Eletromagnetismo

Capacitores

- Podemos armazenar a energia eletrostática!
- Gastamos energia para montar uma configuração de cargas.
- Isto inclui cargas livres e rearranjo de cargas em meios materiais.
- Esta energia está acoplada à própria distribuição do campo elétrico no espaço.
- Como podemos aumentar a eficiência?
- Como aumentar a capacidade de acumular energia?
- Vamos falar de acumuladores, ou *capacitores*.

Acumulando energia

Distribuindo cargas

$$\mathcal{E} = \frac{1}{2} \int_V V(\vec{r})\rho(\vec{r})\, dv$$

$$= \frac{1}{2} \sum V(r_i)\, q_i$$

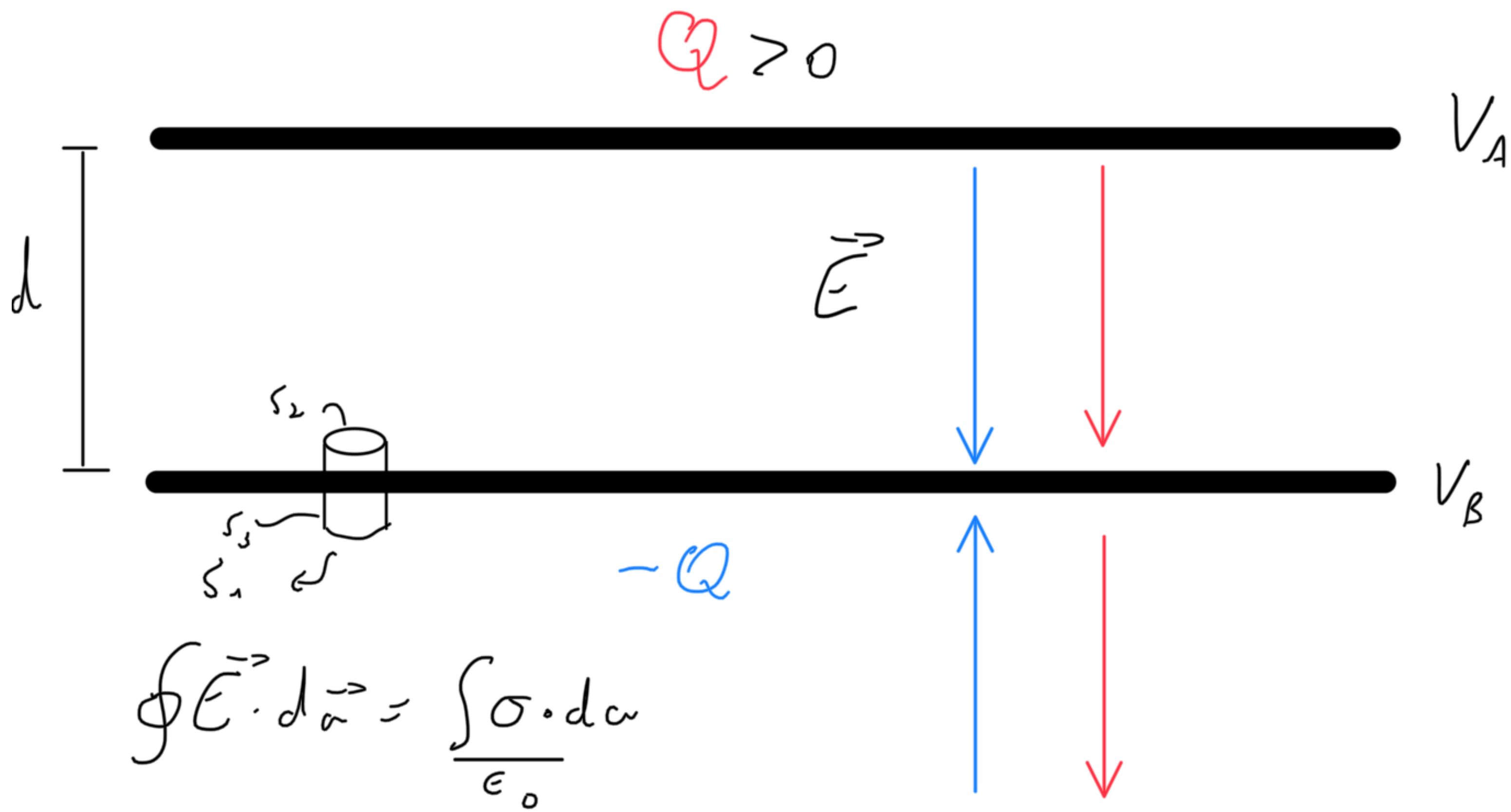
Criando campo

$$\mathcal{E} = \frac{1}{2} \int_V \vec{E} \cdot \vec{D}\, dv$$

$$\vec{E} = -\nabla V$$

$$\nabla \cdot \vec{D} = \rho$$

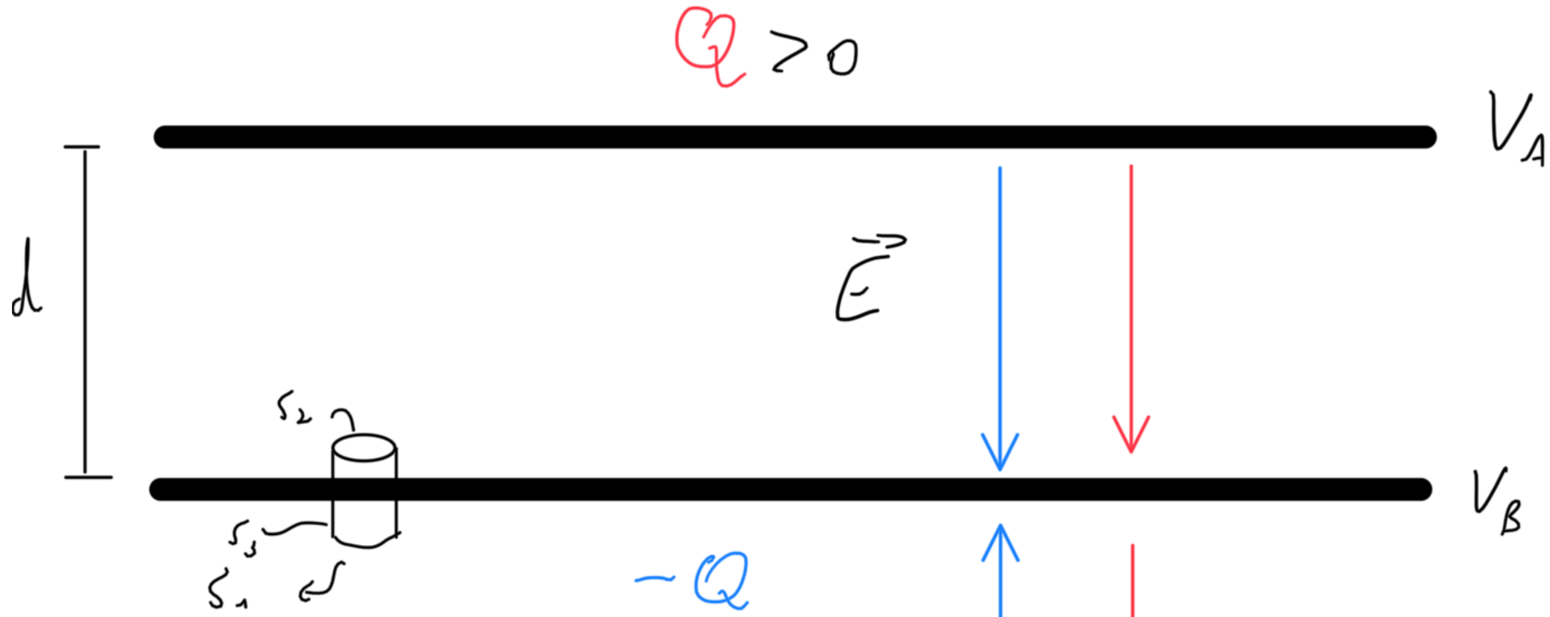
Energia em um arranjo de cargas



$$\oint \vec{E} \cdot d\vec{a} = \frac{\int \sigma \cdot da}{\epsilon_0}$$

$$\int_{S_1} \vec{E} \cdot d\vec{a} = 0 , \quad \int_{S_3} \vec{E} \cdot d\vec{a} = 0$$

Energia em um arranjo de cargas



$$\oint \vec{E}\cdot d\vec{a} = \frac{\int \sigma \cdot da}{\epsilon_0}$$

$$\oint \vec{E}\cdot d\vec{a} = \int E \cdot da = \int \frac{\sigma}{\epsilon_0}\, da \Rightarrow E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$$

$$V_A - V_B = -\int_B^A \vec{E}\cdot d\vec{l} = \frac{Q}{\epsilon_0 A}\, d$$

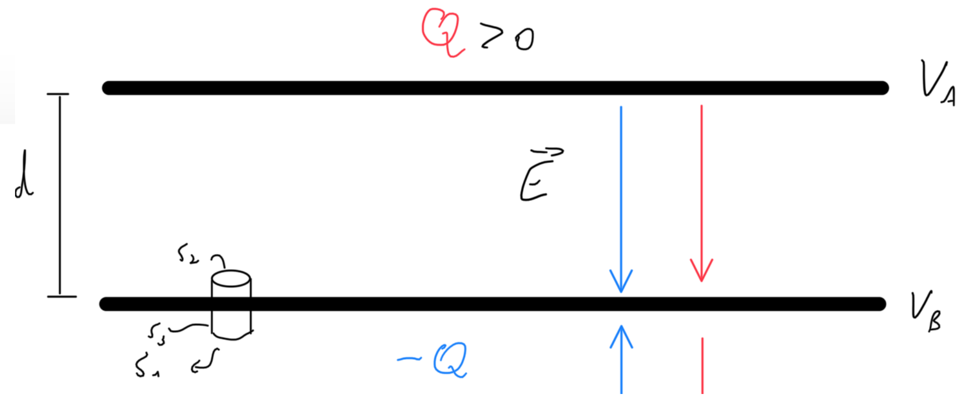
$$V_A - V_B = -\int_B^A \vec{E} \cdot d\vec{l} = \frac{Q}{\epsilon_0 A} d$$

$\Delta V \propto Q \rightarrow$ variações na proporcionalidade advém da geometria

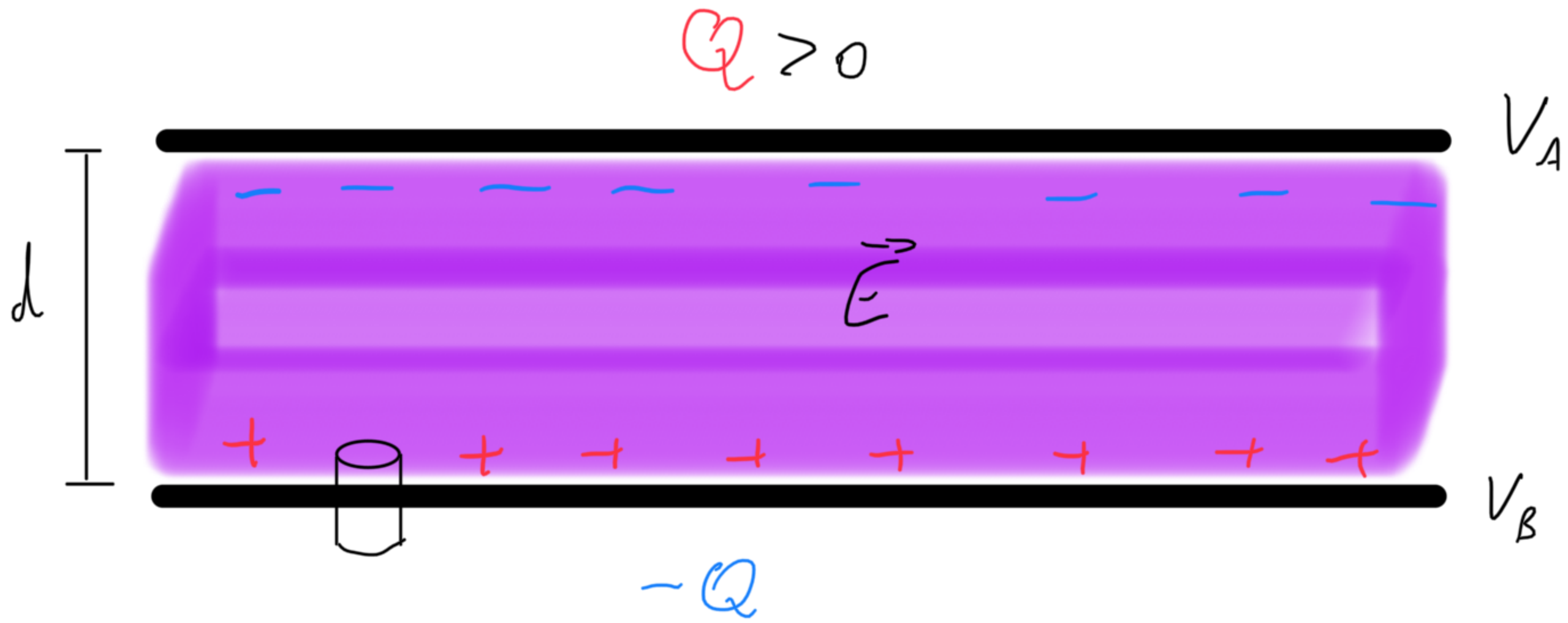
$$\Rightarrow Q = C \Delta V \Rightarrow C = \frac{Q}{\Delta V} \qquad [C] = F = \frac{C}{V}$$

Temos um dispositivo de dois contatos (dipolar) que associa carga acumulada com diferença de potencial!

Duas placas paralelas: $C = \epsilon_0 \frac{A}{d}$



Com um meio dielétrico



Quanto vale $\vec{E}$? Temos a carga de polarização $\sigma_p = \vec{P}\cdot\hat{n}$ reduzindo a carga efetiva

$$\oint \vec{E}\cdot d\vec{a} = \int \frac{\sigma \cdot da}{\epsilon_0} - \int \frac{\sigma_p}{\epsilon_0}\, da = \int \frac{\sigma}{\epsilon_0} \cdot da - \int \frac{\vec{P}}{\epsilon_0}\, \hat{n}\, d\vec{a}$$

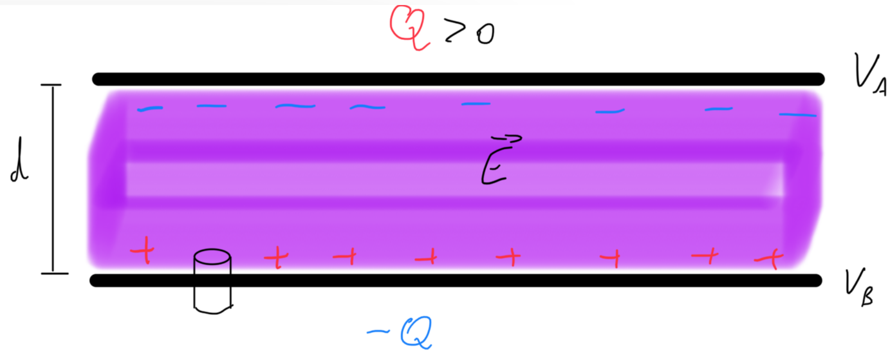
$$\oint \vec{E}\cdot d\vec{a} = \int \frac{\sigma}{\epsilon_0} da - \int \frac{\sigma_p}{\epsilon_0} da = \int \frac{\sigma}{\epsilon_0} da - \int \frac{\vec{P}}{\epsilon_0}\, \hat{n}\, d\vec{a}$$

$$\therefore \int (\epsilon_0 \vec{E} + \vec{P})\, d\vec{a} = \int \sigma\, da \Rightarrow \int \vec{D}\, d\vec{a} = \int \sigma\, da$$

$$\Delta V = E\cdot d \quad ; \quad E = \frac{D}{\epsilon} \quad ; \quad D = \frac{Q}{A}$$

$$\Delta V = \frac{d}{\epsilon A}\cdot Q = \frac{Q}{C_\epsilon} \Rightarrow C_\epsilon = \epsilon \frac{A}{d}$$

Para a mesma carga: $\Delta V \propto 1/C$



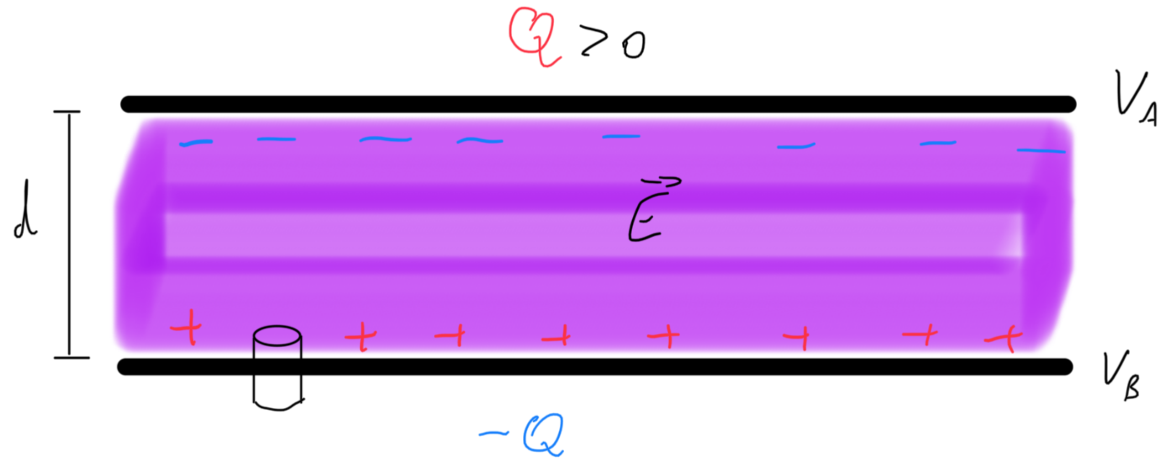
Qual a energia armazenada?

$$\mathcal{E} = \frac{1}{2}\int \vec{E}\cdot\vec{D}\,dv = \frac{\epsilon}{2} d\cdot A\cdot E^2 = \frac{\epsilon\, d.A}{2}\cdot\left(\frac{\Delta V}{d}\right)^2$$

$$\mathcal{E} = \frac{\epsilon A}{d}\frac{\Delta V^2}{2} = \frac{C\Delta V^2}{2} \qquad ; \quad C = \frac{Q}{\Delta V}$$

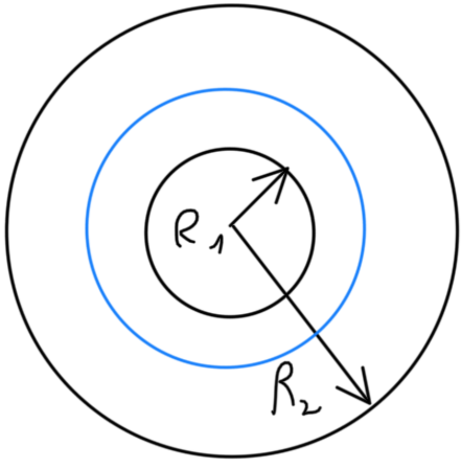
$$= \frac{Q\cdot\Delta V}{2} = \frac{1}{2}\int V\cdot\rho\, dv$$

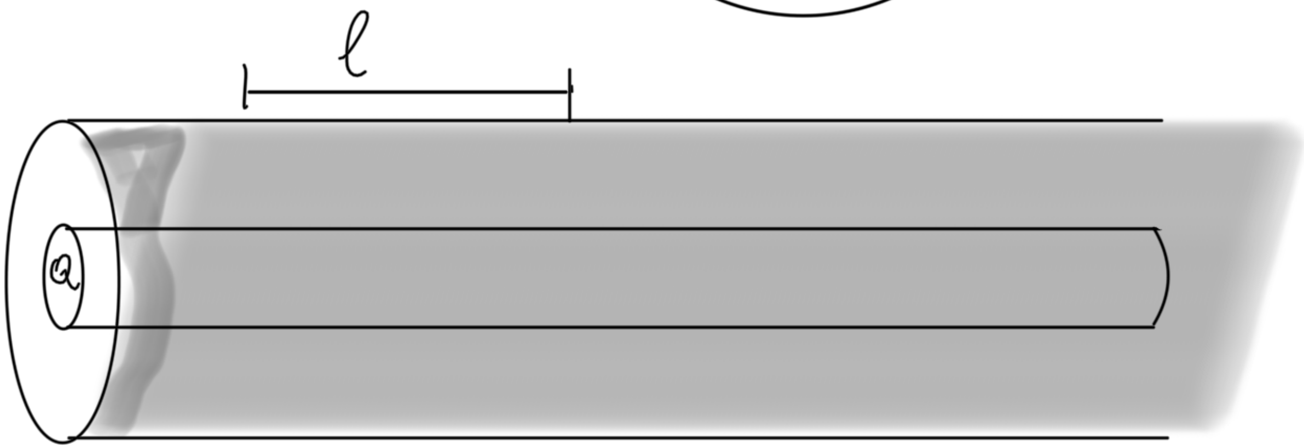
$$\mathcal{E} = \frac{Q^2}{2C}$$



Explorando geometrias

Tubular:





Superfície gaussiana: cilindro de raio r

$$\int \vec{D} \cdot d\vec{a} = D \cdot 2\pi r \cdot \ell = Q$$

$$\int \vec{D} \cdot d\vec{a} = D \cdot 2\pi r \cdot l = Q$$

$$E = \frac{D}{\epsilon} = \frac{Q}{2\pi r l \cdot \epsilon} \longrightarrow \Delta V = -\int_{R_2}^{R_1} \vec{E} \cdot d\vec{r}$$

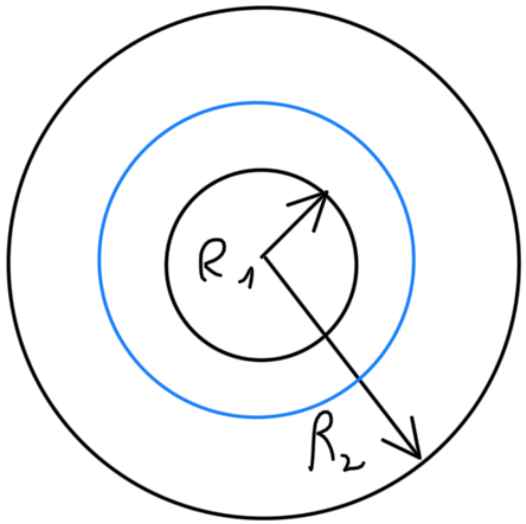
$$\Delta V = \frac{-Q}{2\pi l \epsilon} \int_{R_2}^{R_1} \frac{1}{r}\, dr = \frac{-Q}{2\pi l \epsilon} \left[\ln r \right]_{R_2}^{R_1}$$

$$= \frac{Q}{2\pi l \epsilon} \left[\ln R_2 - \ln R_1 \right]$$

$$\Delta V = V_{R_1} - V_{R_2} = \frac{Q}{2\pi l \epsilon} \ln\left(\frac{R_2}{R_1}\right) \Rightarrow \frac{C}{l} = \frac{2\pi \epsilon}{\ln\left(\frac{R_2}{R_1}\right)}$$

Capacitor esférico

$$\vec{D} = \frac{Q}{4\pi r^2}\,\hat{r} \quad ; \quad R_1 < r < R_2$$



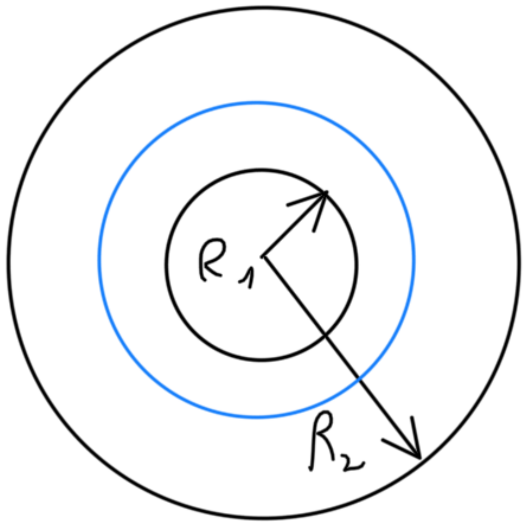
$$\Delta V = V_{R_1} - V_{R_2} = -\int_{R_2}^{R_1} \vec{E}\cdot d\vec{l} = \int_{R_1}^{R_2} \frac{Q}{4\pi\varepsilon}\cdot\frac{1}{r^2}\cdot\hat{r}\cdot\hat{r}\,dr$$

$$= \frac{Q}{4\pi\varepsilon}\int_{R_1}^{R_2}\frac{1}{r^2}\,dr = \frac{-Q}{4\pi\varepsilon}\left[\frac{1}{r}\right]\Bigg|_{R_1}^{R_2}$$

$$= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Capacitor esférico

$$\vec{D} = \frac{Q}{4\pi r^2}\,\hat{r} \; ; \; R_1 < r < R_2$$

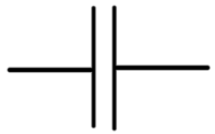


$$C = 4\pi\varepsilon \, \frac{R_1 R_2}{R_2 - R_1}$$

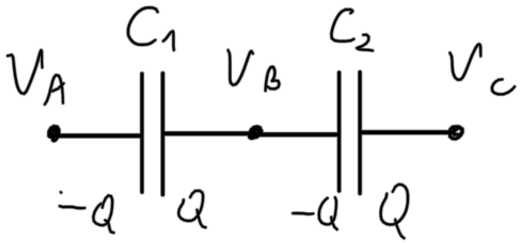
Note: $$\mathcal{E} = \frac{Q^2}{2C} = \frac{Q^2}{8\pi\varepsilon}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

No vácuo, se $\lim\limits_{R_2 \to \infty} \mathcal{E} = \dfrac{Q^2}{8\pi\varepsilon_0 R_1}$ $\rightarrow$ Energia de uma casca esférica carregada!

Associações: Capacitor



Série:



$$V_C - V_B = \frac{Q}{C_2} = V_2 \qquad V_B - V_A = \frac{Q}{C_1} = V_1$$

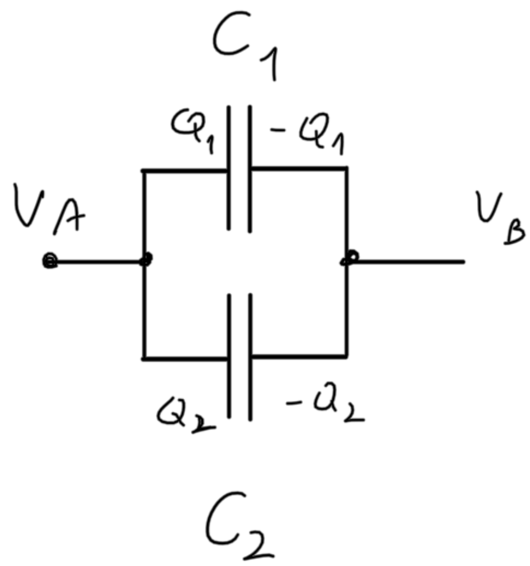
$$\Delta V = V_C - V_A = V_C - V_B + V_B - V_A = Q\left(\frac{1}{C_2} + \frac{1}{C_1}\right) = \frac{Q}{C_{eq}}$$

$$C_{eq} = \frac{C_2 C_1}{C_1 + C_2}$$

$$V_1 = \frac{Q}{C_1} \qquad V_2 = \frac{Q}{C_2} \implies V_1 \cdot C_1 = V_2 \cdot C_2$$

Paralelo:

$$V_A - V_B = \frac{Q_1}{C_1} = \Delta V$$



$$V_A - V_B = \frac{Q_2}{C_2} = \Delta V$$

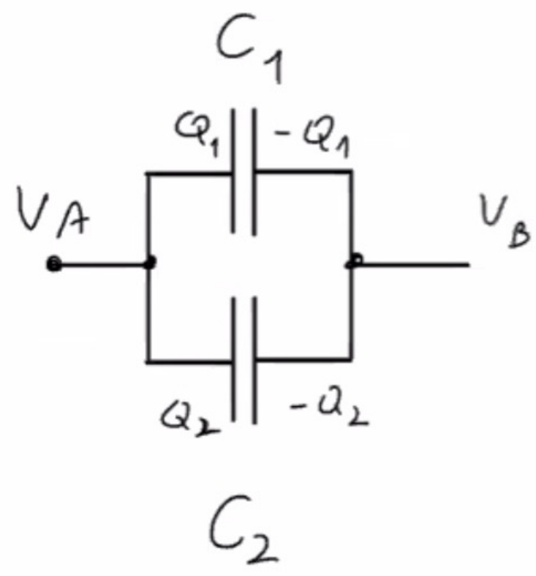
$$Q = Q_1 + Q_2 = C_1 \cdot \Delta V + C_2 \Delta V = \Delta V \cdot (C_1 + C_2)$$

$$C_{eq} = C_1 + C_2$$

$$V_1 = V_2 \qquad \frac{Q_1}{C_1} = \frac{Q_2}{C_2}$$

Paralelo:

$$V_A - V_B = \frac{Q_1}{C_1} = \Delta V$$



$$V_A - V_B = \frac{Q_2}{C_2} = \Delta V$$

$$Q = Q_1 + Q_2 = C_1 \cdot \Delta V + C_2 \Delta V = \Delta V \cdot (C_1 + C_2)$$

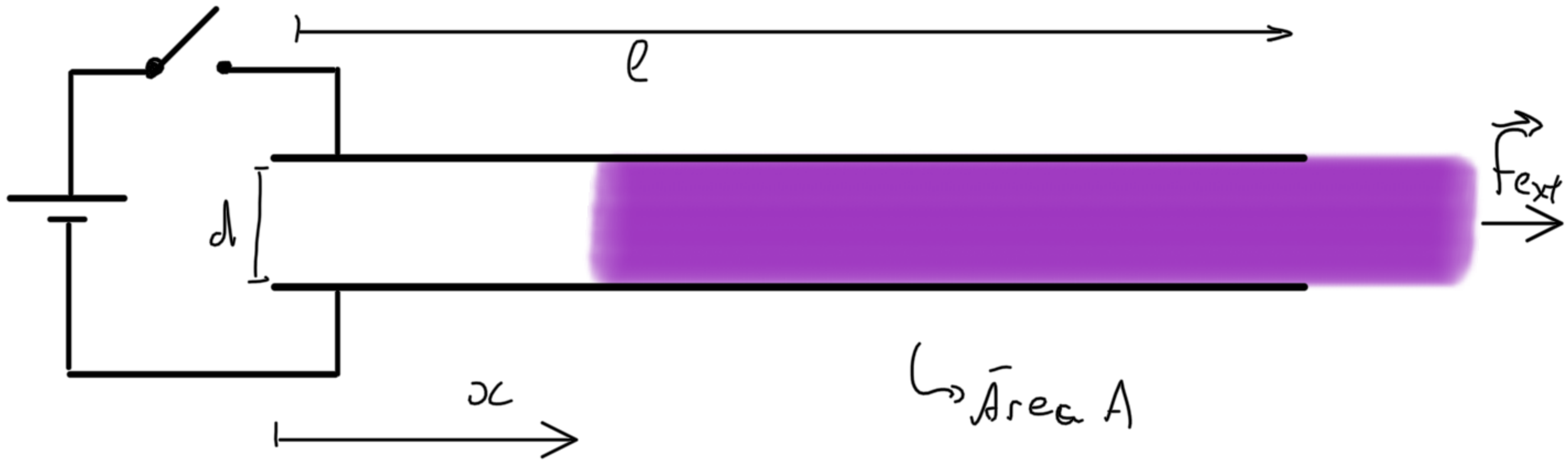
$$C_{eq} = C_1 + C_2$$

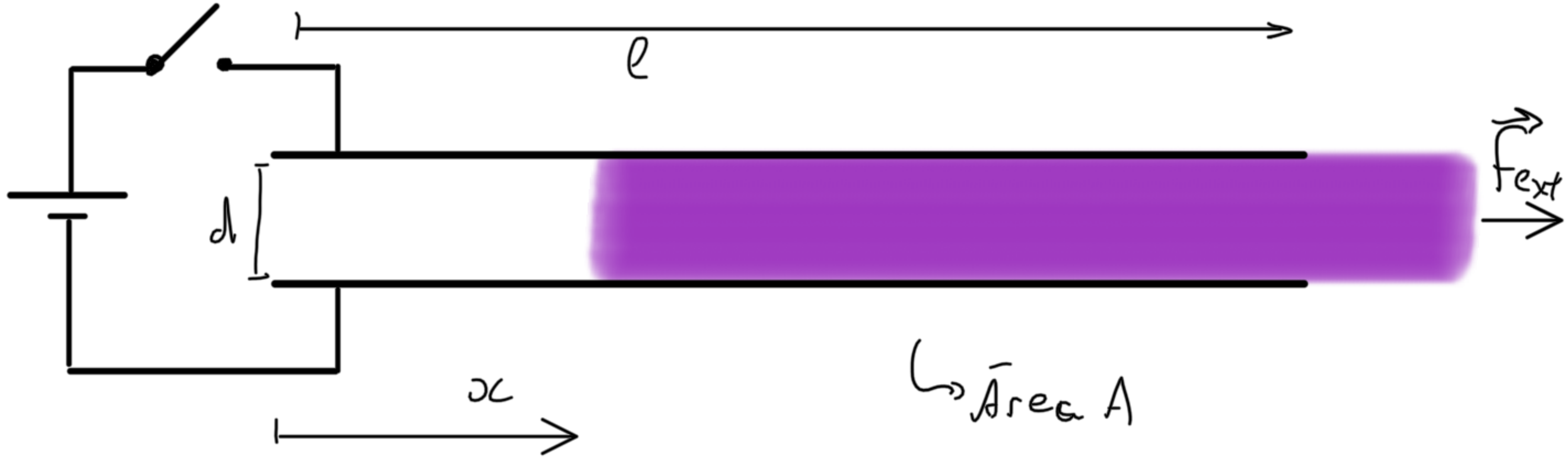
$$V_1 = V_2 \qquad \frac{Q_1}{C_1} = \frac{Q_2}{C_2}$$

$$V_1 = V_A - V_B = V_2$$

$\hookrightarrow C_1$ $\qquad$ $\hookrightarrow C_2$

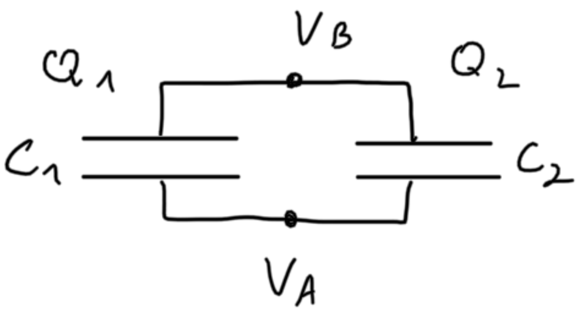
Qual o trabalho da inserção de um dielétrico?





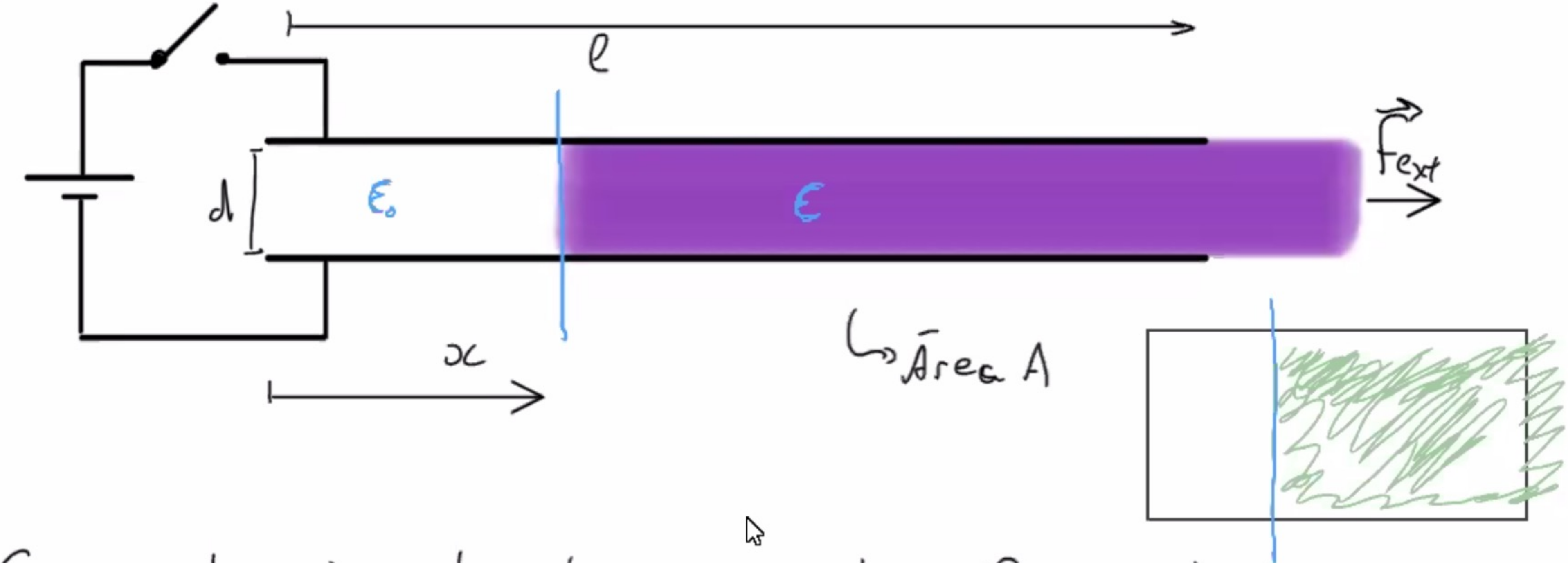
Capacitor inicialmente carregado: Q constante

Temos



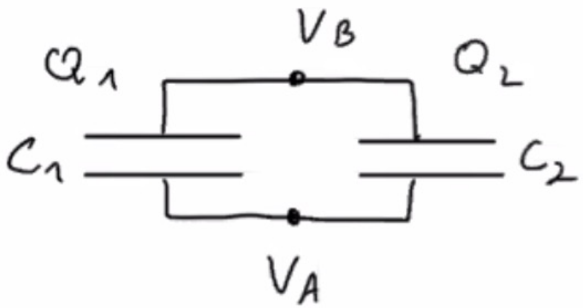
$$V = V_B - V_A = \frac{Q_1}{C_1} = \frac{Q_2}{C_2}$$

$$C_1 = \varepsilon_0 \frac{A}{d} \cdot \frac{x}{\ell} \qquad C_2 = \varepsilon \frac{A}{d} \frac{\ell - x}{\ell}$$



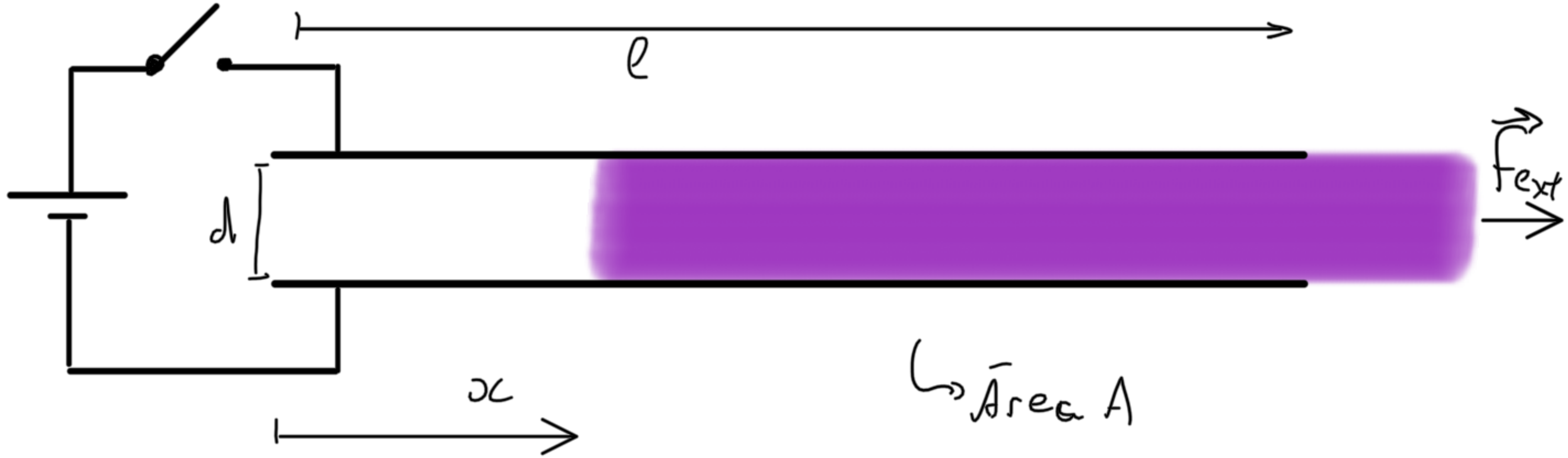
Capacitor inicialmente carregado: Q constante

Temos



$$V = V_B - V_A = \frac{Q_1}{C_1} = \frac{Q_2}{C_2}$$

$$C_1 = \varepsilon_0 \frac{A}{d} \cdot \frac{x}{\ell} \qquad C_2 = \varepsilon \frac{A}{d} \frac{\ell - x}{\ell}$$

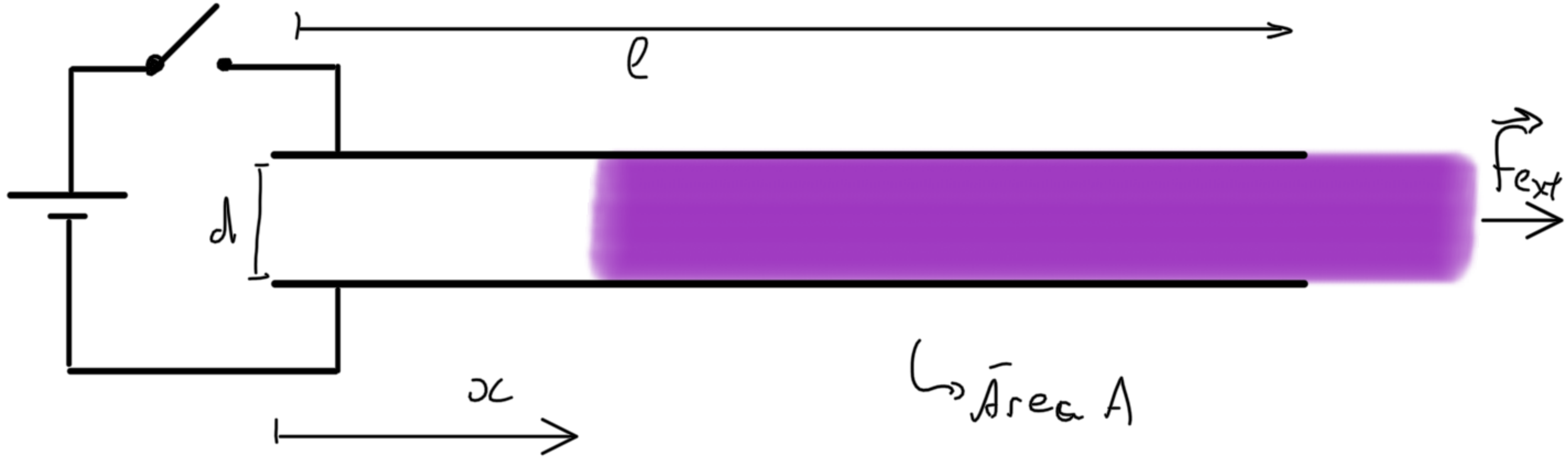


$$C_1 = \varepsilon_0 \frac{A}{d} \cdot \frac{x}{\ell} \qquad C_2 = \varepsilon \frac{A}{d} \frac{\ell - x}{\ell}$$

Energia armazenada: $\mathcal{E} = \dfrac{Q^2}{2C} = \dfrac{CU^2}{2} = \dfrac{Q \cdot V}{2}$

Qual usar?

Energia em $C_1 \to \mathcal{E}_1 = \dfrac{Q_1^2}{2C_1}$ / em $C_2 \to \mathcal{E}_2 = \dfrac{Q_2^2}{2C_2}$



Energia em $C_1 \rightarrow$ $\mathcal{E}_1 = \dfrac{Q_1^2}{2C_1}$ / em $C_2 \rightarrow$ $\mathcal{E}_2 = \dfrac{Q_2^2}{2C_2}$

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 = \left(\frac{Q_1^2}{C_1} + \frac{Q_2^2}{C_2}\right)\frac{1}{2} = \left(\frac{Q_1}{C_1} \cdot Q_1 + \frac{Q_2}{C_2} \cdot Q_2\right)\frac{1}{2}$$

$$= \left(V \cdot Q_1 + V\, Q_2\right)\frac{1}{2} = \frac{V \cdot Q}{2}$$

Trabalho da Força externa $dW = \vec{F}_{ext} \cdot d\vec{l} = F_{ext} \cdot dx$

$dW = d\mathcal{E}$ → variação de energia no sistema

$$\frac{d\mathcal{E}}{dx} = \frac{d}{dx}\left(\frac{Q^2}{2C}\right) = \frac{\partial}{\partial C}\mathcal{E} \cdot \frac{dC}{dx}$$

$$\frac{\partial \mathcal{E}}{\partial C} = -\frac{Q^2}{2C^2} = -\frac{V^2}{2}$$

$$C(x) = \varepsilon_0 \frac{A}{d} \cdot \frac{x}{l} + \varepsilon \frac{A}{d} \frac{l-x}{l}$$

$$C(x) = \varepsilon_0 \frac{A}{d} \cdot \frac{x}{l} + \varepsilon \frac{A}{d} \frac{l-x}{l}$$

$$\frac{dC}{dx} = \varepsilon_0 \frac{A}{d} \frac{1}{l} - \frac{\varepsilon A}{dl} = -(\varepsilon - \varepsilon_0)\frac{A}{dl}$$

$$\Rightarrow \frac{d\mathcal{E}}{dx} = -\frac{V^2}{2} \cdot -(\varepsilon - \varepsilon_0)\frac{A}{dl} = \frac{V^2}{2}\frac{A}{dl}(\varepsilon - \varepsilon_0) > 0$$

$\Rightarrow$ Energia é mínima com $x = 0$

(como vimos na última aula)

$$\Rightarrow \frac{d\mathcal{E}}{dx} = -\frac{V^2}{2} \cdot -(\varepsilon-\varepsilon_0)\frac{A}{dl} = \frac{V^2}{2}\frac{A}{dl}(\varepsilon-\varepsilon_0) > 0$$

$\Rightarrow$ Energia é mínima com $x=0$

(como vimos na última aula)

$\Rightarrow$ Realizamos trabalho para remover o dielétrico $dW>0$

$\Rightarrow$ Força externa compensa a <u>Força atrativa</u> que o capacitor exerce no meio

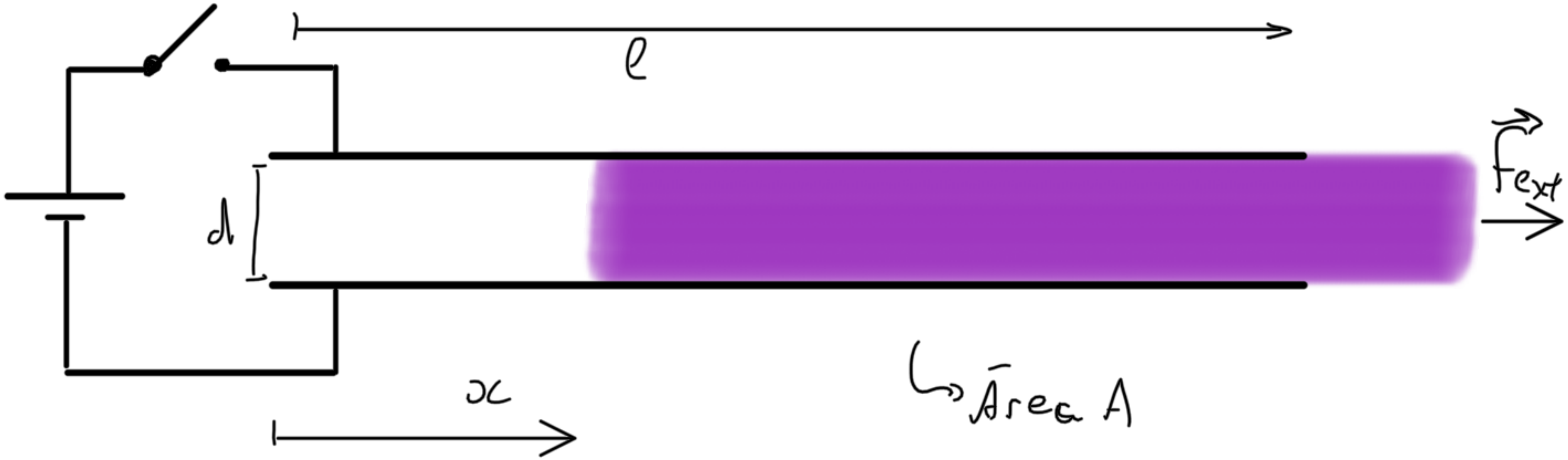
$$\Rightarrow F_e = -F_{ext} = -\frac{d\mathcal{E}}{dx}$$ $\rightarrow$ $-$variação da energia potencial

Trabalho realizado: $$W = \int_0^l \frac{V^2}{2}\frac{A}{d\,l}(\varepsilon-\varepsilon_0)\,dx$$

Carga constante:

Se a carga inicial for $Q \Rightarrow \quad Q = C(x) \cdot V(x) = \varepsilon \frac{A}{d} V_0$

$$V(x) = \varepsilon \frac{A}{d} V_0 \cdot \frac{1}{C(x)}$$



Carga constante:

Se a carga inicial for $Q \Rightarrow \quad Q = C(x) \cdot V(x) = \varepsilon \frac{A}{d} V_0$

$$V(x) = \varepsilon \frac{A}{d} V_0 \cdot \frac{1}{C(x)}$$

$$C(x) = \varepsilon_0 \frac{A}{d} \cdot \frac{x}{\ell} + \varepsilon \frac{A}{d} \frac{\ell - x}{\ell} = \varepsilon \frac{A}{d} \left[1 - \frac{x}{\ell} \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon} \right) \right]$$

$$V(x) = V_0 \cdot \frac{1}{1 - x \cdot \alpha} \qquad \alpha = \frac{\varepsilon - \varepsilon_0}{\varepsilon \ell}$$

$$W_Q = \frac{A V_0^2}{2 d \ell} (\varepsilon - \varepsilon_0) \int_0^{\ell} \frac{1}{(1 - x \cdot \alpha)^2} \cdot dx$$

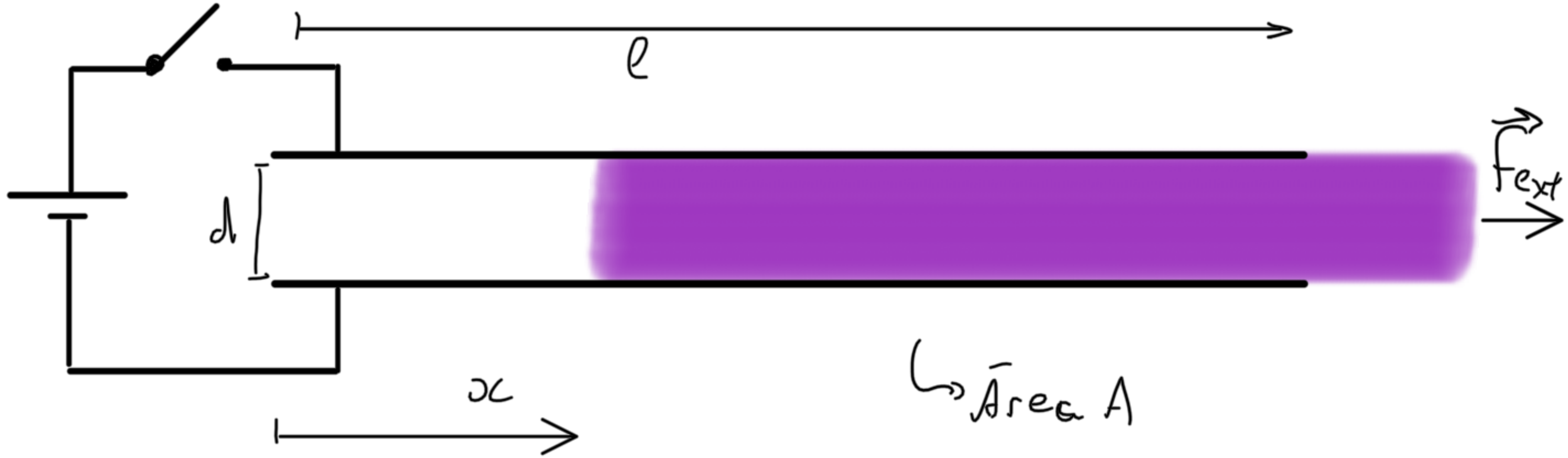
$$W_Q = \frac{A V_0^2}{2 d l} (\varepsilon - \varepsilon_0) \int_0^l \frac{1}{(1 - x\alpha)^2} \cdot dx$$

$$1 - x\alpha = y \qquad dy = -\alpha dx$$

$$\int_0^l \frac{1}{(1-x\alpha)^2} dx = -\frac{1}{\alpha} \int_1^{1-l\alpha} \frac{1}{y^2} dy = \frac{1}{\alpha}\left[\frac{1}{y}\right]\Big|_1^{1-l\alpha} = \frac{1}{\alpha}\left[\frac{1}{1-l\alpha} - 1\right] =$$

$$= \frac{1}{\alpha}\left[\frac{1 - 1 + l\alpha}{1 - l\alpha}\right] = \frac{l}{1 - l\alpha} = l \frac{1}{1 - \frac{(\varepsilon - \varepsilon_0)}{\varepsilon}} = l \frac{\varepsilon}{\varepsilon_0}$$

$$W_Q = \frac{A V_0^2}{2d} (\varepsilon - \varepsilon_0) \cdot \frac{\varepsilon}{\varepsilon_0} = \frac{A V_0^2}{2d} \left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon}\right) \varepsilon^2$$



$$W_Q = \frac{A V_0^2}{2d}(\varepsilon - \varepsilon_0) \cdot \frac{\varepsilon}{\varepsilon_0} = \frac{A V_0^2}{2d}\left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon}\right)\varepsilon^2$$

Colocando em termos da carga $Q = \varepsilon \frac{A}{d} V_0 \Rightarrow V_0^2 = \frac{Q^2}{(\varepsilon A/d)^2}$

$$W_Q = \frac{Q^2}{\varepsilon^2 A^2} \cdot d^2 \cdot \frac{A \varepsilon^2}{2d}\left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon}\right) = \frac{Q^2 d}{2A}\left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon}\right)$$

Diretamente do capacitor: $\mathcal{E} = Q^2/2C \Rightarrow$

$$\mathcal{E}_i = \frac{Q^2}{2C_i} = \frac{Q^2}{2\varepsilon A} \cdot d \quad ; \quad \mathcal{E}_f = \frac{Q^2}{2C_f} = \frac{Q^2}{2\varepsilon_0} \frac{d}{A}$$

$$\mathcal{E}_f - \mathcal{E}_i = Q^2 \cdot \frac{d}{2A} \left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon} \right)$$

Potencial final: $V_f \cdot C_f = V_0 \cdot C_0$

$$V_f = V_0 \cdot \frac{C_0}{C_f} = V_0 \cdot \varepsilon \frac{A}{d} \cdot \frac{1}{\varepsilon_0 A} d = \frac{\varepsilon}{\varepsilon_0} V_0$$

Energia armazenada:

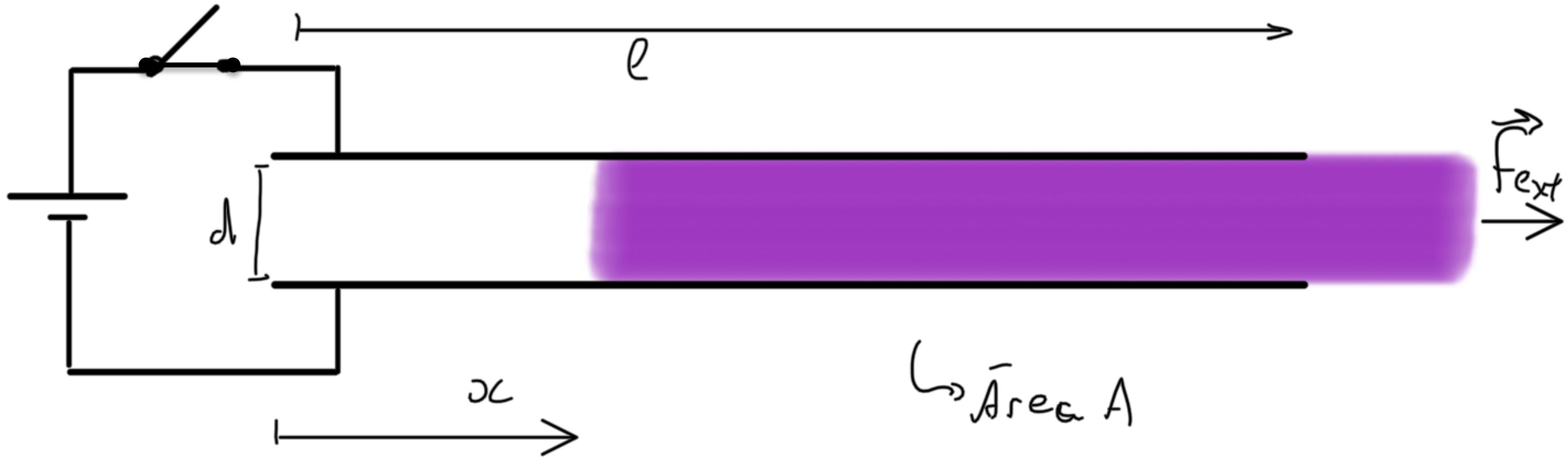
$$\mathcal{E}_f = \frac{Q \cdot V}{2} = \frac{\varepsilon}{\varepsilon_0} \frac{Q V_0}{2} = \frac{\varepsilon}{\varepsilon_0} \mathcal{E}_i$$

$$\frac{\mathcal{E}_f - \mathcal{E}_i}{\mathcal{E}_i} = \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon_0} \right)$$

Bateria ligada $\quad V = cte$

Trabalho realizado: $W_V = \int_0^{\ell} \frac{V_0^2}{2}\frac{A}{d\cdot\ell}(\varepsilon-\varepsilon_0)\,dx = \frac{V_0^2 A}{2d}(\varepsilon-\varepsilon_0)$

Energia final armazenada: $\mathcal{E}_{fV} = \frac{Q_f\cdot V}{2}$



Energia final armazenada: $\mathcal{E}_{fV} = \frac{Q_f \cdot V}{2}$

Como: $Q(x) = C(x) \cdot V \Rightarrow \frac{Q_0}{C_0} = \frac{Q(\ell)}{C\ell}$

$$Q_f = Q_0 \cdot \frac{C_f}{C_0} = Q_0 \frac{d}{\varepsilon A} \cdot \frac{\varepsilon_0 A}{d} = Q_0 \cdot \frac{\varepsilon_0}{\varepsilon}$$

$$\mathcal{E}_{fV} = \frac{Q_0}{2} \frac{\varepsilon_0}{\varepsilon} \cdot V_0 = \frac{\varepsilon_0}{\varepsilon} \mathcal{E}_{iV}$$

$$\mathcal{E}_{fV} - \mathcal{E}_{iV} = \left(\frac{\varepsilon_0 - \varepsilon}{\varepsilon}\right) \mathcal{E}_{iV} < 0$$

$$W_V = \frac{V_0}{2} \cdot \underbrace{V_0 \left(\frac{A\varepsilon}{d}\right)}_{Q_0} \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon}\right) = \mathcal{E}_{iV} \cdot \frac{\varepsilon - \varepsilon_0}{\varepsilon}$$

Comparando: $W_Q = \frac{V_0^2 A}{2d} \cdot (\varepsilon - \varepsilon_0) \cdot \frac{\varepsilon}{\varepsilon_0} = W_V \cdot \frac{\varepsilon}{\varepsilon_0} > W_V$

Energia: $(\mathcal{E}_{fQ} - \mathcal{E}_{iQ}) - W_Q = \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon_0}\right) \mathcal{E}_{iQ} - \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon_0}\right) \mathcal{E}_{iQ} = 0$

$$(\mathcal{E}_{fV} - \mathcal{E}_{iV}) - W_V = \underbrace{-2\,W_V}_{\text{Para onde foi esta energia?}}$$