PSI 5794 - Matrix Analysis — Exam 3 — December 12, 2021

Prof. Cassio Guimaraes Lopes

This exam covers the Homeworks from Lectures 9, 10 and 11. The topics are: Vector and Matrices norms, QR decomposition and the SVD decomposition.

1. Let $A_{N\times N}$ be a matrix with complex entries. If $\|A^*A - AA^*\|_2 = 0$, what can be said about matrix A? Show your derivations to justify your answer.

2. Show that the matrix $p-$norms for a square matrix $A_{M\times M}$, with $p = 1$ and $p = \infty$, may be obtained from

   (a) $\|A\|_1 = \max_{j=1,M}\left(\sum_{i=1}^{M} |a_{ij}|\right)$;

   (b) $\|A\|_\infty = \max_{i=1,M}\left(\sum_{j=1}^{M} |a_{ij}|\right)$.

3. Consider an orthonormal vector $q_1 \in \mathbb{C}^3$. Create an orthonormal basis from q_1 via

   (a) Gram-Schmidt;

   (b) Householder matrices (Hint: Laub).

4. Consider the Frobenious norm $\|\cdot\|_F = \left(\sum_{i,j} |a_{ij}|^2\right)^{1/2}$ for a square matrix $A_{N\times N}$.

   (a) Show that it can be calculated from $\|A\|_F = \left(\mathrm{Tr}(A^*A)\right)^{1/2} = \left(\mathrm{Tr}(AA^*)\right)^{1/2}$;

   (b) Prove that $\|\cdot\|_F$ is, indeed, a consistent norm.

5. Create a *numeric* 3×3 matrix A with integer entries. Propagate quantities as norms and square roots literally (i.e., do not approximate: use $\|a\|$ and $\sqrt{b}$ whenever necessary). In other words, you are not to use the decimal point.

   (a) Find its QR decomposition via Householder transformations;

   (b) Find its QR decomposition via Givens transformations;

   (c) Compute four iterations of the QR algorithm to find the spectrum of A. Are four iterations enough? Comment.

6. Create an inconsistent linear system of equations of the form $Ax = b$, in which A is a 4×3 matrix with column rank $r = 2$ and integer entries; and $b \in \mathbb{Z}^4$.

   (a) Find a least-squares approximation for x via QR decomposition;

   (b) Find a least-squares approximation for x via SVD decomposition.

7. Find the SVD decomposition of the following matrix. Show all you calculations.

$$A = \begin{bmatrix} -1 & 2 & 1 \\ 2 & -4 & -2 \\ -2 & 4 & 2 \\ -1 & 2 & 1 \end{bmatrix}.$$