A note on the propagation of errors used in undergraduate physics experiments

Wei-Liang Qian

January 2019

1 Introduction

This note consists of a short derivation about the calculations of the propagation of errors. Two alternative schemes are discussed. The former views individual measurements as statistically independent events and the latter treats them as maximally correlated ones from a pessimistic viewpoint.

2 Derivations

Firstly, consider the following question concerning error propagation. There are two measured quantities, namely, x and y with respective uncertainties Δx and Δy. By definition, here, Δx and Δy are positive definite. There is a third quantity $z = x + y$. Now, we proceed to evaluate the uncertainty of z.

By definition, the uncertainty of a measurement is its standard deviation σ. For instance, for quantity x

$$\Delta x \equiv \sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\langle (x - \langle x \rangle)^2 \rangle}, \tag{1}$$

where $\langle \cdots \rangle$ indicates the expectation value of a quantity.

It is straightforward to show that

$$\Delta z \equiv \sigma_{(x+y)} = \sqrt{\langle (x+y)^2 - \langle x+y \rangle^2 \rangle} = \sqrt{\sigma_x^2 + \sigma_y^2 - 2\sigma_x\sigma_y\rho_{x,y}}, \tag{2}$$

where

$$\rho_{x,y} = \frac{\mathrm{Cov}(x,y)}{\sigma_x\sigma_y} \tag{3}$$

is known as Pearson correlation, and covariance $\mathrm{Cov}(x,y)$ is defined as

$$\mathrm{Cov}(x,y) = \langle (x - \langle x \rangle)(y - \langle y \rangle) \rangle. \tag{4}$$

It can be shown that the range of Pearson Correlation[1] is $-1 \le \rho_{x,y} \le +1$, where $\rho_{x,y} = \pm 1$ indicates that x and y are completely (anti-)correlated, and $\rho_{x,y} = 0$ indicates that they are uncorrelated.

[1] https://en.wikipedia.org/wiki/Pearson_correlation_coefficient

Subsequently, one may conclude that for independent (and therefore uncorrelated) measurements of x and y,

$$\sigma_z = \sqrt{\sigma_x^2 + \sigma_y^2}. \tag{5}$$

The above formula is what one usually finds in textbooks.

In reality, however, it occurs that measurements are correlated to a certain degree. For instance, an uncalibrated thermometer may always give overestimated measurements. The worst scenario takes place when $\rho_{x,y} = +1$, or in other words,

$$\Delta z = \Delta x + \Delta y. \tag{6}$$

It is not difficult to extend the above arguments to the elementary operations summarized in the following Table.1. We note that for the last two cases

Table 1: The table for error propagation for elementary operations.

Elementary operation	σ_z	Δz
$z = x + y$	$\sqrt{\sigma_x^2 + \sigma_y^2}$	$\Delta x + \Delta y$
$z = x - y$	$\sqrt{\sigma_x^2 + \sigma_y^2}$	$\Delta x + \Delta y$
$z = x \times y$	$\lvert z\rvert\sqrt{\left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2}$	$\lvert z\rvert\left(\frac{\Delta x}{\lvert x\rvert} + \frac{\Delta y}{\lvert y\rvert}\right)$
$z = \frac{x}{y}$	$\lvert z\rvert\sqrt{\left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2}$	$\lvert z\rvert\left(\frac{\Delta x}{\lvert x\rvert} + \frac{\Delta y}{\lvert y\rvert}\right)$

of mulplication and division, one may reformulate the expressions in terms of "relative error" as follows

$$\frac{\sigma_z}{|z|} = \sqrt{\left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2}, \tag{7}$$

and

$$\frac{\Delta z}{|z|} = \left(\frac{\Delta x}{|x|} + \frac{\Delta y}{|y|}\right). \tag{8}$$

For the general case $z = f(x, y)$, by using the expansion

$$z = \langle z\rangle + \frac{\partial z}{\partial x}(x - \langle x\rangle) + \frac{\partial z}{\partial y}(y - \langle y\rangle) + \frac{\partial^2 z}{\partial x \partial y}(x - \langle x\rangle)(y - \langle y\rangle) + \cdots, \tag{9}$$

and therefore one finds, up to first order terms

$$\langle z^2\rangle = \langle z\rangle^2 + \left(\frac{\partial z}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial z}{\partial y}\right)^2 \sigma_y^2 + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\mathrm{Cov}(x, y). \tag{10}$$

As a result, we have

$$\sigma_z = \langle z^2\rangle - \langle z\rangle^2 = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial z}{\partial y}\right)^2 \sigma_y^2 + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\sigma_x\sigma_y\rho_{x,y}}. \tag{11}$$

It is thus straightforward to demonstrate the results shown in Tab.1 by using Eq.(11). In particular, for uncorrelated measurements,

$$\sigma_z = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial z}{\partial y}\right)^2 \sigma_y^2}. \tag{12}$$

For completely correlated measurements

$$\Delta z = \left|\frac{\partial z}{\partial x}\right| \Delta x + \left|\frac{\partial z}{\partial y}\right| \Delta y. \tag{13}$$

Last but not least, consider the case where multiple independent measurements are carried out for the same physical quantity x. In what follows, one shows that the uncertainty of the resulting x obtained by averaging the measured values is inversely proportional to the square root of the number of measurements, n. In other words, for $z' = x_1 + x_2 + \cdots x_n$, by using Eq.(12) we have

$$\sigma_{z'} = \sqrt{\sigma_{x_1}^2 + \sigma_{x_2}^2 \cdots} = \sqrt{n}\sigma_x, \tag{14}$$

where one has simply assumed that the standard error of each measurement is σ_x. Also, since the measurements are independent and therefore their covariances vanish. Subsequently, the resulting uncertainty for the quantity x is actually one in n parts, namely, for the algebraic average $z = \frac{z'}{n}$, we have

$$\sigma_z = \frac{\sigma_{z'}}{n} = \frac{\sigma_x}{\sqrt{n}}. \tag{15}$$

The above result is well-known in literature. We note that, in this context, increasing the total number of measurements for completely correlated measurements does not improve the related precision, as expected.

3 Pedagogical practice

In practice, we would recommend adopting the rightmost column of Tab.1 for the following reasons. Firstly, according to the above analysis, it estimates the worst-case scenario, which is as much as "physical" compared to the *ansatz* of "uncorrelated measurements ."Secondly, the formulation of the last column is easier for the students to memorize. Last but not least, it can be connected directly to the more intuitive illustrations. It is of two folds. Consider the measurement of x and y with $\langle x\rangle = \langle x\rangle = 1$, $\Delta x = 0.1$, and $\Delta y = 0.2$. One immediately finds $0.9 \le x \le 1.1$ and $0.8 \le y \le 1.2$, which leads to $1.7 \le z(= x + y) \le 2.3$. The latter can be rewritten as $\langle z\rangle = \frac{1}{2}(1.7 + 2.3) = 2$ and $\Delta z = \frac{1}{2}(2.3 - 1.7) = 0.3 = \Delta x + \Delta y$, by taking the algebraic average and half of the interval. It is identical to the formulae developed in terms of Δz. Also, Eq.(13) is simply a slightly modified version of Taylor's expansion of a multivariable function in terms of the partial derivatives, and most students are supposed to be familiar with it.

4 Conclusion

The basic formulae for the derivation of error propagation are somehow missing regarding existing materials for college physics experiments. Hopefully, this brief note is meaningful in filling such a gap.

Agradecimentos

I am thankful for discussions with my colleagues Lucas Barboza da Silva and Bertha Cuadros-Melgar.

References