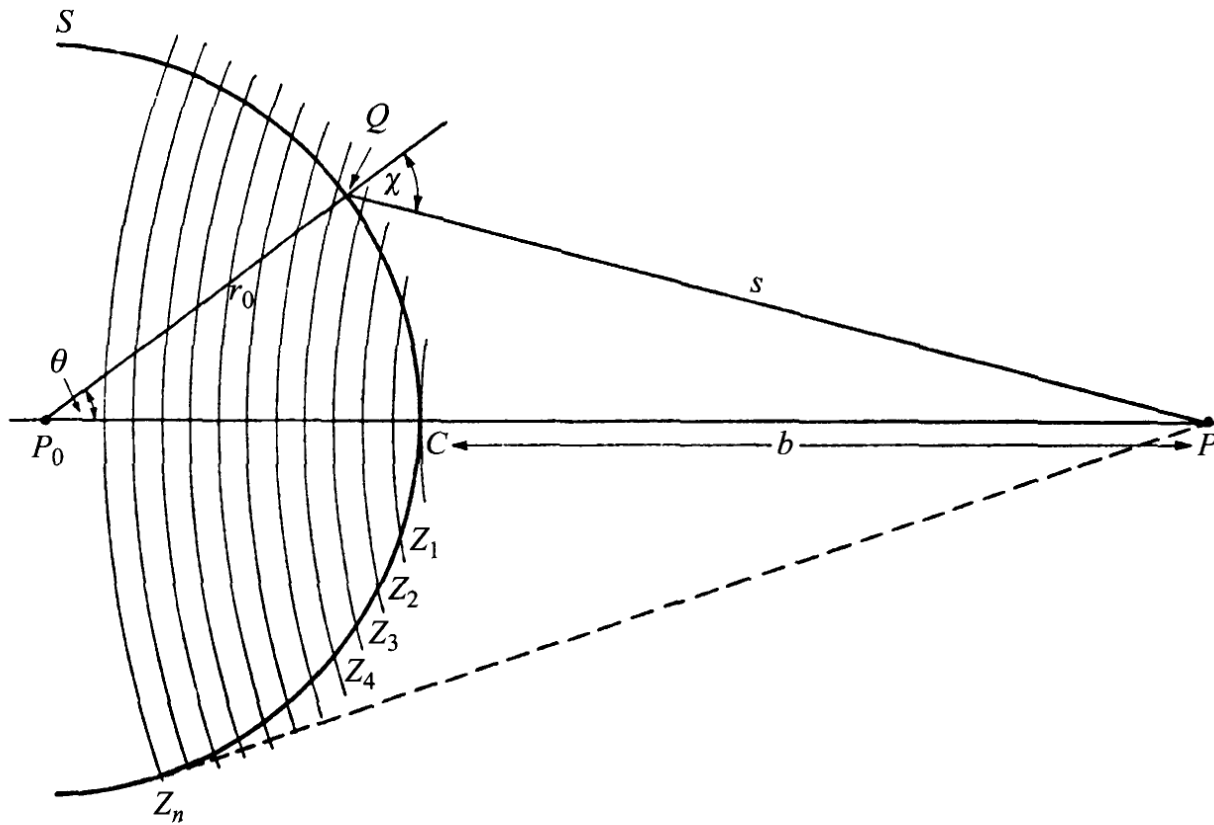
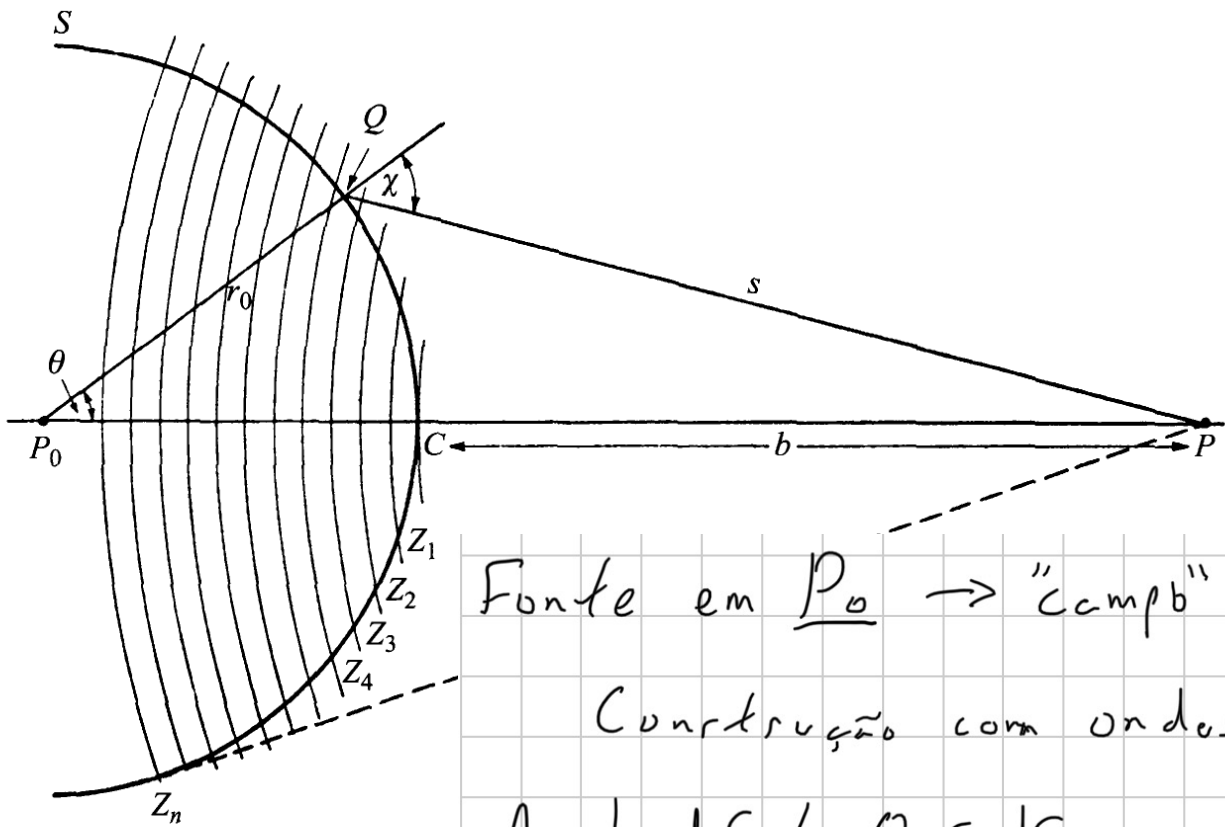


Fig. 8.1 Fresnel's zone construction.



Fonte em $P_0 \rightarrow$ "camp" em P ?

Construção com ondas secundárias de Huygens

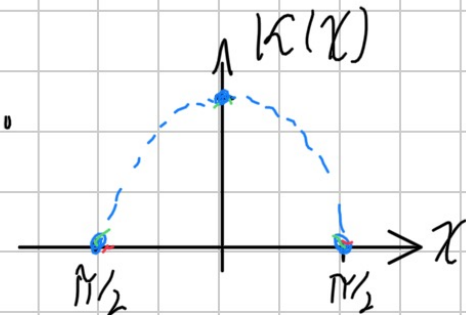
Anel $dS / Q \in dS$

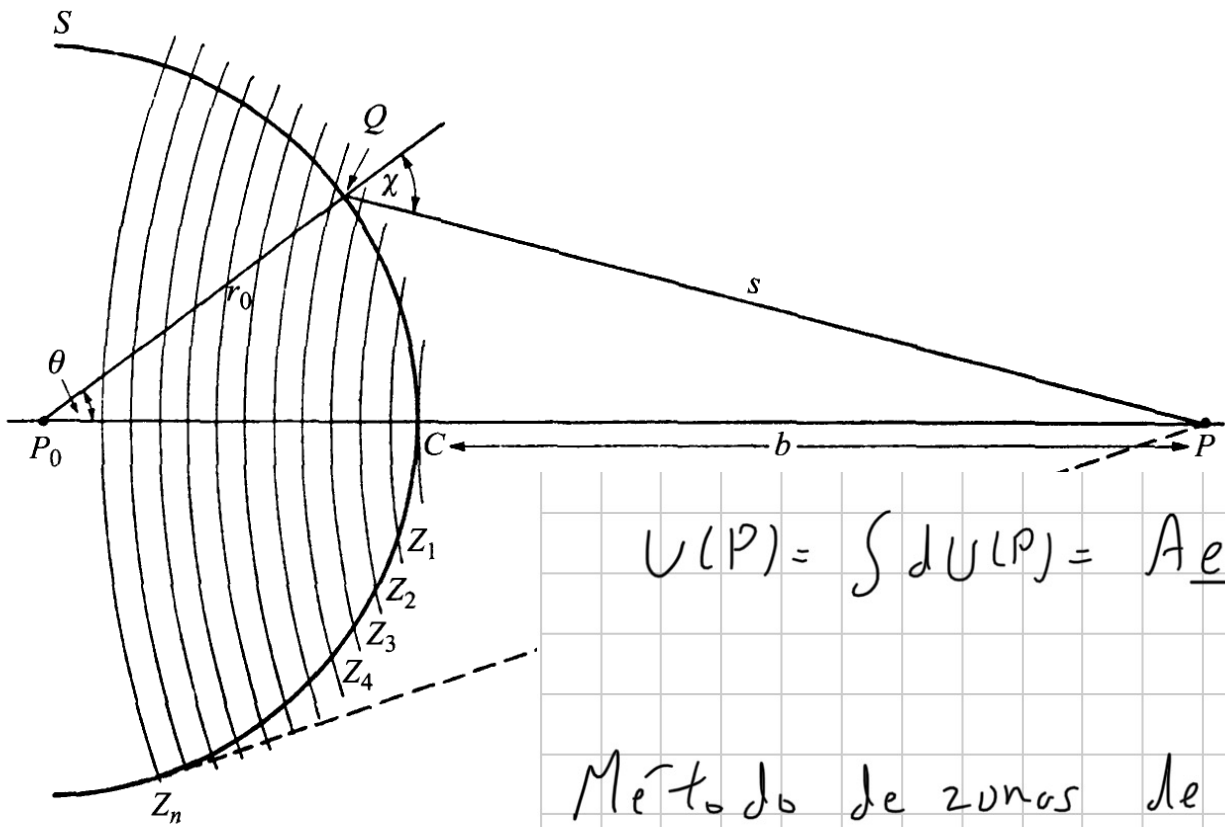
$$\rightarrow \text{contribuição: } dU(P) = K(\chi) \frac{A e^{ikr_0}}{r_0} \frac{e^{iks}}{s} dS$$

$$s = QP, \quad r_0 = P_0Q$$

$K(\chi) \rightarrow$ fator de inclinação

Palpite de Fresnel





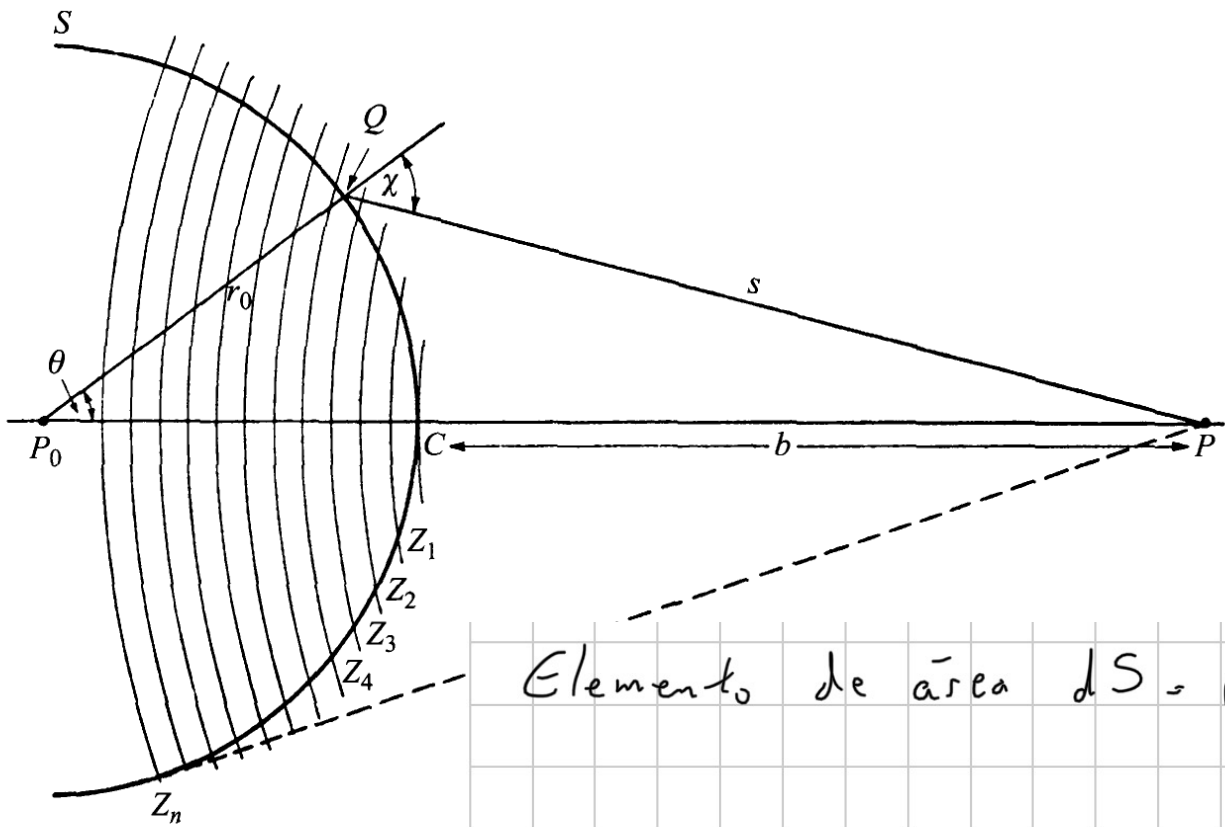
$$U(P) = \int dU(P) = \frac{A e^{i k r_0}}{r_0} \iint \frac{e^{i k s}}{s} \kappa(\chi) ds$$

Método de zonas de Fresnel:

Esferas de raio $b, b + \frac{\lambda}{2}, b + 2\frac{\lambda}{2}, \dots, b + j\frac{\lambda}{2}, \dots$

$b = CP \rightarrow$ dentro do anel, a fase $k s$ varia
menos do que π

Assumindo $r_0, b \gg \lambda$ $\kappa(\chi) \approx \kappa(\chi_j) = \kappa_j$



Elemento de área $dS = r_0^2 \sin \theta d\theta d\varphi$

↳ rotação em torno do eixo PP'

$$\text{Mas } s^2 = r_0^2 + (r_0 + b)^2 - 2r_0(r_0 + b) \cos \theta$$

$$\frac{d}{d\theta} s^2 = 2r_0(r_0 + b) \sin \theta$$

$$\frac{d}{d\theta} s^2 = \frac{ds^2}{ds} \cdot \frac{ds}{d\theta} = 2s \frac{ds}{d\theta}$$

$$s ds = r_0(r_0 + b) \sin \theta d\theta$$

$$\Rightarrow dS = \frac{r_0 \cdot s}{r_0 + b} ds d\varphi$$

Contribuição da zona j :

$$U_j(P) = A \frac{e^{ikr_0}}{r_0} \iint \frac{e^{iks}}{s} \kappa(\kappa) ds$$

$$\cong A \frac{e^{ikr_0}}{r_0} \int_0^{2\pi} \left[\int_{b+(j-1)\lambda/2}^{b+j\lambda/2} \frac{e^{iks}}{s} \cdot \kappa_j \frac{r_0 s}{r_0 + b} ds \right] d\varphi$$

$$= \frac{2\pi A e^{ikr_0}}{r_0 + b} \kappa_j \int_{b+(j-1)\lambda/2}^{b+j\lambda/2} e^{iks} ds =$$

$$= -\frac{i}{k} \frac{2\pi A e^{ikr_0}}{r_0 + b} \kappa_j \left[e^{ikj\lambda/2} e^{ikb} - e^{ikj\lambda/2} e^{ikb} e^{-ik\lambda/2} \right]$$

$$= -i\lambda \kappa_j A \frac{e^{ik(r_0+b)}}{r_0 + b} \underbrace{e^{i\pi j}}_{(-1)^j \cdot 2} \left[1 - e^{-i\pi} \right]$$

$$\frac{k\lambda}{2} = \pi$$

$$e^{-i\pi} = -1$$

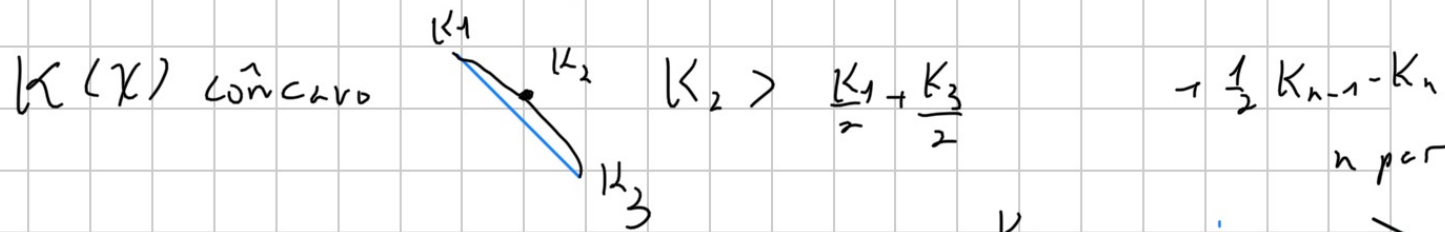
$$U_j(P) = 2i\lambda (-1)^{j+1} K_j A \frac{e^{iK(r_0+b)}}{r_0+b}$$

$$U(P) = \sum_j U_j(P) = 2i\lambda A \frac{e^{iK(r_0+b)}}{r_0+b} \sum_{j=1}^n (-1)^{j+1} K_j$$

Vamos brincar com a série $\Sigma_1 = \sum_{j=1}^n (-1)^{j+1} K_j$

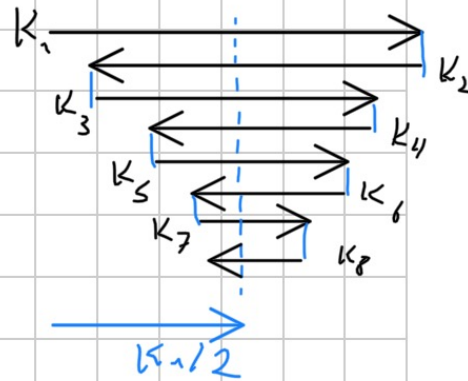
$$\Sigma_1 = \frac{K_1}{2} + \underbrace{\left(\frac{K_1}{2} - K_2 + \frac{K_3}{2} \right)}_{< 0} + \left(\frac{K_3}{2} - K_4 + \frac{K_5}{2} \right) + \dots + \frac{1}{2} K_n$$

n ímpar



$$\Rightarrow \Sigma_1 < \frac{K_1}{2} + \frac{K_n}{2} \text{ ; } n \text{ ímpar}$$

$$\Sigma_1 < \frac{K_1}{2} + \frac{K_{n-1} - K_n}{2} \text{ ; } n \text{ par}$$



Ou jogando com os termos:

$$\sum_1^1 = k_1 - \frac{k_2}{2} - \underbrace{\left(\frac{k_2}{2} - k_3 + \frac{k_4}{2} \right)}_{20} - \left(\frac{k_4}{2} - k_5 + \frac{k_6}{2} \right) - \dots$$

$$\Rightarrow \left. \begin{aligned} \sum_1^1 &> k_1 - \frac{k_2}{2} - \frac{k_{n-1}}{2} + k_n \quad n \text{ par} \\ \sum_1^1 &> k_1 - \frac{k_2}{2} - k_n \quad n \text{ ímpar} \end{aligned} \right\}$$

Temos um limite superior e inferior para \sum_1^1

Como $k_j \approx k_{j-1} \rightarrow n$ par

$$\sum_1^+ > \frac{k_1}{2} + \left(\frac{k_1}{2} - \frac{k_2}{2} \right) - \left(\frac{k_{n-1}}{2} - \frac{k_n}{2} \right) + \frac{k_n}{2}$$

$$\text{e } \sum_1^+ < \frac{k_2}{2} + \frac{k_n}{2}$$

$$\Rightarrow \sum_1^+ \approx \frac{k_1}{2} + \frac{k_n}{2} \quad n \text{ par}$$

Pelo mesmo raciocínio: $\sum_1^- = \frac{k_1}{2} - \frac{k_n}{2} \quad n \text{ ímpar}$

Como a última zona é tangente ($X = \bar{r}/\nu$), $k_n \rightarrow 0$ e

$$U(P) = \frac{1}{2} [U_1(P) + U_n(P)] = i \lambda k_1 A \frac{e^{i k(r_0 + b)}}{r_0 + b} = \frac{1}{2} U_1(P)$$

Para uma onda esférica: $i \lambda k_1 = 1$

Note que se bloquearmos as zonas com $j \geq 2$

$U(P) = U_1(P) \rightarrow$ Amplitude dobra

Intensidade quadruplica!

Mancha de Poisson (reductio ad absurdum)

Experimento de Arago \rightarrow mancha observada!

Grande Prêmio da Académie des Sciences (1819)

Lente de Fresnel: selecionar apenas as zonas com contribuição
aditiva (suprimindo as demais), ou modular as fases das zonas

Integral de Fresnel-Kirchhoff (1882)

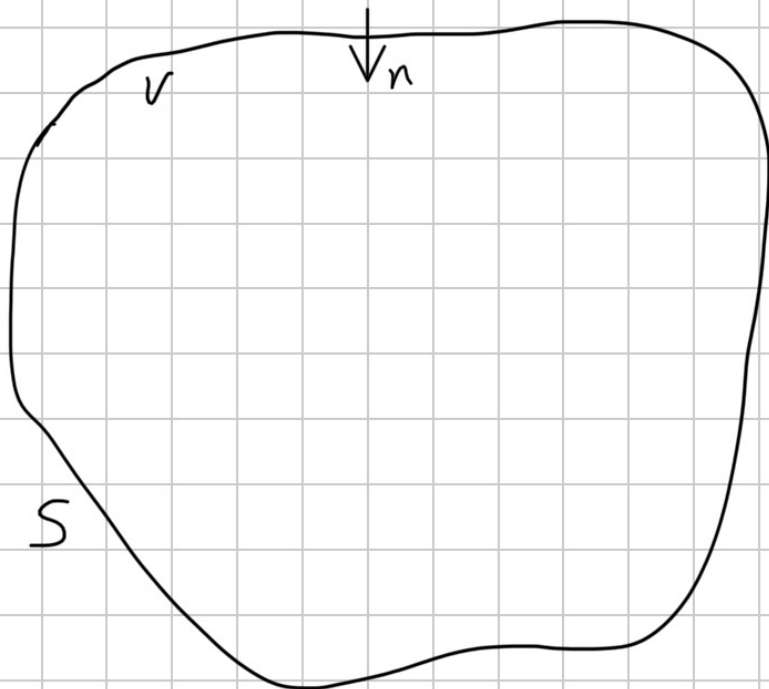
Onda monocromática: $V(\vec{r}, t) = U(\vec{r})e^{-i\omega t}$

$$\therefore \nabla^2 V = \frac{1}{c^2} \frac{d^2 V}{dt^2} = -\frac{\omega^2}{c^2} U(\vec{r})e^{-i\omega t}$$

$$\Rightarrow e^{-i\omega t} \nabla^2 U = -k^2 U(\vec{r})e^{-i\omega t}$$

$$\Rightarrow (\nabla^2 + k^2)U(\vec{r}) = 0 \rightarrow \text{eq. de Helmholtz}$$

Volumen V



$U \rightarrow$ contínua e com derivadas

de 1^ª e 2^ª ordem

Nós vimos na dedução da unicidade de solução da eq. de Laplace para a função potencial, o teorema de Green

$$\int_V (U \cdot \nabla^2 U' - U' \nabla^2 U) dV = - \int_S \left(U \frac{\partial U'}{\partial n} - U' \frac{\partial U}{\partial n} \right) dS$$

$$\text{Se } (\nabla^2 + k^2) U' = 0 \Rightarrow \int_V \underbrace{(U \cdot \nabla^2 U')}_{-k^2 U'} - \underbrace{U' \nabla^2 U}_{-k^2 U} dV = 0$$

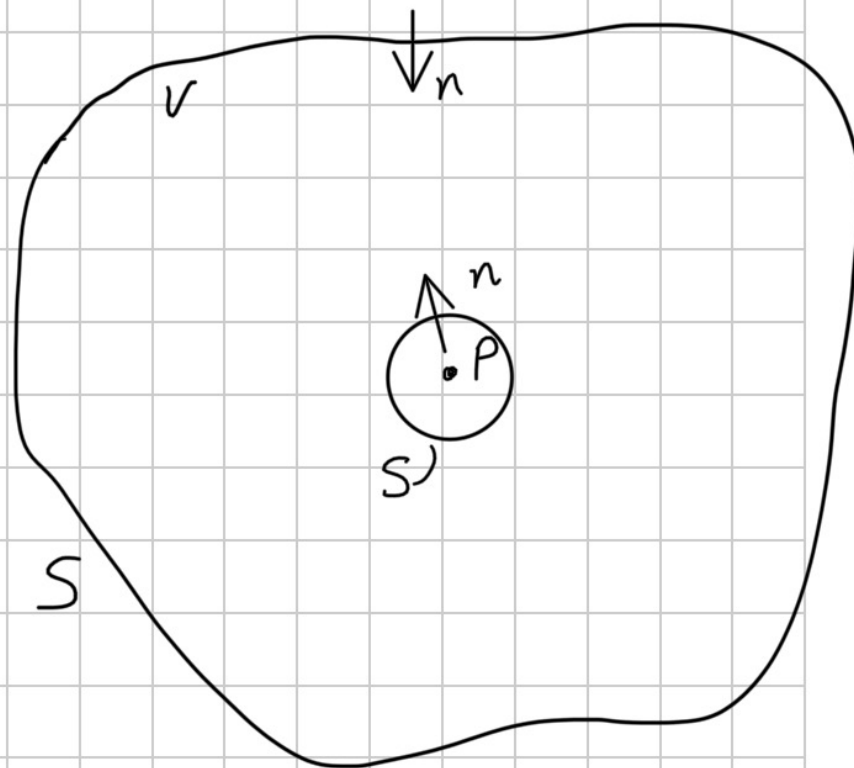
$$\therefore \int_S \left(U \frac{\partial U'}{\partial n} - U' \frac{\partial U}{\partial n} \right) dS = 0$$

Caso particular de $U'(\vec{r}) = \frac{e^{i\vec{k}\vec{r}}}{s}$, $s \rightarrow$ distância a um ponto P à posição \vec{r}

Para mantermos U' derivável, devemos envolver P por uma superfície S'

Neste caso

$$\left[\int_S + \int_{S'} \right] \left[U \frac{\partial}{\partial n} \left(\frac{e^{i\vec{k}\vec{r}}}{s} \right) - \frac{e^{i\vec{k}\vec{r}}}{s} \frac{\partial U}{\partial n} \right] dS = 0$$



$$\int_S \left[U \frac{\partial}{\partial n} \left(\frac{e^{i\vec{k}\vec{r}}}{s} \right) - \frac{e^{i\vec{k}\vec{r}}}{s} \frac{\partial U}{\partial n} \right] dS = - \int_{S'} \left[U \frac{\partial}{\partial n} \left(\frac{e^{i\vec{k}\vec{r}}}{s} \right) - \frac{e^{i\vec{k}\vec{r}}}{s} \frac{\partial U}{\partial n} \right] dS'$$

$$\int_S \left[U \frac{\partial}{\partial n} \left(\frac{e^{i k r}}{s} \right) - \frac{e^{i k r}}{s} \frac{\partial U}{\partial n} \right] dS = - \int_{S'} \left[U \frac{\partial}{\partial n} \left(\frac{e^{i k r}}{s} \right) - \frac{e^{i k r}}{s} \frac{\partial U}{\partial n} \right] dS'$$

$S' \rightarrow$ esfera de raio ϵ

$$= - \int_{\Omega} \left[U \frac{e^{i k r}}{\epsilon} \left(\frac{i k - 1}{\epsilon} \right) - \frac{e^{i k r}}{\epsilon} \frac{\partial U}{\partial s} \right] \epsilon^2 d\Omega$$

Como $\lim_{\epsilon \rightarrow 0} \int U e^{i k r} \epsilon d\Omega = 0$, $\lim_{\epsilon \rightarrow 0} \int \frac{\partial U}{\partial s} e^{i k r} \epsilon d\Omega = 0$

Temos, $\lim_{\epsilon \rightarrow 0} \int U e^{i k r} d\Omega = 4\pi U(P)$

$$\therefore U(P) = \frac{1}{4\pi} \int \left[U \frac{\partial}{\partial n} \left(\frac{e^{i k r}}{s} \right) - \frac{e^{i k r}}{s} \frac{\partial U}{\partial n} \right] dS$$

$$\therefore U(P) = \frac{1}{4\pi} \int \left[U \frac{\partial}{\partial n} \left(\frac{e^{iks}}{s} \right) - \frac{e^{iks}}{s} \frac{\partial U}{\partial n} \right] dS$$

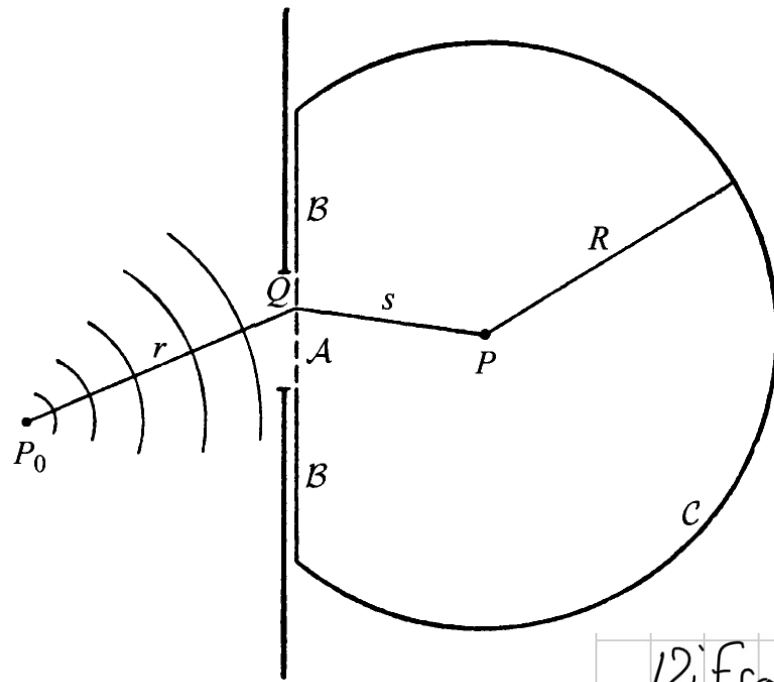
→ para uma onda, o campo em P está ligado ao campo (e sua derivada) na superfície S_0

Teorema integral de Helmholtz - Kirchhoff

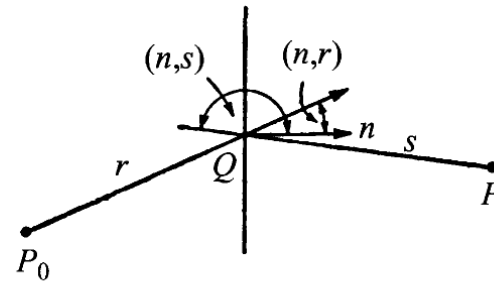
Caso contínuo → $k=0$ → U pode ser uma função potencial

$$U(P) = \frac{1}{4\pi} \int_S \left[U \frac{\partial}{\partial n} \left(\frac{1}{s} \right) - \frac{1}{s} \frac{\partial U}{\partial n} \right] dS$$

Difração de Kirchhoff (1882)



(a)



(b)

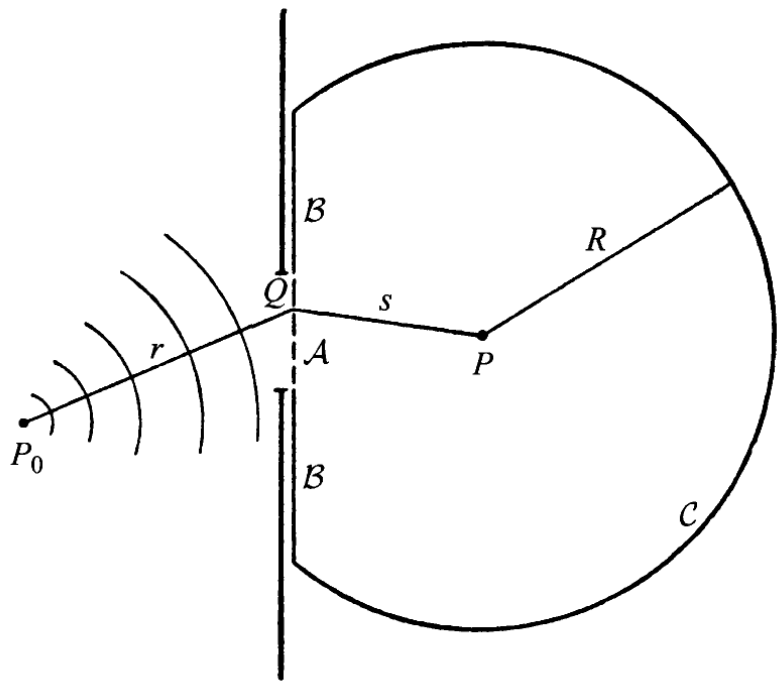
Difração de Kirchhoff Campo em P

$$U(P) = \frac{1}{4\pi} \left[\int_A + \int_B + \int_c \right] \left\{ U \frac{\partial}{\partial n} \left(\frac{e^{iks}}{s} \right) - \frac{e^{iks}}{s} \frac{\partial U}{\partial n} \right\} ds$$

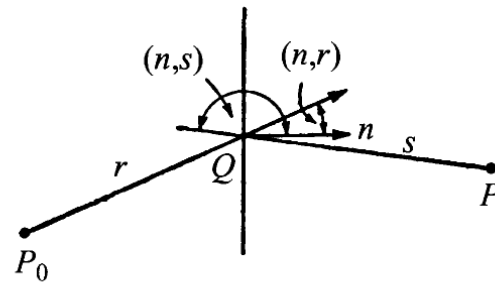
Em B: $U=0$, $\frac{\partial U}{\partial n}=0$

Em A: $U = \frac{Ae^{ikr}}{r}$; $\frac{\partial U}{\partial n} = \frac{Ae^{ikr}}{r} \left[ik - \frac{1}{r} \right] \cos(\alpha, n)$

↓
ângulo com a normal



(a)



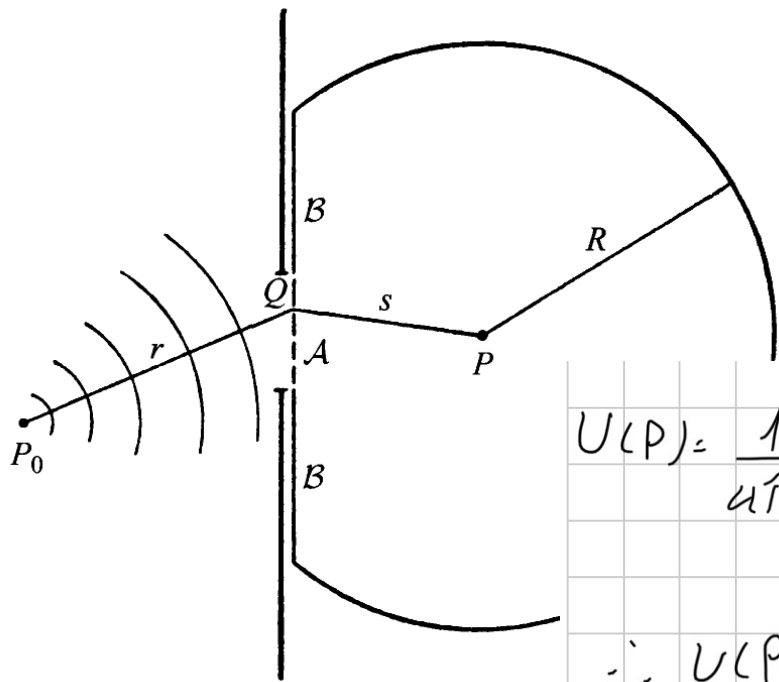
(b)

Em C :? Mesmo que $U_c \rightarrow 0$ $\frac{\partial}{\partial n} U_c \rightarrow 0$ $R \rightarrow \infty$

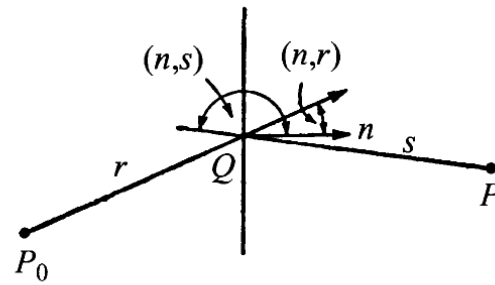
A área integrada diverge.

Por causalidade: tempo finito de propagação, tomamos

$$\int_C [] ds = 0$$



(a)



(b)

$$U(P) = \frac{1}{4\pi} \int_A \left\{ A \frac{e^{ikr}}{r} \frac{\partial}{\partial n} \left(\frac{e^{iks}}{s} \right) - \frac{e^{iks}}{s} \cdot A \frac{e^{ihr}}{r} \left[i \frac{h-1}{r} \right] \cos(n, r) \right\} ds$$

$$\therefore U(P) = \frac{-iA}{2\lambda} \int_A \frac{e^{i k(r+s)}}{r \cdot s} [\cos(n, r) - \cos(n, s)] ds$$

usando $\left(\frac{1}{r}, \frac{1}{s}\right) \ll k \Rightarrow s, r \gg \lambda$

Para o problema de Fresnel $\rightarrow \cos(n, r) = 1$

$$U(P) = -i \frac{A e^{ikh_0}}{2\lambda r_0} \int \frac{e^{iks}}{s} [1 + \cos(\chi)] ds$$

onde identificamos $U(\chi) = -\frac{i}{2\lambda} (1 + \cos \chi)$

Difração de Fraunhofer (1821)

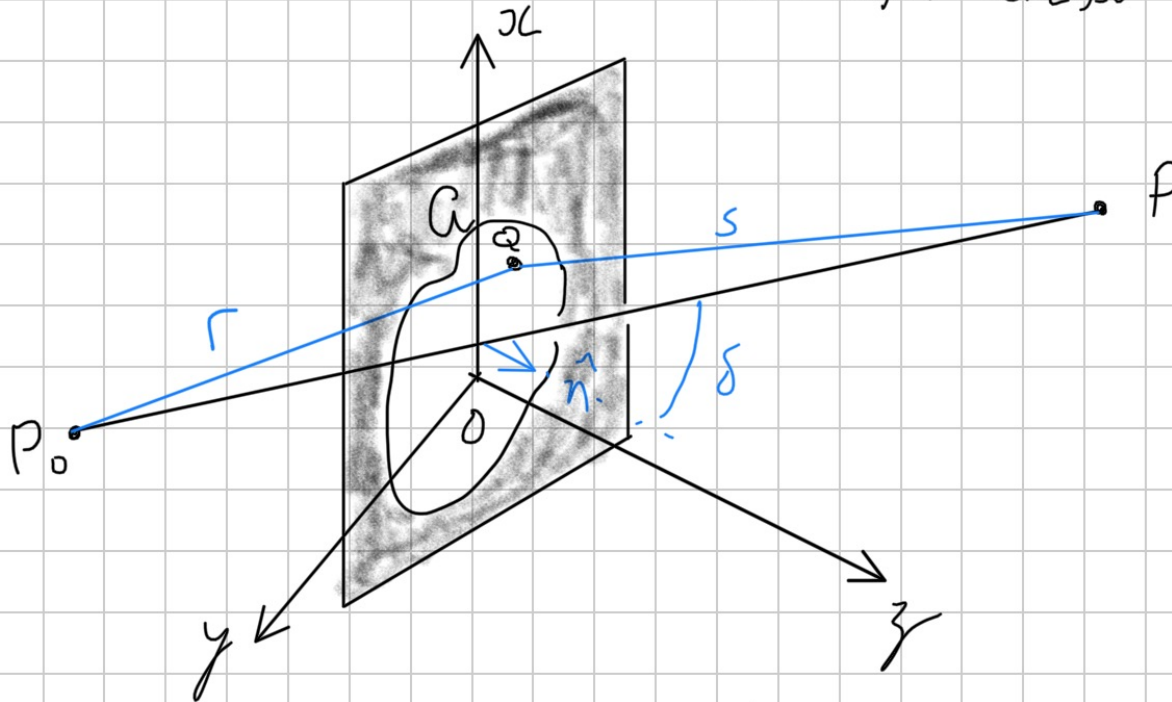
Partindo do Integral de Fresnel-Kirchhoff

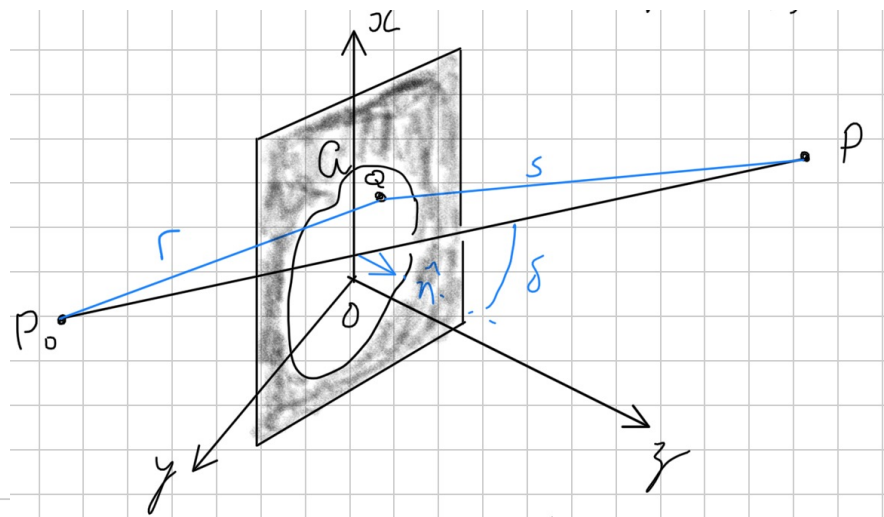
$$U(P) = -\frac{Ai}{2\lambda} \int_a \frac{e^{i k(r+s)}}{r \cdot s} [\cos(n,r) - \cos(n,s)] dS$$

$(r+s) \gg \lambda \rightarrow$ oscilações rápidas

$r^2, s^2 \gg a^2 \rightarrow [\cos(n,r) - \cos(n,s)] \sim \text{cte}$
↳ abertura

$$\cos(n,r) - \cos(n,s) \approx 2 \cos \delta$$





$$P_0 Q \rightarrow r \approx r' \rightarrow P_0 O \quad / \quad P Q \rightarrow s \approx s' \rightarrow P O$$

$$\frac{1}{r} \frac{1}{s} \approx \frac{1}{r' s'}$$

$$\Rightarrow U(P) = -\frac{A_i}{\lambda} \frac{\omega \delta}{r' s'} \int_A e^{i k(r+s)} dS$$

Coordinates de Q $\rightarrow (\xi, \eta, 0)$, $P_0 \rightarrow (x_0, y_0, z_0)$, $P = (x, y, z)$

$$r^2 = (x_0 - \xi)^2 + (y_0 - \eta)^2 + z_0^2$$

$$s^2 = (x - \xi)^2 + (y - \eta)^2 + z^2$$

$$r'^2 = x_0^2 + y_0^2 + z_0^2$$

$$s'^2 = x^2 + y^2 + z^2$$

$$r^2 = r'^2 - 2(x_0 \xi + y_0 \eta) + \xi^2 + \eta^2$$

$$s^2 = s'^2 - 2(x \xi + y \eta) + \xi^2 + \eta^2$$

Expandindo em série de potências de ξ/r' , η/r' , ξ/s' , η/s'

$$r \approx r' - \frac{x_0 \xi + y_0 \eta}{r'} + \frac{\xi^2 + \eta^2}{2r'} - \frac{(x_0 \xi + y_0 \eta)^2}{2r'^3} + \dots$$

$$s \approx s' - \frac{x \xi + y \eta}{s'} + \frac{\xi^2 + \eta^2}{2s'} - \frac{(x \xi + y \eta)^2}{2s'^3} + \dots$$

Substituindo:

$$U(P) = -i \frac{\cos \delta}{\lambda} A \frac{e^{i k (r+s')}}{r' s'} \int_a e^{i k f(\xi, \eta)} d\xi d\eta$$

$$f(\xi, \eta) = - \frac{x_0 \xi + y_0 \eta}{r'} - \frac{x \xi + y \eta}{s'} + \frac{\xi^2 + \eta^2}{2r'} + \frac{\xi^2 + \eta^2}{2s'} - \frac{(x_0 \xi + y_0 \eta)^2}{2r'^3} - \frac{(x \xi + y \eta)^2}{2s'^3} + \dots$$

Aproximação linear de $f(\xi, \eta) \rightarrow$ Difração de Fraunhofer
quadrática \rightarrow Difração de Fresnel

$\lim r' \rightarrow \infty, s' \rightarrow \infty \rightarrow$ fonte colimada, projeção no infinito
(nada que não fazemos com lentes)

$$k \left| \frac{\xi^2 + \eta^2}{r'} + \frac{\xi^2 + \eta^2}{s'} - \frac{(\cos \xi + \gamma_0 \eta)^2}{r'^3} - \frac{(\cos \xi + \gamma \eta)^2}{s'^3} \right| \ll 2\pi$$

$$\Rightarrow |r'| \gg \frac{(\xi^2 + \eta^2)}{\lambda}; \quad |s'| \gg \frac{(\xi^2 + \eta^2)}{\lambda}$$

Difração de Fraunhofer

$$U(P) = C \int_a e^{-i k (p\xi + q\eta)} d\xi d\eta$$

$$p = l - l_0, \quad q = m - m_0$$

$$\text{Com } l_0 = -\frac{x_0}{r_0}; \quad l = \frac{x}{r'}; \quad m_0 = -\frac{y_0}{r_0}; \quad m = \frac{y}{r'}$$

Cursos: Abertura retangular

$$U(P) = C \int_{-a}^a \int_{-b}^b e^{-ik(\rho\xi + \eta\eta)} d\xi d\eta = C \int_{-a}^a e^{-ik\rho\xi} d\xi \int_{-b}^b e^{-ik\eta\eta} d\eta$$

$$\int_{-a}^a e^{-ik\rho\xi} d\xi = -\frac{1}{ik\rho} [e^{-ik\rho a} - e^{ik\rho a}] = 2 \frac{\text{sen } k\rho a}{k\rho}$$

$\theta_0 = 0 \rightarrow \rho = \frac{2\alpha}{S} \rightarrow$ ângulo de saída

$$I(P) = |U(P)|^2 = \left(\frac{\text{sen } k\rho a}{k\rho a} \right)^2 \left(\frac{\text{sen } k\eta b}{k\eta b} \right)^2 \cdot I_0$$

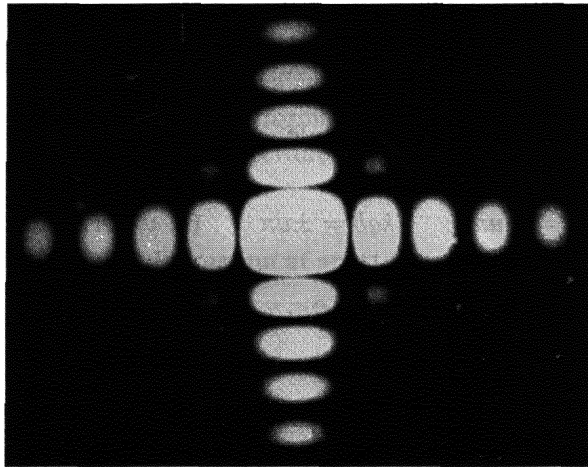
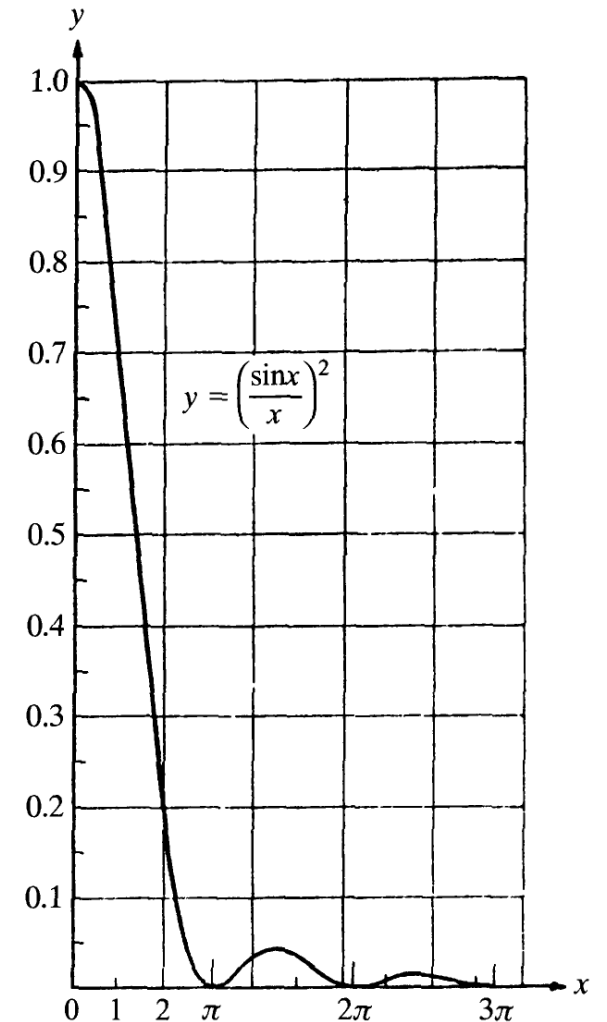
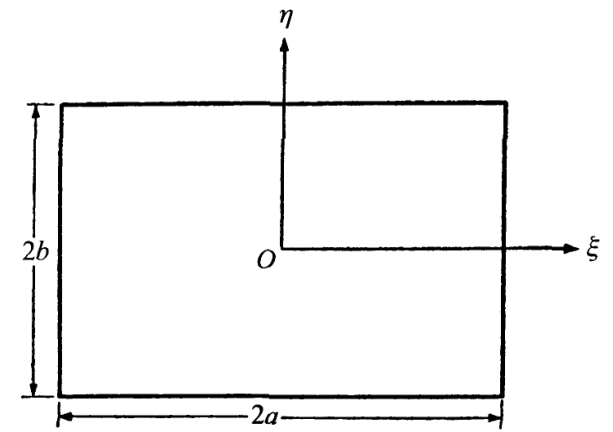


Fig. 8.10 Fraunhofer diffraction pattern of a rectangular aperture $8 \text{ mm} \times 7 \text{ mm}$, magnification $50\times$, mercury yellow light $\lambda = 5790 \text{ \AA}$. To show the existence of the weak secondary maxima the central portion was overexposed. (Photograph courtesy of H. Lipson, C. A. Taylor, and B. J. Thompson.)



Caso: Abertura circular

$$\xi = \rho \cos \theta, \quad \eta = \rho \sin \theta; \quad \rho = m \cos \psi, \quad q = m \sin \psi$$

$$U(P) = C \int_0^a \int_0^{2\pi} \text{escp} [-ik\rho m \cos(\theta - \psi)] d\rho d\theta$$

$$= C 2\pi \int_0^a J_0(k\rho m) \rho d\rho$$

↳ função de Bessel

$$= \left[\frac{2 J_1(kam)}{kam} \right]^2 I_0 \rightarrow \text{Airy (1835)}$$

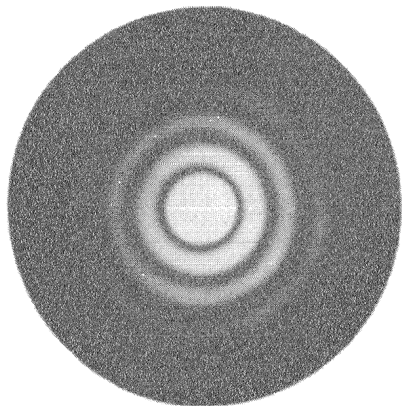


Fig. 8.12 Fraunhofer diffraction pattern of a circular aperture (the Airy pattern) 6 mm in diameter, magnification 50 \times , mercury yellow light $\lambda = 5790 \text{ \AA}$. To show the existence of the weak subsidiary maxima, the central portion was overexposed. (Photograph courtesy of H. Lipson, C. A. Taylor, and B. J. Thompson.)

