PGF5003: Classical Electrodynamics I

Problem Set 5

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(Due to June 22, 2021)

Guidelines: *write down the most relevant passages in your calculations, not only the final results. Do not forget to write the mathematical expressions that you are using in order to solve the questions. We strongly recommended the use of the International System of Units.*

1 Question (1 point)

Use the electric stress tensor formalism to prove that no isolated charge distribution $\rho(r)$ can exert a net force on itself. Distinguish the cases when $\rho(r)$ has a net charge and when it does not.

1.1 Solution

Being $\mathbf{E}(\mathbf{r})$ the electric field produced by this charge density $\rho(r)$, the force exerted on itself ($\rho(r)$) can be computed as

$$\mathbf{F} = \int d^3r\rho(\mathbf{r})\mathbf{E}(\mathbf{r}). \tag{1}$$

Now, converting this expression to the formalism of the electric stress tensor, we have the components of the force given by

$$F_j = \int d^3r\partial_i T_{ij}(\mathbf{E}) = \int ds\hat{n}_i T_{ij}(\mathbf{E}) \tag{2}$$

$$T_{ij}(\mathbf{E}) = \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij}E^2 \right), \tag{3}$$

in the way that the net force on the charge inside the closed surface s is

$$\mathbf{F} = \int ds\hat{n} \cdot \mathbf{T}(\mathbf{E}) \tag{4}$$

$$= \epsilon_0 \int ds \left[(\hat{n} \cdot \mathbf{E})\, \mathbf{E} - \frac{1}{2} (\mathbf{E} \cdot \mathbf{E})\, \hat{n} \right] = 0, \tag{5}$$

for any $\rho(r)$:

- **There is net charge:** implies that the distribution behaves like punctual charges, i.e., $E \propto 1/r^2 \Rightarrow dsE^2 \propto 1/r^2$. In the limit where $r \to \infty$, the integral is zero!
- **There is NO net charge:** means that the field goes to zero;

Therefore, no charge distribution is able to produce a net force on itself.

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2 Question (1 point)

The charge and current densities for a single point charge q can be written formally as

$$\rho(\mathbf{x}', t') = q\delta\left[\mathbf{x}' - \mathbf{r}(t')\right] \tag{6}$$

$$\mathbf{J}(\mathbf{x}', t') = q\mathbf{v}(t')\delta\left[\mathbf{x}' - \mathbf{r}(t')\right] \tag{7}$$

where $\mathbf{r}(t')$ is the charge's position at time t' and $\mathbf{v}(t')$ is its velocity. In evaluating expressions involving the retarded time, one must put $t' = t - R(t')/c$, where $\mathbf{R} = \mathbf{x} - \mathbf{r}(t')$ (but $\mathbf{R} = \mathbf{x} - \mathbf{x}'(t')$ inside the delta functions).

a) *As a preliminary to deriving the Heaviside-Feynman expression for the electric and magnetic fields of a point charge, show that*

$$\int d^3x'\delta\left[\mathbf{x}' - \mathbf{r}(t_{ret})\right] = \frac{1}{\kappa}, \tag{8}$$

where $\kappa = 1 - \mathbf{v} \cdot \hat{R}/c$. Note that κ is evaluated at the retarded time.

b) *Starting with the Jefimenko generalizations of the Coulomb and Biot-Savart laws, use the expressions for the charge and current densities for a point charge and the result of part a to obtain the*

Heaviside-Feynman expressions for the electric and magnetic fields of a point charge,

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left\{\left[\frac{\hat{R}}{\kappa R^2}\right]_{ret} + \frac{\partial}{c\partial t}\left[\frac{\hat{R}}{\kappa R}\right]_{ret} - \frac{\partial}{c^2\partial t}\left[\frac{\mathbf{v}}{\kappa R}\right]_{ret}\right\} \tag{9}$$

$$\mathbf{B} = \frac{\mu_0 q}{4\pi}\left\{\left[\frac{\mathbf{v}\times\hat{R}}{\kappa R^2}\right]_{ret} + \frac{\partial}{c\partial t}\left[\frac{\mathbf{v}\times\hat{R}}{\kappa R}\right]_{ret}\right\}. \tag{10}$$

2.1 Solution

a) To show the Heaviside-Feynman expression we could use the identity

$$\delta[f(x)] = \frac{\delta(x-x_0)}{\partial f/\partial x|_{x=x_0}}, \tag{11}$$

where x_0 is a root of $f(x)$. Here we need to set $f(\mathbf{x}') = [\mathbf{x}' - \mathbf{r}(t)]$, considering the root $\mathbf{x}_0 = \mathbf{r}(t_{ret})$ in the way that

$$\delta\left[\mathbf{x}' - \mathbf{r}(t_{ret})\right] = \delta\left[f(\mathbf{x}')\right] = \delta\left(\mathbf{x}' - \mathbf{x}_0\right)\left\{\frac{\partial}{\partial x'}\left[\mathbf{x}' - \mathbf{r}(t_{ret})\right]\right\}^{-1}_{\mathbf{x}'=\mathbf{x}_0} \tag{12}$$

$$= \delta\left(\mathbf{x}' - \mathbf{x}_0\right)\left\{1 - \frac{\partial \mathbf{r}(t_{ret})}{\partial x'}\right\}^{-1}_{\mathbf{x}'=\mathbf{x}_0} = \delta\left(\mathbf{x}' - \mathbf{x}_0\right)\left\{1 - \frac{\partial \mathbf{r}}{\partial t}\frac{\partial t}{\partial x'}\right\}^{-1}_{\mathbf{x}'=\mathbf{x}_0} \tag{13}$$

$$= \delta\left(\mathbf{x}' - \mathbf{x}_0\right)\left\{1 - \frac{\partial \mathbf{r}}{\partial t}\frac{\partial}{\partial x'}\left[t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right]\right\}^{-1}_{\mathbf{x}'=\mathbf{x}_0} \tag{14}$$

$$= \delta\left(\mathbf{x}' - \mathbf{x}_0\right)\left\{1 - \frac{\partial \mathbf{r}}{\partial t}\left[\frac{\mathbf{x}-\mathbf{x}'}{c|\mathbf{x}-\mathbf{x}'|}\right]\right\}^{-1}_{\mathbf{x}'=\mathbf{x}_0} \tag{15}$$

$$= \delta\left(\mathbf{x}' - \mathbf{x}_0\right)\left\{1 - \frac{\partial \mathbf{r}}{\partial t}\left[\frac{\mathbf{R}}{c|\mathbf{R}|}\right]\right\}^{-1}_{\mathbf{x}'=\mathbf{x}_0} = \delta\left(\mathbf{x}' - \mathbf{x}_0\right)\left[1 - \frac{\mathbf{v}\cdot\hat{R}}{c}\right]^{-1} \tag{16}$$

$$= \frac{\delta\left(\mathbf{x}' - \mathbf{x}_0\right)}{\left[1 - \frac{\mathbf{v}\cdot\hat{R}}{c}\right]} = \frac{\delta\left(\mathbf{x}' - \mathbf{x}_0\right)}{\kappa}. \tag{17}$$

Therefore, that is why

$$\int d^3x'\delta\left[\mathbf{x}' - \mathbf{r}(t_{ret})\right] = \int d^3x'\frac{\delta\left(\mathbf{x}' - \mathbf{x}_0\right)}{\kappa} = \frac{1}{\kappa}\square \tag{18}$$

b) The Jefimenko generalizations for the fields $\mathbf{E}$ and $\mathbf{B}$ are written as

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int d^3x'\left\{\frac{\hat{R}}{R^2}\left[\rho\left(\mathbf{x}', t'\right)\right]_{ret} + \frac{\hat{R}}{cR}\left[\frac{\partial\rho(\mathbf{x}', t')}{\partial t'}\right]_{ret} - \frac{1}{c^2 R}\left[\frac{\partial\mathbf{J}(\mathbf{x}', t')}{\partial t'}\right]_{ret}\right\}, \tag{19}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int d^3x'\left\{\left[\mathbf{J}\left(\mathbf{x}', t'\right)\right]_{ret}\times\frac{\hat{R}}{R^2} + \left[\frac{\partial\mathbf{J}(\mathbf{x}', t')}{\partial t'}\right]_{ret}\times\frac{\hat{R}}{cR}\right\}. \tag{20}$$

To compute the fields for a punctual electrical charge let's define:

$$[\rho(\mathbf{x}',t')]_{ret} = q\delta\left[\mathbf{x}' - \mathbf{r}(t')\right]_{ret}, \tag{21}$$

$$[\mathbf{J}(\mathbf{x}',t')]_{ret} = q\mathbf{v}(t')\delta\left[\mathbf{x}' - \mathbf{r}(t')\right]_{ret}. \tag{22}$$

Then, computing $\mathbf{E}$ we have:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int d^3x'\left\{\frac{\hat{R}}{R^2}[\rho(\mathbf{x}',t')]_{ret} + \frac{\hat{R}}{cR}\left[\frac{\partial\rho(\mathbf{x}',t')}{\partial t'}\right]_{ret} - \frac{1}{c^2R}\left[\frac{\partial\mathbf{J}(\mathbf{x}',t')}{\partial t'}\right]_{ret}\right\} \tag{23}$$

$$\begin{aligned}
&= \frac{1}{4\pi\epsilon_0}\int d^3x'\left\{\frac{\hat{R}}{R^2}q\delta\left[\mathbf{x}' - \mathbf{r}(t')\right]_{ret} + \frac{\hat{R}}{cR}\left[\frac{\partial q\delta\left[\mathbf{x}' - \mathbf{r}(t')\right]}{\partial t'}\right]_{ret} - \frac{1}{c^2R}\left[\frac{\partial q\mathbf{v}(t')\delta\left[\mathbf{x}' - \mathbf{r}(t')\right]}{\partial t'}\right]_{ret}\right\} \\
&= \frac{q}{4\pi\epsilon_0}\left\{\int d^3x'\left[\frac{\hat{R}}{R^2}\delta\left[\mathbf{x}' - \mathbf{r}(t')\right]\right]_{ret} + \frac{\partial}{\partial t}\int d^3x'\left[\frac{\hat{R}}{cR}\delta\left[\mathbf{x}' - \mathbf{r}(t')\right]\right]_{ret}\right. \\
&\qquad\left. - \frac{\partial}{\partial t}\int d^3x'\left[\frac{1}{c^2R}\mathbf{v}(t')\delta\left[\mathbf{x}' - \mathbf{r}(t')\right]\right]_{ret}\right\}
\end{aligned} \tag{24}$$

$$\begin{aligned}
&= \frac{q}{4\pi\epsilon_0}\left\{\left[\frac{\hat{R}}{R^2\kappa}\right]_{ret} + \frac{\partial}{\partial t}\left[\frac{\hat{R}}{cR\kappa}\right]_{ret} - \frac{\partial}{\partial t}\left[\frac{\mathbf{v}(t')}{c^2R\kappa}\right]_{ret}\right\} \\
&= \frac{q}{4\pi\epsilon_0}\left\{\left[\frac{\hat{R}}{\kappa R^2}\right]_{ret} + \frac{\partial}{c\partial t}\left[\frac{\hat{R}}{\kappa R}\right]_{ret} - \frac{\partial}{c^2\partial t}\left[\frac{\mathbf{v}}{\kappa R}\right]_{ret}\right\}\square
\end{aligned} \tag{25}$$

And the same follows for $\mathbf{B}$:

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int d^3x'\left\{[\mathbf{J}(\mathbf{x}',t')]_{ret} \times \frac{\hat{R}}{R^2} + \left[\frac{\partial\mathbf{J}(\mathbf{x}',t')}{\partial t'}\right]_{ret} \times \frac{\hat{R}}{cR}\right\} \tag{26}$$

$$= \frac{\mu_0}{4\pi}\int d^3x'\left\{q\mathbf{v}(t')\delta\left[\mathbf{x}' - \mathbf{r}(t')\right]_{ret} \times \frac{\hat{R}}{R^2} + \left[\frac{\partial q\mathbf{v}(t')\delta\left[\mathbf{x}' - \mathbf{r}(t')\right]}{\partial t'}\right]_{ret} \times \frac{\hat{R}}{cR}\right\} \tag{27}$$

$$= \frac{\mu_0 q}{4\pi}\left\{\int d^3x'\left[\mathbf{v}(t') \times \frac{\hat{R}}{R^2}\delta\left[\mathbf{x}' - \mathbf{r}(t')\right]\right]_{ret} + \frac{\partial}{\partial t}\int d^3x'\left[\mathbf{v}(t') \times \frac{\hat{R}}{cR}\delta\left[\mathbf{x}' - \mathbf{r}(t')\right]\right]_{ret}\right\} \tag{28}$$

$$= \frac{\mu_0 q}{4\pi}\left\{\left[\mathbf{v}(t') \times \frac{\hat{R}}{R^2\kappa}\right]_{ret} + \frac{\partial}{\partial t}\left[\mathbf{v}(t') \times \frac{\hat{R}}{cR\kappa}\right]_{ret}\right\} \tag{29}$$

$$= \frac{\mu_0 q}{4\pi}\left\{\left[\frac{\mathbf{v} \times \hat{R}}{\kappa R^2}\right]_{ret} + \frac{\partial}{c\partial t}\left[\frac{\mathbf{v} \times \hat{R}}{\kappa R}\right]_{ret}\right\}\square \tag{30}$$

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3 Question (1 point)

a) *When the current density* $\mathbf{J}$ *is independent of the time, the charge density is given by*

$$\rho(\mathbf{r},t) = \rho(\mathbf{r},0) + \dot{\rho}(\mathbf{r},0)t. \tag{31}$$

In this case, show that the electric field is given by

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3r'\rho(\mathbf{r},t)\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{32}$$

where ρ is computed at the time t and not in the retarded time t_{ret}, which is identical to the electrostatic situation. Could you give an example where this situation happens?

b) *Show that the Biot-Savart law*

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int d^3r\frac{\mathbf{J}(\mathbf{r}',t)\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{33}$$

is still valid even in the case the the density current $\mathbf{J}$ changes with time, being the time variation sufficiently small to use the first approximation order

$$\mathbf{J}(t_{ret}) = \mathbf{J}(t) + (t_{ret}-t)\dot{\mathbf{J}}(t) \tag{34}$$

what leads to this quantity be calculated in the time t and not in the retarded one.

3.1 Solution

a) To compute the electrical field we can start from the Jefimenko equation for it

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int d^3r'\left\{\frac{\hat{R}}{R^2}\left[\rho\left(\mathbf{r}',t'\right)\right]_{ret} + \frac{\hat{R}}{cR}\left[\frac{\partial\rho(\mathbf{r}',t')}{\partial t'}\right]_{ret} - \frac{1}{c^2R}\left[\frac{\partial\mathbf{J}(\mathbf{r}',t')}{\partial t'}\right]_{ret}\right\}. \tag{35}$$

The quantities are then

$$\frac{\partial\mathbf{J}(\mathbf{r}',t')}{\partial t'} = 0, \tag{36}$$
$$\frac{\partial\rho(\mathbf{r}',t')}{\partial t'} = \dot{\rho}(\mathbf{r},0) \tag{37}$$

and

$$t_{ret} = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}. \tag{38}$$

Putting it on the field

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3r'\left\{\frac{\hat{R}}{R^2}\left[\rho\left(\mathbf{r}',t'\right)\right]_{ret} + \frac{\hat{R}}{cR}\left[\frac{\partial\rho(\mathbf{r}',t')}{\partial t'}\right]_{ret} - \frac{1}{c^2R}\left[\frac{\partial\mathbf{J}(\mathbf{r}',t')}{\partial t'}\right]_{ret}\right\} \tag{39}$$

$$= \frac{1}{4\pi\epsilon_0}\int d^3r'\left\{\frac{\hat{R}}{R^2}\left[\rho(\mathbf{r},0)+\dot{\rho}(\mathbf{r},0)t\right]_{ret} + \frac{\hat{R}}{cR}\left[\dot{\rho}(\mathbf{r},0)\right]_{ret}\right\} \tag{40}$$

$$= \frac{1}{4\pi\epsilon_0}\int d^3r'\left\{\frac{\hat{R}}{R^2}\left[\rho(\mathbf{r},0)+\dot{\rho}(\mathbf{r},0)\left(t-\frac{R}{c}\right)\right] + \frac{\hat{R}}{cR}\dot{\rho}(\mathbf{r},0)\right\} \tag{41}$$

$$= \frac{1}{4\pi\epsilon_0}\left[\int d^3r'\frac{\hat{R}}{R^2}\rho(\mathbf{r},0) + \int d^3r'\frac{\hat{R}}{R^2}\dot{\rho}(\mathbf{r},0)t - \cancel{\int d^3r'\frac{\hat{R}}{cR}\dot{\rho}(\mathbf{r},0)} + \cancel{\int d^3r'\frac{\hat{R}}{cR}\dot{\rho}(\mathbf{r},0)}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\int d^3r'\frac{\hat{R}}{R^2}\left[\rho(\mathbf{r},0)+\dot{\rho}(\mathbf{r},0)t\right] = \frac{1}{4\pi\epsilon_0}\int d^3r'\frac{\hat{R}}{R^2}\rho(\mathbf{r},t) \tag{42}$$

$$= \frac{1}{4\pi\epsilon_0}\int d^3r'\rho(\mathbf{r},t)\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\square \tag{43}$$

This behavior occurs, for example, in the charge of a capacitor.

b) The Jefimenko equation for the magnetic field is

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int d^3x'\left\{\left[\mathbf{J}\left(\mathbf{r}',t'\right)\right]_{ret}\times\frac{\hat{R}}{R^2} + \left[\frac{\partial\mathbf{J}(\mathbf{r}',t')}{\partial t'}\right]_{ret}\times\frac{\hat{R}}{cR}\right\}. \tag{44}$$

As we have $\mathbf{J}$ we need

$$\frac{\partial\mathbf{J}(t_{ret})}{\partial t} = \frac{\partial\mathbf{J}(t_{ret})}{\partial t_{ret}}\frac{\partial t_{ret}}{\partial t} = \frac{\partial\mathbf{J}(t_{ret})}{\partial t_{ret}} \tag{45}$$

$$t_{ret} - t = -\frac{R}{c}. \tag{46}$$

Then, we got

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int d^3x'\left\{\left[\mathbf{J}\left(\mathbf{r}',t'\right)\right]_{ret}\times\frac{\hat{R}}{R^2} + \left[\frac{\partial\mathbf{J}(\mathbf{r}',t')}{\partial t'}\right]_{ret}\times\frac{\hat{R}}{cR}\right\} \tag{47}$$

$$= \frac{\mu_0}{4\pi}\int d^3x'\left\{\left[\mathbf{J}(\mathbf{r}',t)+(t_{ret}-t)\dot{\mathbf{J}}(\mathbf{r}',t)\right]\times\frac{\hat{R}}{R^2} + \dot{\mathbf{J}}(\mathbf{r}',t)\times\frac{\hat{R}}{cR}\right\} \tag{48}$$

$$= \frac{\mu_0}{4\pi}\left\{\int d^3x'\frac{\mathbf{J}(\mathbf{r}',t)\times\hat{R}}{R^2} - \cancel{\int d^3x'\frac{\dot{\mathbf{J}}(\mathbf{r}',t)\times\hat{R}}{cR}} + \cancel{\int d^3x'\frac{\dot{\mathbf{J}}(\mathbf{r}',t)\times\hat{R}}{cR}}\right\} \tag{49}$$

$$= \frac{\mu_0}{4\pi}\int d^3r\frac{\mathbf{J}(\mathbf{r}',t)\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\square \tag{50}$$

This is exactly the Biot-Savart law, valid to magneto-statics. This shows that, up to the first order in the approximation the magneto-statics approach is very good.

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4 Question (1.5 point)

Two halves of a spherical metallic shell of radius R and infinite conductivity are separated by a very small insulating gap. An alternating potentials is applied between the two halves of the sphere, so that the potentials are $\pm V\cos(\omega t)$. Find:

a) *the electrical potential (inside and outside the sphere) when the voltages are in their peak. It is just other way to ask you to solve the static version of this problem using Legendre polynomials;*

b) *the momentum of dipole (extending the previous result for the time dependent potential and doing the comparison with the potential due to an electric dipole);*

c) *the first and second derivative in time of the moment of dipole;*

d) *the potential vector $\mathbf{A}$, the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ using dipole approximation;*
Hint: *you can use:*

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\dot{\mathbf{p}}(t_{ret})}{r}, \;\; \mathbf{B} = \vec{\nabla}\times\mathbf{A} = \frac{\mu_0}{4\pi c}\frac{\ddot{\mathbf{p}}\times\hat{r}}{r} \text{ and } \mathbf{E} = c\mathbf{B}\times\hat{r}. \tag{51}$$

e) *Find the Poynting vector and the radiated power from the sphere.*

4.1 Solution

a) When the voltages are in their peaks we have the analogous problem of a sphere with $\pm V$ on it surface in the static version. Then, we solve

$$\nabla^2\Phi = 0, \tag{52}$$

which, due to azimuthal and spherical symmetry has the general solution given by

$$\Phi(r,\theta) = \begin{cases} \sum_\ell A_\ell r^\ell P_\ell(\cos\theta), & r < R \\ \sum_\ell \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta), & r > R \end{cases}. \tag{53}$$

Using the continuity of the potential

$$\Phi_{r<}(R) = \Phi_{r>}(R) \tag{54}$$

$$\sum_\ell A_\ell R^\ell P_\ell(\cos\theta) = \sum_\ell \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta) \tag{55}$$

$$A_\ell R^\ell = \frac{B_\ell}{R^{\ell+1}}. \tag{56}$$

Using Dirichlet condition

$$\Phi(R,\theta) = \begin{cases} V, & \theta \in [0,\pi/2] \\ -V, & \theta \in [\pi/2,\pi] \end{cases}, \tag{57}$$

for the first we have

$$\Phi_{r<}(R) = \sum_\ell A_\ell r^\ell P_\ell(\cos\theta) \tag{58}$$

$$\int_0^\pi d\theta \sin\theta P_{\ell'}(\cos\theta)\Phi_{r<}(R) = \sum_\ell \int_0^\pi d\theta \sin\theta P_{\ell'}(\cos\theta) A_\ell R^\ell P_\ell(\cos\theta) \tag{59}$$

and for the second

$$\Phi_{r>}(R) = \sum_\ell \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \tag{60}$$

$$-\int_0^\pi d\theta \sin\theta P_{\ell'}(\cos\theta)\Phi_{r>}(R) = \sum_\ell \int_0^\pi d\theta \sin\theta P_{\ell'}(\cos\theta) \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta). \tag{61}$$

Using the property

$$\int_0^\pi d\theta \sin\theta P_\ell(\cos\theta) P_{\ell'}(\cos\theta) = \frac{2\delta_{\ell,\ell'}}{(2\ell+1)}, \tag{62}$$

the values for the potential on the sphere surface and changing variables $x \to \cos\theta$ we get

$$A_\ell = \frac{(2\ell+1)V}{2R^\ell}\left[\int_0^1 dx P_\ell(x) - \int_{-1}^0 dx P_\ell(x)\right] \tag{63}$$

$$= \frac{(2\ell+1)V}{2R^\ell}\left[\int_0^1 dx P_\ell(x) - (-1)^\ell \int_0^1 dx P_\ell(x)\right] \tag{64}$$

$$= \frac{(2\ell+1)V}{R^\ell}\left[\int_0^1 dx P_\ell(x)\right], \text{for odd } \ell, \tag{65}$$

$$B_\ell = \frac{(2\ell+1)V}{2} R^{\ell+1}\left[\int_0^1 dx P_\ell(x) - \int_{-1}^0 dx P_\ell(x)\right] \tag{66}$$

$$= (2\ell+1)V R^{\ell+1}\left[\int_0^1 dx P_\ell(x)\right], \text{for odd } \ell. \tag{67}$$

Otherwise, even ℓ, the above expressions are zero. Noticing here that

$$\int_0^1 dx P_\ell(x) = \frac{\left[P_{(\ell-1)}(0) - P_{(\ell+1)}(0)\right]}{(2\ell+1)}, \text{for } \ell \geq 1, \tag{68}$$

and keeping only the terms with $\ell = 1$

$$A_\ell = \frac{V}{R^\ell}\left[P_{(\ell-1)}(0) - P_{(\ell+1)}(0)\right] \Rightarrow A_1 = \frac{V}{R}\left[1 - \frac{(3x^2-1)}{2}\right]_{x=0} = \frac{3V}{2R}, \tag{69}$$

$$B_\ell = VR^{\ell+1}\left[P_{(\ell-1)}(0) - P_{(\ell+1)}(0)\right] \Rightarrow B_1 = VR^2\left[1 - \frac{(3x^2-1)}{2}\right]_{x=0} = \frac{3VR^2}{2}. \tag{70}$$

in the way that the potential stay as

$$\Phi(r,\theta) = \begin{cases} \frac{3V}{2R} r P_1(\cos\theta) = \frac{3V}{2R} r\cos\theta, & r \leq R \\ \frac{3}{2}V\frac{R^2}{r^2}P_1(\cos\theta) = \frac{3}{2}V\frac{R^2}{r^2}\cos\theta, & r \geq R \end{cases}. \tag{71}$$

b) We know that an electric dipole, pointing in the z direction should look like

$$\Phi = \frac{1}{4\pi\epsilon_0}\frac{p}{r^2}\cos\theta. \tag{72}$$

Thus, when it is equal to the previous potential (for $r > R$), taking the case where $V \to V\cos(\omega t)$, we have

$$\frac{1}{4\pi\epsilon_0}\frac{p}{r^2}\cos\theta = \frac{3}{2}V\cos(\omega t)\frac{R^2}{r^2}P_1(\cos\theta) \tag{73}$$

$$\frac{1}{4\pi\epsilon_0}\frac{p}{r^2}\cos\theta = \frac{3}{2}V\cos(\omega t)\frac{R^2}{r^2}\cos\theta \tag{74}$$

$$\mathbf{p} = 6\pi\epsilon_0 R^2 V\cos(\omega t)\hat{z}. \tag{75}$$

c) Doing the derivative, we have

$$\dot{\mathbf{p}} = -\omega 6\pi\epsilon_0 R^2 V\sin(\omega t)\hat{z} \tag{76}$$

$$\ddot{\mathbf{p}} = -\omega^2 6\pi\epsilon_0 R^2 V\cos(\omega t)\hat{z} \tag{77}$$

d) Using the hints, we have

$$\begin{aligned}
\mathbf{A} &= \frac{\mu_0}{4\pi}\frac{\dot{\mathbf{p}}(t_{ret})}{r} = \frac{\mu_0}{4\pi}\frac{-\omega 6\pi\epsilon_0 R^2 V \sin(\omega t_{ret})\hat{z}}{r} = -\frac{3}{2}\frac{\omega\mu_0\epsilon_0 R^2 V \sin(\omega t - kR)}{r}\hat{z} && (78)\\
&= -\frac{3}{2}\frac{\omega\mu_0\epsilon_0 R^2 V \sin(\omega t - kr)}{r}\hat{z} && (79)\\
\mathbf{B} &= \frac{\mu_0}{4\pi c}\frac{\ddot{\mathbf{p}} \times \hat{r}}{r} = \frac{\mu_0}{4\pi c}\frac{(-\omega^2 6\pi\epsilon_0 R^2 V \cos(\omega t_{ret})\hat{z}) \times \hat{r}}{r} && (80)\\
&= \frac{\mu_0}{4\pi c}\frac{\left[-\omega^2 6\pi\epsilon_0 R^2 V \cos(\omega t_{ret})(\cos\theta\hat{r} - \sin\theta\hat{\theta})\right] \times \hat{r}}{r} = -\frac{3\mu_0\epsilon_0}{2c}\frac{\omega^2 R^2 V \cos(\omega t_{ret})\sin\theta}{r}\hat{\phi} && (81)\\
&= -\frac{3}{2c}\frac{k^2 R^2 V \cos[\omega t - kR]\sin\theta}{r}\hat{\phi} = -\frac{3}{2c}\frac{k^2 R^2 V \cos[\omega t - kr]\sin\theta}{r}\hat{\phi} && (82)\\
\mathbf{E} &= c\mathbf{B} \times \hat{r} = cB\hat{\theta} && (83)\\
&= -\frac{3}{2c^2}\frac{\omega^2 R^2 V \cos(\omega t_{ret})\sin\theta}{r}\hat{\theta} = -\frac{3}{2}\frac{k^2 R^2 V \cos[\omega t - kr]\sin\theta}{r}\hat{\theta}, && (84)
\end{aligned}$$

where I have used $\omega = kc$, $c^2 = 1/(\mu_0\epsilon_0)$, $R = r - r'$ and the approach that $r \gg r'$.

e) The Poyinting vector is as follows

$$\begin{aligned}
\mathbf{S} &= \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} = \frac{EB}{\mu_0}\hat{\theta} \times \hat{\phi} = \frac{EB}{\mu_0}\hat{r} && (85)\\
&= \frac{1}{\mu_0}\left(\frac{3}{2}\frac{k^2 R^2 V \cos(\omega t - kr)\sin\theta}{r}\right)\left(\frac{3}{2c}\frac{k^2 R^2 V \cos(\omega t - kr)\sin\theta}{r}\right)\hat{r} && (86)\\
&= \frac{9k^4 R^4 V^2 \cos^2(\omega t - kr)\sin^2\theta}{4c\mu_0 r^2}\hat{r}. && (87)
\end{aligned}$$

Therefore, the radiated power is

$$\begin{aligned}
P &= \oint_S d\mathbf{s} \cdot \mathbf{S} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \cancel{r^2}\frac{9k^4 R^4 V^2 \cos^2(\omega t - kr)\sin^2\theta}{4c\mu_0 \cancel{r^2}} && (88)\\
&= 2\pi\frac{\cancel{4}}{\cancel{3}}\frac{\cancel{9}k^4 R^4 V^2 \cos^2(\omega t)}{\cancel{4}c\mu_0} = \frac{6\pi k^4 R^4 V^2 \cos^2(\omega t - kr)}{c\mu_0}. && (89)
\end{aligned}$$

And the time-averaged gives $1/2$, the time-averaged power radiatedis then

$$P = \frac{3\pi k^4 R^4 V^2}{c\mu_0}. \qquad (90)$$

---

5 Question (1 point)

A spherical shell of radius R uniformly charged has sinusoidal oscillations purely in the radial axis.

What is the radiated power?

5.1 Solution

Here we have a spherical shell uniformly charged that has oscillations in the radial axis. This means that the current density could be written as

$$\mathbf{J}(\mathbf{r}', t') = J_0(\mathbf{r}', t')\hat{r}. \tag{91}$$

Using this current, the potential vector follow as

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} e^{-i\omega t} \int d^3r' \frac{\mathbf{J}(\mathbf{r}') e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \propto A_0(\mathbf{r}, t)\hat{r}. \tag{92}$$

Therefore, as the magnetic field is $\mathbf{B} = \vec{\nabla} \times \mathbf{A} = 0$, the emitted radiation, which is $P \propto \mathbf{S} \propto \mathbf{B} = 0$. So, there is no emission of radiation in this case!

---

6 Question (1.5 point)

A thin linear antenna of length d is excited in such a way that the sinusoidal current makes a full wavelength of oscillation as shown in the Figure.

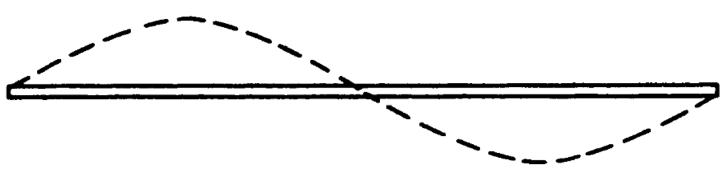


Figure 1: Figure for the exercise.

a) *Calculate exactly the power radiated per unit of solid angle.* **Hint:** *compute the potential vector, find the electrical and magnetic field, compute the Poyinting vector and then, the radiated power.*

b) *Plot the angular distribution of radiation from item (a). Show your code to me!*

c) *Determine the total power radiated and find a numerical value for the radiation resistance.* **Hint:** *you may need the following result: $I = \int_0^\pi dx \frac{\sin^2(\pi \cos x)}{\sin x} = 1.55718$.*

6.1 Solution

a) To compute the radiated power, first we need to write the current density of it. Choosing the z-axis along the antenna and, for simplicity, let $z = 0$ be the center of it, we can write

$$\mathbf{J}(\mathbf{r},t) = I_0 \sin(kz)\, e^{-i\omega t}\delta(x)\delta(y)\hat{z} = I_0 \sin\left(\frac{2\pi z}{d}\right) e^{-i\omega t}\delta(x)\delta(y)\hat{z}. \tag{93}$$

The potential vector is

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int d^3r' \int dt' \frac{\mathbf{J}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t' + \frac{|\mathbf{r}-\mathbf{r}'|}{c} - t\right). \tag{94}$$

The sinusoidal time dependence leads to

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}e^{-i\omega t}\int d^3r' \frac{\mathbf{J}(\mathbf{r}')e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tag{95}$$

and the *radiation zone* approximates $\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \simeq \frac{e^{ikx}}{x}e^{-ikz'\cos\theta}$ in the way that we get

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}e^{-i\omega t}\int d^3r' \frac{\mathbf{J}(\mathbf{r}')e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tag{96}$$

$$= \frac{\mu_0}{4\pi}e^{-i\omega t}\frac{e^{ikr}}{r}\int_{-\infty}^{-\infty} dx' \int_{-\infty}^{-\infty} dy' \int_{-d/2}^{d/2} dz' I_0 \sin(kz')\,\delta(x')\delta(y')e^{-ikz'\cos\theta}\hat{z} \tag{97}$$

$$= \frac{\mu_0}{4\pi}e^{-i\omega t}I_0\frac{e^{ikr}}{r}\int_{-d/2}^{d/2} dz' \sin(kz')\, e^{-ikz'\cos\theta}\hat{z}. \tag{98}$$

The integral can be written as

$$I = \int_{-d/2}^{d/2} dz' \sin(kz')e^{-ikz'} = \left\{-\frac{\csc^2\theta e^{-ikz\cos\theta}\left[\cos(kz) + i\cos\theta\sin(kz)\right]}{k}\right\}_{-d/2}^{d/2} \tag{99}$$

$$= \frac{\csc^2\theta}{k}2i\sin\left(\frac{k}{2}d\cos\theta\right)\cos\left(\frac{kd}{2}\right) = \frac{-id\sin(\pi\cos\theta)}{\pi\sin^2\theta}, \tag{100}$$

where I have used that $k = 2\pi/d$.

Then, coming back to the potential vector, we have

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}e^{-i\omega t}I_0\frac{e^{ikr}}{r}\int_{-d/2}^{d/2} dz' \sin(kz')\, e^{-ikz'\cos\theta}\hat{z} \tag{101}$$

$$= -\frac{id\mu_0}{4\pi^2}e^{-i\omega t}I_0\frac{e^{ikr}}{r}\frac{\sin(\pi\cos\theta)}{\sin^2\theta}\hat{z}. \tag{102}$$

Finally, we can compute the fields

$$\mathbf{B} = \vec{\nabla} \times \mathbf{A} = ik\hat{r} \times \mathbf{A} = -ikA\sin\theta\hat{\phi} = -\frac{kd\mu_0}{4\pi^2} e^{-i\omega t} I_0 \frac{e^{ikr}}{r} \frac{\sin(\pi\cos\theta)}{\sin\theta}\hat{\phi} \tag{103}$$

$$\mathbf{E} = c\mathbf{B} \times \hat{r} = cB\hat{\theta} = -\frac{kd\mu_0 c}{4\pi^2} e^{-i\omega t} I_0 \frac{e^{ikr}}{r} \frac{\sin(\pi\cos\theta)}{\sin\theta}\hat{\theta}. \tag{104}$$

Then, the Poyinting vector becomes

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}^* = \frac{EB}{\mu_0}\hat{r} \tag{105}$$

$$= \frac{k^2 d^2 \mu_0 c}{4^2\pi^4} \frac{I_0^2}{r^2} \frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}\hat{r}. \tag{106}$$

The radiated power per unit solid angle (not taking the mean over time) is

$$\frac{dP}{d\Omega} = r^2 Re[|\mathbf{S}|] = \frac{k^2 d^2 \mu_0 c}{2^4\pi^4} I_0^2 \frac{\sin^2(\pi\cos\theta)}{\sin^2\theta} \tag{107}$$

$$= \frac{\mu_0 c}{2^2\pi^2} I_0^2 \frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}. \tag{108}$$

b) To plot the radiated power per unit of solid angle I have plotted

$$f(\theta) = \frac{\sin^2(\pi\cos\theta)}{\sin^2\theta} \tag{109}$$

in polar coordinates in Python. See the figure bellow:

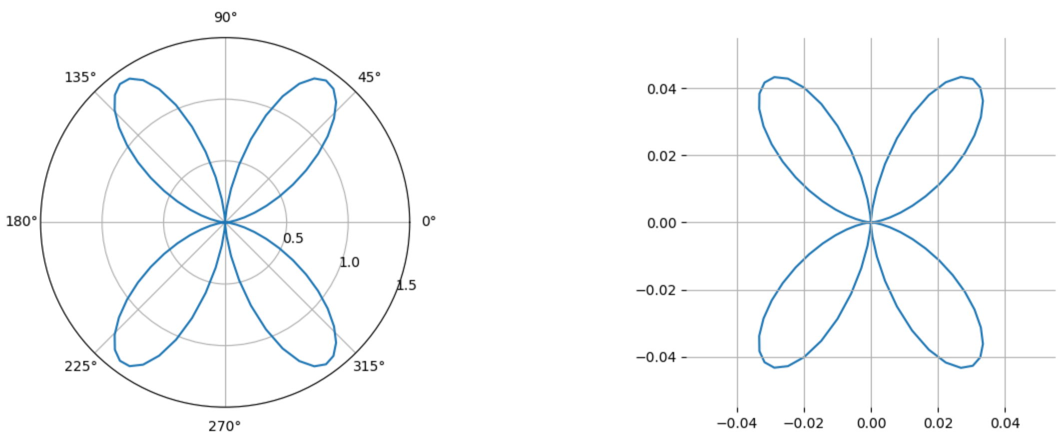


Figure 2: Radiated power (per solid angle) for the antenna: polar plot (on the left) and Cartesian plot (on the right).

And the code that I have used is as following:

```
import numpy as np
import matplotlib.pyplot as plt

#Function
theta = np.linspace(0.0001, 2.*np.pi - 0.0001, 100)
```

```
r = (np.sin(np.pi*np.cos(theta))/np.sin(theta))**2

#Plot in polar space
fig, ax = plt.subplots(subplot_kw={'projection': 'polar'})
ax.plot(theta, r)
ax.set_rmax(1.5)
ax.set_rticks([0.5, 1, 1.5])
ax.set_rlabel_position(-22.5)
ax.grid(True)
plt.savefig('polar-power_antenna.png')

#Plot in cartesian space
fig, ax = plt.subplots(subplot_kw={'projection': 'polar'})
plt.axis('off')
ax.plot(theta, r)
new_axis = fig.add_axes(ax.get_position(), frameon = False)
new_axis.plot()
new_axis.grid(True)
plt.savefig('cartesian-power_antenna.png')
```

c) The total power radiated (not time averaged) is given by

$$P = \int d\Omega \frac{dP}{d\Omega} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \frac{\mu_0 c}{2^2\pi^2} I_0^2 \frac{\sin^2(\pi\cos\theta)}{\sin^2\theta} \tag{110}$$

$$= \frac{\mu_0 I_0^2 c}{2\pi} \int_0^{\pi} d\theta \frac{\sin^2(\pi\cos\theta)}{\sin\theta} = \frac{\mu_0 c I_0^2}{2\pi} 1.55718. \tag{111}$$

The resistance could be computed using that $P = I^2 R$, then

$$R = \frac{P}{I_0^2} = \frac{\mu_0 c}{2\pi} 1.55718 \simeq 90\Omega. \tag{112}$$

---

7 Question (1 point)

Consider the electric dipole radiation fields. **Hint:** *see, e.g., Jackson, Ch. 9.2 and 9.3.*

a) *What are these dipole fields in terms of the Transverse Electric and Transverse Magnetic radiation fields?*

b) *Assume that an oscillating electric dipole is aligned along the z direction. Express the electric dipole radiation field in terms of the scalar and vector spherical harmonics.*

7.1 Solution

a) Here, we want to write the dipole radiation in terms of the transverse fields. The definition of the transverse fields comes from the solution of the Helmholtz equations written as

$$\left(\nabla^2 + k^2\right) \cdot \mathbf{E} = 0, \text{ with } \vec{\nabla} \times \mathbf{H} = -ik\mathbf{E}/Z_0, \tag{113}$$

$$\left(\nabla^2 + k^2\right) \cdot \mathbf{H} = 0, \text{ with } \mathbf{E} = \frac{iZ_0}{k}\vec{\nabla} \times \mathbf{H}, \tag{114}$$

in the way that they obey

$$\left(\nabla^2 + k^2\right)(\mathbf{r} \cdot \mathbf{E}) = 0, \tag{115}$$

$$\left(\nabla^2 + k^2\right)(\mathbf{r} \cdot \mathbf{H}) = 0. \tag{116}$$

Then, this defines the transverse fields according to

$$(\mathbf{r} \cdot \mathbf{E}) = 0 \text{ and } (\mathbf{r} \cdot \mathbf{H}) \neq 0 \Rightarrow \text{ Transverse Magnetic (TM)}, \tag{117}$$

$$(\mathbf{r} \cdot \mathbf{H}) = 0 \text{ and } (\mathbf{r} \cdot \mathbf{E}) \neq 0 \Rightarrow \text{ Transverse Electric (TE)}, \tag{118}$$

$$\tag{119}$$

in the way that we can write the electric and the magnetic field for the **electric dipole** and **magnetic dipole** as function of the transverse magnetic and electric fields as

$$\mathbf{E}_{dip} = \mathbf{E}^{\mathcal{E}} + \mathbf{E}^{\mathcal{M}} \tag{120}$$

$$\mathbf{H}_{dip} = \mathbf{H}^{\mathcal{E}} + \mathbf{H}^{\mathcal{M}}. \tag{121}$$

To obtain these fields we can use the *multipole expansion*: the **Transverse Eletric (TE)** is given by

$$\mathbf{H}^{\mathcal{E}} = e^{-i\omega t} \sum_{\ell,m} a^{\mathcal{E}}_{\ell,m} h^{(1)}_{\ell}(kr)\mathbf{X}_{\ell,m}(\mathbf{x}) \tag{122}$$

$$\mathbf{E}^{\mathcal{E}} = e^{-i\omega t}\frac{i}{k} \sum_{\ell,m} a^{\mathcal{E}}_{\ell,m} \nabla \times \left[h^{(1)}_{\ell}(kr)\mathbf{X}_{\ell,m}(\mathbf{x})\right]. \tag{123}$$

and the same follows to the **Transverse Magnetic (TM)**

$$\mathbf{E}^{\mathcal{M}} = e^{-i\omega t} Z_0 \sum_{\ell,m} a^{\mathcal{M}}_{\ell,m} h^{(1)}_{\ell}(kr)\mathbf{X}_{\ell,m}(\mathbf{x}) \tag{124}$$

$$\mathbf{H}^{\mathcal{M}} = -e^{-i\omega t}\frac{i}{k} \sum_{\ell,m} a^{\mathcal{M}}_{\ell,m} \nabla \times \left[h^{(1)}_{\ell}(kr)\mathbf{X}_{\ell,m}(\mathbf{x})\right]. \tag{125}$$

Then, the coefficients are given by

$$a^{\mathcal{E}}_{\ell,m} = -i\frac{k^2 c}{\ell(\ell+1)}\int d^3x Y^{\star}_{\ell,m}(\hat{x})\left\{\frac{\partial\left[r j_\ell(kr)\right]}{\partial r}\rho(\mathbf{x}) + i\frac{\omega}{c^2}j_\ell(kr)\mathbf{r}\cdot\mathbf{J}(\mathbf{x})\right\}, \tag{126}$$

$$a^{\mathcal{M}}_{\ell,m} = -i\frac{k^2 c}{\ell(\ell+1)}\int d^3x Y^{\star}_{\ell,m}(\hat{x}) j_\ell(kr)\vec{\nabla}\cdot\left[\mathbf{r}\times\mathbf{J}(\mathbf{x})\right]. \tag{127}$$

i) Electric Dipole:

We can use the eletric dipole fields in the *near zone* as

$$\mathbf{H} = \frac{i\omega}{4\pi}\left(\hat{r}\times\mathbf{p}\right)\frac{1}{r^2} \tag{128}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\left[3\hat{r}\left(\hat{r}\cdot\mathbf{p}\right) - \mathbf{p}\right]\frac{1}{r^3}. \tag{129}$$

then, as the charge density and the vector current are given by the Maxwell equations

$$\vec{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \Rightarrow \rho = \epsilon_0\vec{\nabla}\cdot\mathbf{E} \tag{130}$$

$$\vec{\nabla}\times\mathbf{H} = \epsilon_0\frac{\partial\mathbf{E}}{\partial t} + \mathbf{J} \Rightarrow \mathbf{J} = \vec{\nabla}\times\mathbf{H} - \epsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{131}$$

we get

$$\rho = \epsilon_0\vec{\nabla}\cdot\left\{\frac{1}{4\pi\epsilon_0}\left[3\hat{r}\left(\hat{r}\cdot\mathbf{p}\right) - \mathbf{p}\right]\frac{1}{r^3}\right\} \tag{132}$$

$$= \frac{1}{4\pi}\left\{\frac{1}{r^3}\vec{\nabla}\cdot\left[3\hat{r}\left(\hat{r}\cdot\mathbf{p}\right) - \mathbf{p}\right] + \left[3\hat{r}\left(\hat{r}\cdot\mathbf{p}\right) - \mathbf{p}\right]\vec{\nabla}\cdot\left(\frac{1}{r^3}\right)\right\} \tag{133}$$

$$= \frac{1}{4\pi}\left\{\frac{1}{r^3}\left[3\vec{\nabla}\cdot\left(\hat{r}\left(\hat{r}\cdot\mathbf{p}\right)\right) - \vec{\nabla}\cdot\mathbf{p}\right] - \left[3\hat{r}\left(\hat{r}\cdot\mathbf{p}\right) - \mathbf{p}\right]\cdot\left(\frac{3}{r^4}\hat{r}\right)\right\} \tag{134}$$

$$= \frac{1}{4\pi}\left\{\frac{1}{r^3}\left[3\hat{r}\cdot\left(\hat{r}\cdot\vec{\nabla}\right)\mathbf{p} - \vec{\nabla}\cdot\mathbf{p}\right] - \left[9\left(\hat{r}\cdot\mathbf{p}\right) - 3(\hat{r}\cdot\mathbf{p})\right]\frac{1}{r^4}\right\} \tag{135}$$

$$= \frac{1}{4\pi}\left\{\frac{1}{r^3}\left[3\frac{\partial}{\partial r}\left(\hat{r}\cdot\mathbf{p}\right) - \vec{\nabla}\cdot\mathbf{p}\right] - \left[6\left(\hat{r}\cdot\mathbf{p}\right)\right]\frac{1}{r^4}\right\} \tag{136}$$

$$= \frac{1}{4\pi r^3}\left[3\frac{\partial}{\partial r}\left(\hat{r}\cdot\mathbf{p}\right) - \vec{\nabla}\cdot\mathbf{p} - \frac{6\hat{r}\cdot\mathbf{p}}{r}\right] \tag{137}$$

$$\mathbf{J} = \vec{\nabla}\times\left[\frac{i\omega}{4\pi}\left(\hat{r}\times\mathbf{p}\right)\frac{1}{r^2}\right] - \epsilon_0\frac{\partial}{\partial t}\left\{\frac{1}{4\pi\epsilon_0}\left[3\hat{r}\left(\hat{r}\cdot\mathbf{p}\right) - \mathbf{p}\right]\frac{1}{r^3}\right\} \tag{138}$$

$$= \frac{i\omega}{4\pi}\left[\frac{1}{r^2}\vec{\nabla}\times\left(\hat{r}\times\mathbf{p}\right) + \vec{\nabla}\left(\frac{1}{r^2}\right)\times\left(\hat{r}\times\mathbf{p}\right)\right] - \frac{1}{4\pi}\left\{\left[3\hat{r}\left(\hat{r}\cdot\dot{\mathbf{p}}\right) - \dot{\mathbf{p}}\right]\frac{1}{r^3}\right\} \tag{139}$$

$$= \frac{i\omega}{4\pi}\left\{\frac{1}{r^2}\left[\hat{r}\left(\vec{\nabla}\cdot\mathbf{p}\right) - \left(\hat{r}\cdot\vec{\nabla}\right)\mathbf{p}\right] - \frac{2}{r^3}\hat{r}\times\left(\hat{r}\times\mathbf{p}\right)\right\} - \frac{1}{4\pi}\left\{\left[3\hat{r}\left(\hat{r}\cdot\dot{\mathbf{p}}\right) - \dot{\mathbf{p}}\right]\frac{1}{r^3}\right\} \tag{140}$$

$$= \frac{i\omega}{4\pi}\left\{\frac{1}{r^2}\left[\hat{r}\left(\vec{\nabla}\cdot\mathbf{p}\right) - \frac{\partial\mathbf{p}}{\partial r}\right] - \frac{2}{r^3}\left[\left(\hat{r}\cdot\mathbf{p}\right)\hat{r} - \left(\hat{r}\cdot\hat{r}\right)\mathbf{p}\right]\right\} - \frac{1}{4\pi}\left\{\left[3\hat{r}\left(\hat{r}\cdot\dot{\mathbf{p}}\right) - \dot{\mathbf{p}}\right]\frac{1}{r^3}\right\}$$

$$= \frac{1}{4\pi r^2}\left\{i\omega\left[\hat{r}\left(\vec{\nabla}\cdot\mathbf{p}\right) - \frac{\partial\mathbf{p}}{\partial r} - \frac{2}{r}\left[\left(\hat{r}\cdot\mathbf{p}\right)\hat{r} - \mathbf{p}\right]\right] - \frac{1}{r}\left[3\hat{r}\left(\hat{r}\cdot\dot{\mathbf{p}}\right) - \dot{\mathbf{p}}\right]\right\}. \tag{141}$$

Now we need to replace those results into the coefficients. But, to simplify the computations, let's compute the operations on them, separately

$$\mathbf{r}\cdot\mathbf{J} = r\hat{r}\cdot\mathbf{J} = \frac{1}{4\pi r}\left\{i\omega\left[\left(\vec{\nabla}\cdot\mathbf{p}\right) - \frac{\partial\left(\hat{r}\cdot\mathbf{p}\right)}{\partial r} - \frac{2}{r}\left[\cancel{(\hat{r}\cdot\mathbf{p})} - \cancel{(\hat{r}\cdot\mathbf{p})}\right]\right] - \frac{1}{r}\left[3\left(\hat{r}\cdot\dot{\mathbf{p}}\right) - \left(\hat{r}\cdot\dot{\mathbf{p}}\right)\right]\right\}$$

$$= r\hat{r}\cdot\mathbf{J} = \frac{1}{4\pi r}\left\{i\omega\left[\left(\vec{\nabla}\cdot\mathbf{p}\right) - \frac{\partial\left(\hat{r}\cdot\mathbf{p}\right)}{\partial r}\right] - \frac{2}{r}\left(\hat{r}\cdot\dot{\mathbf{p}}\right)\right\} \tag{142}$$

$$\mathbf{r}\times\mathbf{J} = r\hat{r}\times\mathbf{J} = \frac{1}{4\pi r}\left\{i\omega\left[-\hat{r}\times\frac{\partial\mathbf{p}}{\partial r} + \frac{2}{r}\hat{r}\times\mathbf{p}\right] + \frac{1}{r}\hat{r}\times\dot{\mathbf{p}}\right\} \tag{143}$$

$$= \frac{1}{4\pi r}\left(\frac{1}{r}\hat{r}\times\dot{\mathbf{p}} - i\omega\hat{r}\times\frac{\partial\mathbf{p}}{\partial r} + \frac{2i\omega}{r}\hat{r}\times\mathbf{p}\right) \tag{144}$$

$$\vec{\nabla}\cdot(\mathbf{r}\times\mathbf{J}) = \frac{1}{4\pi}\left\{\vec{\nabla}\cdot\left[\left(\frac{1}{r^2}\right)\hat{r}\times\dot{\mathbf{p}}\right] - i\omega\vec{\nabla}\cdot\left[\left(\frac{1}{r}\right)\hat{r}\times\frac{\partial\mathbf{p}}{\partial r}\right] + 2i\omega\vec{\nabla}\cdot\left[\left(\frac{1}{r^2}\right)\hat{r}\times\mathbf{p}\right]\right\} \tag{145}$$

$$= \frac{1}{4\pi}\left\{\left[2i\omega\left(\frac{\hat{r}}{r^3}\right)\cdot(\hat{r}\times\mathbf{p}) + i\omega\left(\frac{\hat{r}}{r^2}\right)\cdot\left(\vec{\nabla}\times\mathbf{p}\right)\right]\right. \tag{146}$$

$$+\left[i\omega\left(\frac{\hat{r}}{r^2}\right)\cdot(\hat{r}\times\partial_r\mathbf{p}) + i\omega\left(\frac{\hat{r}}{r}\right)\cdot\left(\vec{\nabla}\times\partial_r\mathbf{p}\right)\right] \tag{147}$$

$$\left.+\left[-4i\omega\left(\frac{\hat{r}}{r^3}\right)\cdot(\hat{r}\times\mathbf{p}) - 2i\omega\left(\frac{\hat{r}}{r^2}\right)\cdot\left(\vec{\nabla}\times\mathbf{p}\right)\right]\right\} \tag{148}$$

$$= \frac{1}{4\pi}\left[-2i\omega\left(\frac{\hat{r}}{r^3}\right)\cdot(\hat{r}\times\mathbf{p}) + i\omega\left(\frac{\hat{r}}{r^2}\right)\cdot(\hat{r}\times\partial_r\mathbf{p})\right. \tag{149}$$

$$\left.+i\omega\left(\frac{\hat{r}}{r}\right)\cdot\left(\vec{\nabla}\times\partial_r\mathbf{p}\right) - i\omega\left(\frac{\hat{r}}{r^2}\right)\cdot\left(\vec{\nabla}\times\mathbf{p}\right)\right], \tag{150}$$

where I have used $\dot{\mathbf{p}} = -i\omega\mathbf{p}$.

Finally,

$$a^{\mathcal{E}}_{\ell,m} = -i\frac{k^2c}{\ell(\ell+1)}\int d^3x Y^\star_{\ell,m}(\hat{x})\left\{\frac{\partial\left[rj_\ell(kr)\right]}{\partial r}\left(\frac{1}{4\pi r^3}\left[3\frac{\partial}{\partial r}\left(\hat{r}\cdot\mathbf{p}\right) - \vec{\nabla}\cdot\mathbf{p} - \frac{6\hat{r}\cdot\mathbf{p}}{r}\right]\right)\right. \tag{151}$$

$$\left.+i\frac{\omega}{c^2}j_\ell(kr)\left(\frac{1}{4\pi r}\left\{i\omega\left[\left(\vec{\nabla}\cdot\mathbf{p}\right) - \frac{\partial\left(\hat{r}\cdot\mathbf{p}\right)}{\partial r}\right] - \frac{2}{r}\left(\hat{r}\cdot\dot{\mathbf{p}}\right)\right\}\right)\right\}, \tag{152}$$

$$a^{\mathcal{M}}_{\ell,m} = -i\frac{k^2c}{\ell(\ell+1)}\int d^3x Y^\star_{\ell,m}(\hat{x})j_\ell(kr)\left\{\frac{1}{4\pi}\left[-2i\omega\left(\frac{\hat{r}}{r^3}\right)\cdot(\hat{r}\times\mathbf{p}) + i\omega\left(\frac{\hat{r}}{r^2}\right)\cdot(\hat{r}\times\partial_r\mathbf{p})\right.\right.$$

$$\left.\left.+i\omega\left(\frac{\hat{r}}{r}\right)\cdot\left(\vec{\nabla}\times\partial_r\mathbf{p}\right) - i\omega\left(\frac{\hat{r}}{r^2}\right)\cdot\left(\vec{\nabla}\times\mathbf{p}\right)\right]\right\}.$$

b) Assuming that the dipole is in the direction $\hat{z}$, we can write

$$\mathbf{p} = p_0e^{-i\omega t}\hat{z} = p_0e^{-i\omega t}\left(\cos\theta\hat{r} - \sin\theta\hat{\theta}\right). \tag{153}$$

Writing the terms in the coefficients expressions for this dipole we get

$$
\begin{aligned}
\hat{r}\cdot\mathbf{p} &= p_0 e^{-i\omega t}\cos\theta && (154)\\
\vec{\nabla}\cdot\mathbf{p} &= p_0 e^{-i\omega t}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cos\theta\right) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin^2\theta\right)\right] = 0 && (155)\\
\hat{r}\times\mathbf{p} &= -p_0 e^{-i\omega t}\sin\theta\hat{\phi} && (156)\\
\partial_r\mathbf{p} &= 0 && (157)\\
\vec{\nabla}\times\mathbf{p} &= \frac{\hat{\phi}}{r}\left[-\frac{\partial(r\sin\theta)}{\partial r} - \frac{\partial\cos\theta}{\partial\theta}\right] = 0. && (158)
\end{aligned}
$$

In the way that the coefficients are

$$
\begin{aligned}
a^{\mathcal{E}}_{\ell,m} = &-i\frac{k^2c}{\ell(\ell+1)}\int d^3x Y^{\star}_{\ell,m}(\hat{x})\left\{\frac{\partial\left[rj_\ell(kr)\right]}{\partial r}\left(\frac{1}{4\pi r^3}\left[3\cancel{\frac{\partial}{\partial r}(\hat{r}\cdot\mathbf{p})} - \cancel{\vec{\nabla}\cdot\mathbf{p}} - \frac{6p_0e^{-i\omega t}\cos\theta}{r}\right]\right)\right.\\
&\left. + i\frac{\omega}{c^2}j_\ell(kr)\left(\frac{1}{4\pi r}\left\{i\omega\left[\cancel{\left(\vec{\nabla}\cdot\mathbf{p}\right)} - \cancel{\frac{\partial\left(\hat{r}\cdot\mathbf{p}\right)}{\partial r}}\right] + \frac{2i\omega p_0 e^{-i\omega t}\cos\theta}{r}\right\}\right)\right\},\\
= &\, i\frac{k^2c}{\ell(\ell+1)}\int d^3x Y^{\star}_{\ell,m}(\hat{x})\left\{\frac{p_0e^{-i\omega t}\cos\theta}{2\pi r^2}\left[\frac{3}{r^2}\frac{\partial\left[rj_\ell(kr)\right]}{\partial r} + \frac{\omega^2}{c^2}j_\ell(kr)\right]\right\} && (159)\\
= &\, i\frac{k^2c}{\ell(\ell+1)}\int_0^\infty drr^2\left\{\frac{p_0e^{-i\omega t}}{2\pi r^2}\left[\frac{3}{r^2}\frac{\partial\left[rj_\ell(kr)\right]}{\partial r} + \frac{\omega^2}{c^2}j_\ell(kr)\right]\right\}\int d\Omega Y^{\star}_{\ell,m}(\hat{x})\cos\theta && (160)\\
= &\, i\frac{k^2c}{2}\int_0^\infty dr\left\{\frac{p_0e^{-i\omega t}}{2\pi}\left[\frac{3}{r^2}\frac{\partial\left[rj_1(kr)\right]}{\partial r} + \frac{\omega^2}{c^2}j_1(kr)\right]\right\} && (161)\\
a^{\mathcal{E}}_{1,0} = &\, i\frac{k^2c}{2}\int_0^\infty dr\left\{\frac{p_0e^{-i\omega t}}{2\pi}\left[\frac{3}{r^2}\left[j_1(kr) + \frac{kr}{2}\left(j_0(kr) - j_2(kr)\right) - \frac{j_1(kr)}{(2kr)}\right] + \frac{\omega^2}{c^2}j_1(kr)\right]\right\}\\
& && (162)\\
a^{\mathcal{M}}_{\ell,m} = &-i\frac{k^2c}{\ell(\ell+1)}\int d^3x Y^{\star}_{\ell,m}(\hat{x})j_\ell(kr)\left\{\frac{1}{4\pi}\left[-2i\omega\cancel{\left(\frac{\hat{r}}{r^3}\right)\cdot(\hat{r}\times\mathbf{p})} + i\omega\left(\frac{\hat{r}}{r^2}\right)\cdot\cancel{(\hat{r}\times\partial_r\mathbf{p})}\right.\right.\\
&\left.\left. + i\omega\left(\frac{\hat{r}}{r}\right)\cdot\cancel{\left(\vec{\nabla}\times\partial_r\mathbf{p}\right)} - i\omega\left(\frac{\hat{r}}{r^2}\right)\cdot\cancel{\left(\vec{\nabla}\times\mathbf{p}\right)}\right]\right\} = 0.
\end{aligned}
$$

So, the fields for the electric dipole are written as

$$
\begin{aligned}
\mathbf{E}_{dip} &= \mathbf{E}^{\mathcal{E}} = e^{-i\omega t}\frac{i}{k}a^{\mathcal{E}}_{1,0}\nabla\times\left[h^{(1)}_1(kr)\mathbf{X}_{1,0}(\mathbf{x})\right] && (163)\\
\mathbf{H}_{dip} &= \mathbf{H}^{\mathcal{E}} = e^{-i\omega t}a^{\mathcal{E}}_{1,0}h^{(1)}_1(kr)\mathbf{X}_{1,0}(\mathbf{x}). && (164)
\end{aligned}
$$

with the scalar and vector spherical harmonics given respectively by

$$
\begin{aligned}
&Y_{1,0} && (165)\\
&\mathbf{X}_{1,0}(\mathbf{x})\frac{1}{\sqrt{2}}\mathbf{L}Y_{1,0}(\hat{r}). && (166)
\end{aligned}
$$

8 Question (1 point)

A scalar spherical wave ψ is emitted from a source at the origin in such a way that the Fourier transforms are given by:

$$\tilde{\psi}(t, k, \hat{k}) = A\, \exp\left(-i\omega t\right)\, \exp\left(-ak\right)\, \cos\theta_k$$

a) *Show that this is a pure dipole pattern in real and in Fourier space;*

b) *Show that this corresponds to a pulse of width $\Delta r = a$ that propagates from the origin outwards. Plot the spatial dependence of that pulse as it propagates in space and in time.*

c) *Show that this pulse has a finite (and fixed) energy at any time. You can interpret the energy density as $|\psi|^2$.*

Hint: *For this problem you will need the following integral:*

$$\int_0^\infty dk\, k^2\, j_1(kx)\, e^{-ak} = 2x/(x^2 + a^2)^2$$

8.1 Solution

a) We can expand this spherical wave according to Rayleigh as

$$\tilde{\Psi}(t, k, \hat{k}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \tilde{f}_{\ell,m}(t, k) Y_{\ell,m}(\hat{k}) \tag{167}$$

in the way that

$$\begin{aligned}
\tilde{f}_{\ell,m}(t, k) &= \int d^2k Y^\star_{\ell,m}(\hat{k}) \tilde{\Psi}(t, k, \hat{k}) && (168)\\
&= \int d\Omega Y^\star_{\ell,m}(\hat{k}) A\, \exp\left(-i\omega t\right)\, \exp\left(-ak\right)\, \cos\theta_k && (169)\\
&= A\, \exp\left(-i\omega t\right)\, \exp\left(-ak\right) \int d\Omega Y^\star_{\ell,m}(\hat{k}) \cos\theta_k && (170)\\
&= A\, \exp\left(-i\omega t\right)\, \exp\left(-ak\right) \int d\Omega Y^\star_{\ell,m}(\hat{k}) 2\sqrt{\frac{\pi}{3}} Y_{1,0}(\hat{k}) && (171)\\
&= A\, \exp\left(-i\omega t\right)\, \exp\left(-ak\right) \delta_{\ell,1}\delta_{m,0}, && (172)\\
\tilde{f}_{1,0}(t, k) &= 2A\sqrt{\frac{\pi}{3}}\, \exp\left(-i\omega t\right)\, \exp\left(-ak\right). && (173)
\end{aligned}$$

Then, we can use this result in the previous expression, seeing that

$$\tilde{\Psi}(t, k, \hat{k}) = \tilde{f}_{1,0}(t, k) Y_{1,0}(\hat{k}) = A\, \exp\left(-i\omega t\right)\, \exp\left(-ak\right) \cos\theta, \tag{174}$$

which is, therefore, a dipole in the Fourier space, due to the term $\cos\theta$!

Using the same formalism, we can correlate this term with the same (in the real space expansion)

$$f_{\ell,m}(t,r) = \frac{4\pi i^\ell}{(2\pi)^3}\int_0^\infty dk k^2 \tilde{f}_{\ell,m}(t,k) j_\ell(kr) \tag{175}$$

$$f_{1,0}(t,r) = \frac{4\pi i}{(2\pi)^3}\int_0^\infty dk k^2 \tilde{f}_{1,0}(t,k) j_1(kr) \tag{176}$$

$$= \frac{4\pi i}{(2\pi)^3} 2A\sqrt{\frac{\pi}{3}}\int_0^\infty dk k^2 e^{-i\omega t} \exp(-ak) j_1(kr) \tag{177}$$

$$= \frac{i}{\pi^2} A\sqrt{\frac{\pi}{3}}\int_0^\infty dk k^2 e^{-ikt/c} \exp(-ak) j_1(kr) \tag{178}$$

$$= \frac{i}{\pi^2} A\sqrt{\frac{\pi}{3}}\int_0^\infty dk k^2 e^{-(it/c+a)k} j_1(kr) \tag{179}$$

$$= \frac{i}{\pi^2} A\sqrt{\frac{\pi}{3}}\frac{2r}{\left[\frac{(ac+it)^2}{c^2} + r^2\right]^2}, \tag{180}$$

where we need to use the following integral

$$I = \int_0^\infty dk k^2 e^{-\alpha k} j_1(kr) = \frac{2r}{\left[\alpha^2 + r^2\right]^2} \Rightarrow \alpha = \frac{(ac+it)}{c}. \tag{181}$$

As the expansion in the real space is given by

$$\Psi(t,r,\hat{r}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_{\ell,m}(t,r) Y_{\ell,m}(\hat{r}) \tag{182}$$

$$= f_{1,0}(t,r) Y_{1,0}(\hat{r}) \tag{183}$$

$$= \frac{i}{\pi^2} A\frac{r}{\left[\frac{(ac+it)^2}{c^2} + r^2\right]^2}\cos\theta, \tag{184}$$

which, again, has the $\cos\theta$ in the expansion and, thus, is a pure dipole in the real space as well! An animation of the real and imaginary parts of this wave, according to time and in function of r, is in the link.

b) Let's compute the Fourier transform of this spherical wave, obtaining it in the real space

$$\psi(t,r,\hat{r}) = \int \frac{d^3k}{(2\pi)^3} e^{-i\hat{k}\cdot\hat{r}} \tilde{\psi}(t,k,\hat{k}) = \int \frac{d^3k}{(2\pi)^3} e^{-i\hat{k}\cdot\hat{r}} A \exp(-i\omega t) \exp(-ak) \cos\theta_k \tag{185}$$

$$= \frac{A}{(2\pi)^2}\int_0^\pi d\theta \sin\theta \int_0^\infty dk k^2 \exp(-ikt/c) e^{-ikr\cos\theta} e^{-ak} \cos\theta. \tag{186}$$

Changing variables using

$$\begin{cases} u = \cos\theta \\ du = -\sin\theta d\theta \end{cases} \tag{187}$$

we got

$$\psi(t,r,\hat{r}) = \frac{A}{(2\pi)^2}\int_{-1}^{1} duu \int_0^\infty dkk^2 \exp\left(-ikt/c\right)e^{-ikru}e^{-ak}. \tag{188}$$

Now, using the Wolfram's integrals

$$I = \int_0^\infty dkk^2 \exp\left(-ikt/c\right)e^{-ikru}e^{-ak} = \frac{2}{\left[a+\frac{i(t+cru)}{c}\right]^3}, \tag{189}$$

$$II = \int_{-1}^{1} du \frac{2u}{\left[a+\frac{i(t+cru)}{c}\right]^3} = \frac{-4ic^4 r}{(iac+cr-t)^2(-iac+cr+t)^2}, \tag{190}$$

we arrive at

$$\psi(t,r,\hat{r}) = \frac{A}{(2\pi)^2}\frac{-4ic^4 r}{(iac+cr-t)^2(-iac+cr+t)^2}. \tag{191}$$

An animation of this function is in the link. It is a **pulse**

$$r^2|\psi(t,r,\hat{r})|^2 \tag{192}$$

of width $\Delta r = a$. An animation for the pulse is in the link.

You can plot this pulse using the following code:

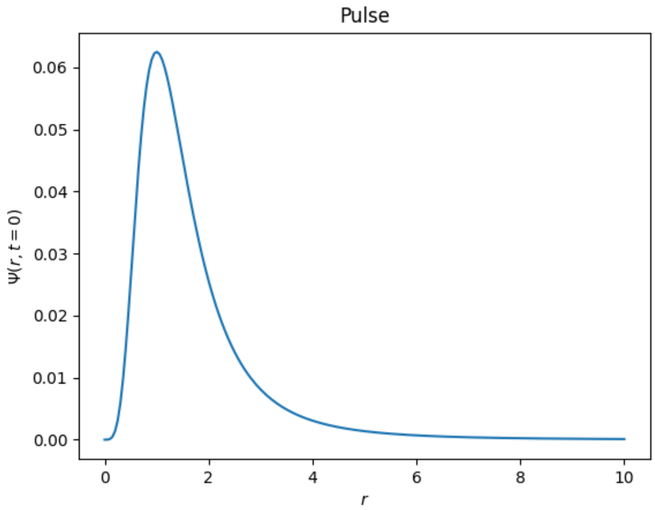
```
import numpy as np
import matplotlib.pyplot as plt

def f(t, r):
    return r**2*abs(1j*r/(r**2 + (1. + 1j*t)**2)**2)**2

r = np.linspace(0, 10, 200)

#Plot in polar space
plt.figure(dpi = 100)
plt.title('Pulse')
plt.plot(r, f(0, r))
plt.ylabel(r'$\Psi (r, t = 0)$')
plt.xlabel(r'$r$')
plt.savefig('dipole.png')
```

The example for the time propagation can be seen in the link. Notice that the animation is for $\cos\theta = 1$ and constants are defined to 1.



c) The pulse has the energy given, in the Fourier space, by

$$E = \int d^3k |\tilde{\psi}(t, k, \hat{k})|^2 \tag{193}$$

$$= A^2 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \cos^2\theta \int_0^{\infty} k^2 \exp(-2ak) \tag{194}$$

$$= A^2 (2\pi) \frac{2}{3} \frac{1}{4a^3} = \frac{\pi A^2}{3a^3}. \tag{195}$$

If you compute it in the real space, for $t = 0$, you get

$$E = \int d^3r |\tilde{\psi}(t, r, \hat{r})|^2 \tag{196}$$

$$= \frac{A^2}{\pi^4} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \cos^2\theta \int_0^{\infty} r^2 Abs \left[\frac{ir}{(a^2 + r^2)^2} \right]^2 \tag{197}$$

$$= \frac{A^2}{\pi^4} (2\pi) \frac{2}{3} \frac{\pi}{32a^3} = \frac{A^2}{24\pi^2 a^3}. \tag{198}$$

---

9 Question (1 point)

A particle with mass m and charge e moves in a uniform, static, electric field $\mathbf{E}_0$.

a) *Solve the velocity and position of the particle as explicit functions of time. Assume the initial velocity $\mathbf{v}_0$ is perpendicular to the electric field.*

b) *Eliminate the time to obtain the trajectory of the particle in space. Discuss the shape of the path for short and long times (define what you call as "short" and as "long" times!).*

9.1 Solution

a) The equation of motion for a particle of charge e in external fields could be written in covariant form as

$$\frac{dU^\alpha}{d\tau} = \frac{e}{mc}F^{\alpha\beta}U_\beta, \tag{199}$$

where m is the mass of the particle, τ is the proper time and $U^\alpha = (\gamma c, \gamma \mathbf{u}) = p^\alpha/m$ (**Attention:** in this notation (Jackson does not use E_i/c, but just E_i). For a constant electric field $\mathbf{E}_0$ we can split it into

$$\frac{dU^0}{d\tau} = \frac{e}{mc}\mathbf{E}_0 \cdot \mathbf{u} \tag{200}$$

$$\frac{d\mathbf{u}}{d\tau} = \frac{e}{mc}\mathbf{E}_0 U^0. \tag{201}$$

We can solve these equations taking

$$\frac{d}{d\tau}\left(\frac{dU^0}{d\tau}\right) = \frac{d^2U^0}{d\tau^2} = \frac{e}{mc}\mathbf{E}_0 \cdot \frac{d\mathbf{u}}{d\tau} = \left(\frac{eE_0}{mc}\right)^2 U^0, \tag{202}$$

where the general solution is

$$U^0 = Ae^{(eE_0/mc)\tau} + Be^{-(eE_0/mc)\tau} \tag{203}$$

and it take us to

$$\mathbf{u} = \mathbf{u}_0 + \left[Ae^{(eE_0/mc)\tau} - Be^{-(eE_0/mc)\tau}\right]\hat{E}_0. \tag{204}$$

But we know that: $\mathbf{E}_0 \cdot \mathbf{u}_0 = 0$ and we need to satisfy invariant like

$$U^\mu U_\mu = -c^2 \tag{205}$$

$$-(U^0)^2 + (\mathbf{u})^2 = -c^2 \tag{206}$$

$$-4AB + (\mathbf{u})^2 = 0c^2. \tag{207}$$

For reason of simplicity we can chose $\tau = t = 0$ in the way that we need $A = B$, to have the initial velocity perpendicular to the electric field. Then,

$$2A = \sqrt{c^2 + u_0^2} \tag{208}$$

and

$$U^0 = A\left[e^{(eE_0/mc)\tau} + e^{-(eE_0/mc)\tau}\right] = 2A\cosh\left(\frac{eE_0\tau}{c}\right) = \sqrt{c^2 + u_0^2}\cosh\left(\frac{eE_0\tau}{mc}\right) \tag{209}$$

$$\mathbf{u} = \mathbf{u}_0 + A\left[e^{(eE_0/mc)\tau} - e^{-(eE_0/mc)\tau}\right]\hat{E}_0 = \mathbf{u}_0 + 2A\sinh\left(\frac{eE_0\tau}{mc}\right)\hat{E}_0 \tag{210}$$

$$= \mathbf{u}_0 + \sqrt{c^2 + u_0^2}\sinh\left(\frac{eE_0\tau}{mc}\right)\hat{E}_0 \tag{211}$$

As

$$\mathbf{u}_0 = \gamma_0\mathbf{v}_0 \tag{212}$$

$$\gamma_0 = \frac{1}{\sqrt{1 - \frac{v_0^2}{c^2}}} \tag{213}$$

we get the velocity like

$$U^0 = c\gamma_0\cosh\left(\frac{eE_0\tau}{mc}\right), \tag{214}$$

$$\mathbf{u} = \gamma_0\mathbf{v}_0 + c\gamma_0\sinh\left(\frac{eE_0\tau}{mc}\right)\hat{E}_0. \tag{215}$$

Now, using the definition of the 4-velocity $\frac{dx^\mu}{d\tau} = U^\mu$ we obtain the position integrating it as following

$$x^0 = ct = \int_0^\tau d\tau' U^0 = \int_0^\tau d\tau' c\gamma_0\cosh\left(\frac{eE_0\tau'}{mc}\right) \tag{216}$$

$$= \frac{mc^2\gamma_0}{eE_0}\sinh\left(\frac{eE_0\tau}{mc}\right) \tag{217}$$

$$\mathbf{x} = \int_0^\tau d\tau'\mathbf{u} = \gamma_0\mathbf{v}_0\tau + \frac{mc^2\gamma_0}{eE_0}\left[\cosh\left(\frac{eE_0\tau}{mc}\right) - 1\right]\hat{E}_0, \tag{218}$$

where, of course, the initial position was defined as $x^\mu = (0, \vec{0})$.

However, all these results are in terms of τ and not t. To obtain it in terms of the time t we need to invert the first of the last expressions, obtaining

$$\tau = \frac{mc}{eE_0}\sinh^{-1}\left(\frac{eE_0 t}{mc\gamma_0}\right). \tag{219}$$

Replacing this in the expression for U^0, by definition of U^μ we get γ as

$$U^0 = c\gamma_0 \cosh\left(\frac{eE_0\tau}{mc}\right) = c\gamma_0\sqrt{1+\sinh^2\left(\frac{eE_0\tau}{mc}\right)} \tag{220}$$

$$= c\gamma_0\sqrt{1+\sinh^2\left[\left(\frac{eE_0}{mc}\right)\frac{mc}{eE_0}\sinh^{-1}\left(\frac{eE_0t}{mc\gamma_0}\right)\right]} \tag{221}$$

$$\gamma = \frac{U^0}{c} = \gamma_0\sqrt{1+\left(\frac{eE_0t}{mc\gamma_0}\right)^2}. \tag{222}$$

Thus, we can replace these in the expressions for velocity and position obtaining them in function of time

$$\mathbf{u} = \gamma_0\mathbf{v}_0 + c\gamma_0\sinh\left(\frac{eE_0\tau}{mc}\right)\hat{E}_0 \tag{223}$$

$$= \gamma_0\mathbf{v}_0 + c\gamma_0\sinh\left[\frac{eE_0\tau}{mc}\frac{mc}{eE_0}\sinh^{-1}\left(\frac{eE_0t}{mc\gamma_0}\right)\right]\hat{E}_0 \tag{224}$$

$$= \gamma_0\left(\mathbf{v}_0 + \frac{e\mathbf{E}_0t}{m\gamma_0}\right) \tag{225}$$

$$\mathbf{v}(t) = \frac{\mathbf{u}}{\gamma} = \left[1+\left(\frac{eE_0t}{mc\gamma_0}\right)^2\right]^{-1/2}\left(\mathbf{v}_0 + \frac{et\mathbf{E}_0}{m\gamma_0}\right), \tag{226}$$

$$\mathbf{x}(t) = \frac{mc\gamma_0}{eE_0}\left\{\sinh^{-1}\left(\frac{eE_0t}{mc\gamma_0}\right)\mathbf{v}_0 + c\left[\sqrt{1+\left(\frac{eE_0t}{mc\gamma_0}\right)^2}-1\right]\hat{E}_0\right\}. \tag{227}$$

b) As the direct expressions for the position and velocity have an intricate dependence with the time t, we can work with the expression with the proper time τ

$$\mathbf{x} = \gamma_0\mathbf{v}_0\tau + \frac{mc^2\gamma_0}{eE_0}\left[\cosh\left(\frac{eE_0\tau}{mc}\right)-1\right]\hat{E}_0 \tag{228}$$

and split the solution in the positions parallel and perpendicular to the electrical field as

$$x_\perp = \gamma_0 v_0 \tau \tag{229}$$

$$x_{//} = \frac{mc^2\gamma_0}{eE_0}\left[\cosh\left(\frac{eE_0\tau}{mc}\right)-1\right]. \tag{230}$$

Then, isolating τ, we get

$$x_{//} = \frac{mc^2\gamma_0}{eE_0}\left[\cosh\left(\frac{eE_0x_\perp}{mc\gamma_0v_0}\right)-1\right]. \tag{231}$$

To say anything about the limits of the above expression we need to recall the definition for τ as

$$\tau = \frac{mc}{eE_0}\sinh^{-1}\left(\frac{eE_0t}{mc\gamma_0}\right) \tag{232}$$

which means

$$\tau = \frac{x_\perp}{\gamma_0 v_0} \begin{cases} \ll 1, & t \ll \frac{mc\gamma_0}{eE_0} \quad \text{short times} \\ \gg 1, & t \gg \frac{mc\gamma_0}{eE_0} \quad \text{long times} \end{cases} . \tag{233}$$

Using the following expansions

$$\cosh(x) \simeq 1 + \frac{x^2}{2} + O(x^4),\ x \ll 1 \tag{234}$$

$$\lim_{x\to\infty} \cosh(x) \simeq \lim_{x\to\infty} \frac{e^x + e^{-x}}{2} = \frac{e^x}{2},\ x \gg 1, \tag{235}$$

the solution becomes

$$x_{//} = \begin{cases} \frac{eE_0}{2m\gamma_0 v_0^2} x_\perp^2, & \text{parabolic solution}, \quad t \ll \frac{mc\gamma_0}{eE_0} \\ \frac{mc^2\gamma_0}{2eE_0} \exp\left(\frac{eE_0}{2mc\gamma_0} \frac{x_\perp}{v_0}\right), & \text{exponential solution}, \quad t \gg \frac{mc\gamma_0}{eE_0} \end{cases} . \tag{236}$$