# Ondas gravitacionais

- **Produção**:

$$\begin{cases} \Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} & \text{(Eq. de onda c/ fonte)} \\ \partial_\mu \bar{h}^{\mu\nu} = 0 & \text{(condição de "gauge")} \end{cases}$$

Notemos que a conservação de $T_{\mu\nu}$ implica que $\Box(\partial_\mu \bar{h}^{\mu\nu}) = 0$. Logo, se $\partial_\mu \bar{h}^{\mu\nu} = 0$ na condição inicial ( $\partial_\mu \bar{h}^{\mu\nu}\big|_{t\to\infty} = 0$, $\partial_t \partial_\mu \bar{h}^{\mu\nu}\big|_{t\to-\infty} = 0$ ), então a Eq. de onda garante a validade da condição de "gauge".

## - Domínio de frequências

$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega \, e^{-i\omega t} \, \hat{\bar{h}}_{\mu\nu}(\omega;\mathbf{x})$$

$$T_{\mu\nu}(t,\mathbf{x}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega \, e^{-i\omega t} \, \hat{T}_{\mu\nu}(\omega;\mathbf{x})$$

- $$\partial_\mu \bar{h}^{\mu\nu} = 0 \iff \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega \left[ -i\omega \hat{\bar{h}}^{0\nu} + \partial_j \hat{\bar{h}}^{j\nu} \right] e^{-i\omega t} = 0 \iff \begin{cases} \hat{\bar{h}}^{00}(\omega;\mathbf{x}) = -\dfrac{i}{\omega} \partial_j \hat{\bar{h}}^{j0}(\omega;\mathbf{x}) \\ \hat{\bar{h}}^{0j}(\omega;\mathbf{x}) = -\dfrac{i}{\omega} \partial_k \hat{\bar{h}}^{kj}(\omega;\mathbf{x}) \end{cases}$$

Logo:

$$\bar{h}_{ij}(t,\mathbf{x}) \xrightarrow{\text{determina}} \hat{\bar{h}}_{ij}(\omega;\mathbf{x}) \xrightarrow{\text{determina}} \hat{\bar{h}}_{0j}(\omega;\mathbf{x}) \xrightarrow{\text{determina}} \begin{cases} \hat{\bar{h}}_{00}(\omega;\mathbf{x}) \\ \bar{h}_{0j}(t,\mathbf{x}) \end{cases} \xrightarrow{\text{determina}} \bar{h}_{00}(t,\mathbf{x})$$

A mesma manipulação pode ser feita p/ $\hat{T}_{\mu\nu}$:

$$\begin{cases} \hat{T}^{00} = -\frac{i}{\omega}\partial_j \hat{T}^{j0} \\ \hat{T}^{0j} = -\frac{i}{\omega}\partial_k \hat{T}^{kj} \end{cases}$$

Então:

F. GREEN RETARDADA

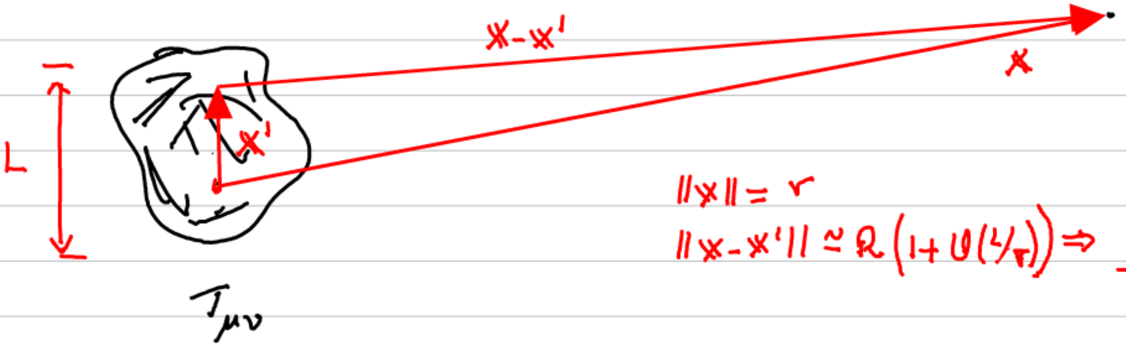
$$\Box \bar{h}_{jk} = -16\pi G T_{jk} \Leftarrow \bar{h}_{jk}(t,\mathbf{x}) = 4G\int d^3x' \left[\frac{T_{jk}(t',\mathbf{x}')}{\|\mathbf{x}-\mathbf{x}'\|}\right]\Bigg|_{t'=t-\frac{\|\mathbf{x}-\mathbf{x}'\|}{c}} \Leftrightarrow$$

$$\Leftrightarrow \quad \bar{h}_{jk}(t,\mathbf{x}) = \frac{4G}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} d\omega\, e^{-i\omega t}\int d^3x'\, e^{i\omega\|\mathbf{x}-\mathbf{x}'\|}\frac{\hat{T}_{jk}(\omega;\mathbf{x}')}{\|\mathbf{x}-\mathbf{x}'\|} \Leftrightarrow$$

$$\Leftrightarrow \quad \hat{\bar{h}}_{jk}(\omega;\mathbf{x}) = 4G\int d^3x'\, e^{i\omega\|\mathbf{x}-\mathbf{x}'\|}\frac{\hat{T}_{jk}(\omega;\mathbf{x}')}{\|\mathbf{x}-\mathbf{x}'\|} \simeq \frac{4Ge^{i\omega r}}{r}\int d^3x'\, \hat{T}_{jk}(\omega;\mathbf{x}')$$

$L \ll r$

$L \ll \lambda$




$\|\mathbf{x}\| = r$

$$\|\mathbf{x}-\mathbf{x}'\| \simeq R\left(1+\mathcal{O}(L/r)\right) \Rightarrow \frac{e^{i\omega\|\mathbf{x}-\mathbf{x}'\|}}{\|\mathbf{x}-\mathbf{x}'\|} = \frac{e^{i\omega r}}{r}\left(1+\mathcal{O}(\omega L)\right)\left(1+\mathcal{O}(L/r)\right)$$

Além disso: $\partial_j(\hat{T}^{jk}x^l) = x^l\partial_j\hat{T}^{jk} + \hat{T}^{jk}\partial_j x^l = x^l i\omega\hat{T}^{0k} + \hat{T}^{lk} \Leftrightarrow$

$$\Leftrightarrow \hat{T}^{jk} = \partial_l(\hat{T}^{lk}x^j) - i\omega\hat{T}^{0k}x^j = \frac{1}{2}\partial_l(\hat{T}^{lk}x^j + \hat{T}^{lj}x^k) - \frac{i\omega}{2}(\hat{T}^{0k}x^j + \hat{T}^{0j}x^k)$$

$$\partial_l(x^jx^k\hat{T}^{0l}) = (\partial_l x^j)x^k\hat{T}^{0l} + x^j(\partial_l x^k)\hat{T}^{0l} + x^jx^k\partial_l\hat{T}^{0l} = x^k\hat{T}^{0j} + x^j\hat{T}^{0k} + i\omega x^jx^k\hat{T}^{00}$$

$$\Leftrightarrow \underbrace{(x^j\hat{T}^{0k} + x^k\hat{T}^{0j})} = \partial_l(x^jx^k\hat{T}^{0l}) - i\omega x^jx^k\hat{T}^{00}$$

$$\boxed{\hat{T}^{jk} = \frac{1}{2}\partial_l\left(x^j\hat{T}^{lk} + x^k\hat{T}^{lj} - i\omega x^jx^k\hat{T}^{0l}\right) - \frac{\omega^2}{2}x^jx^k\hat{T}^{00}}$$

Substituindo na expressão aproximada p/ $\hat{h}_{jk}$:

$$\hat{\bar{h}}_{jk}(\omega;\mathbf{x}) = -2G\omega^2\frac{e^{i\omega r}}{r}\int d^3x'\, x'_j x'_k\, \hat{T}_{00}(\omega;\mathbf{x}') \Leftrightarrow$$

$$\Leftrightarrow \bar{h}_{jk}(t,\mathbf{x}) = -\frac{2G}{r}\int\frac{d^3x'}{\sqrt{2\pi}}x'_jx'_k\int_{-\infty}^{+\infty}d\omega\,\omega^2 e^{i\omega r}e^{-i\omega t}\hat{T}_{00}(\omega;\mathbf{x}') = \frac{2G}{r}\int\frac{d^3x'}{\sqrt{2\pi}}x'_jx'_k\frac{\partial^2}{\partial t'^2}\left[\int_{-\infty}^{+\infty}d\omega\, e^{-i\omega t'}\hat{T}_{00}(\omega,\mathbf{x}')\right]\Bigg|_{t'=t-r}$$

$$\Leftrightarrow \bar{h}_{jk}(t,\mathbf{x}) = \frac{2G}{r}\frac{d^2}{dt'^2}\Bigg[\overbrace{\int d^3x'\, x'_jx'_k\, T_{00}(t',\mathbf{x}')}^{q_{jk}(t')/3}\Bigg]\Bigg|_{t'=t-r} =: \frac{2G}{3r}\ddot{q}_{jk}(t')\Big|_{t'=t-r}, \quad r = \|\mathbf{x}\|$$

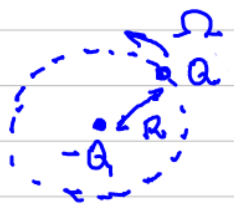
## In electromagnetism

Charge conservation $\Rightarrow$ Absence of monopole radiation

No current-density conservation $\Rightarrow$ Dipole radiation

$$A_j \propto \frac{1}{r}\frac{d}{dt'}\left(\underbrace{\int d^3x'\, \rho_e(t',x')\, x_j}_{p_j(t')}\right)\Bigg|_{t'=t-r}$$

Radiated Power $\sim (\partial_0 A_j)^2 \cdot 4\pi r^2 \sim \ddot{p}^2$




Radiated Power $\sim \frac{k}{c^3} Q^2 \Omega^2 (\Omega L)^2$

$$\frac{\Delta \mathcal{E}}{\mathcal{E}}\Big|_{\text{per circle}} \sim \left(\frac{\Omega L}{c}\right)^3$$

## In gravity

Mass-energy conservation $\Rightarrow$ Absence of monopole radiation

Momentum conservation $\Rightarrow$ Absence of dipole radiation

No higher-moment cons. $\Rightarrow$ Quadrupole radiation

$$\bar{h}_{ij} \propto \frac{G}{r}\ddot{q}_{ij}(t')\Big|_{t'=t-r}$$

Radiated Power $\sim \frac{1}{G}(\partial_0 \bar{h}_{ij})^2 4\pi r^2 \sim G\,(\dddot{q})^2$


Radiated Power $\sim \frac{G}{c^5} M^2 \Omega^2 (\Omega L)^4$

$$\frac{\Delta \mathcal{E}}{\mathcal{E}}\Big|_{\text{per circle}} \sim \left(\frac{\Omega L}{c}\right)^5$$

**Exercício:** Estime qual a ordem de grandeza na mudança do raio da órbita da Lua em torno da Terra ao longo de toda a história do sistema solar ($\sim 10$ bilhões de anos) se essa mudança fosse exclusivamente devida à emissão de ondas gravitacionais pelo sistema Terra-Lua

**Exercício:** Num sistema binário com estrelas com massa da ordem da do Nosso Sol, estime o tamanho típico da órbita desse sistema e de seu período para que, devido à emissão de ondas gravitacionais, houvesse uma variação da ordem de milissegundos em seu período ao longo de 1 década (terrestre) de observação.