PSI 5794 - Matrix Analysis

NOV/25/2021
LEC 09

7. Vectors and Matrices Norms

7.1. Vector Norms

DEF1: norm is a function $\|\cdot\|: V \to \mathbb{R}$ defined over a vec space $(V, \mathbb{F})$ that must obey some properties ($\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$)

1) $\|X\| \geq 0$ Non-negative
2) $\|X\| = 0$ iff $X = 0$ positive*
3) $\|cX\| = |c|\,\|X\| \;\forall c \in \mathbb{F}$ homogeneous
4) $\|X+Y\| \leq \|X\| + \|Y\|$ Δ-inequality

* In semi-norms, property (2) is not required: some non-zero vecs are allowed to have zero norms.

Remarks 1) A vec space equipped with a norm is called a normed vector space; it has a measure of size, of distance between vectors

2) the selected function $\|\cdot\|$ should reflect the nature of the problem under study, e.g., $\|X\|_1$ versus $\|X\|_2$

3) In engineering & Applied sciences is also paramount how "easy" is to compute the norm.

7.2: Vector norms via inner products

A very typical norm is that induced (derived) from inner products.

An inner product is a bilinear function of two vectors. Bilinear: linear in each of the two vector arguments.

Def 2: An inner product is a bilinear function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ defined over a vec space $(V, \mathbb{F})$. For all $x, y, z \in V$, the function must obey $\qquad \mathbb{F} = \mathbb{C}$ or $\mathbb{R}$

1) $\langle x, x \rangle \geq 0$ Non-negative

2) $\langle x, x \rangle = 0$ iff $x = 0$ positive

3) $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ Additive

4) $\langle cx, y \rangle = c\langle x, y \rangle \quad \forall\, c \in \mathbb{F}$ Homogeneous

5) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (Conjugate) symmetry

Remarks: 1) A vec space equipped with an inner-product is called a inner product space. It allows for the introduction of formal definitions of intuitive geometric notions, such as lengths, angles and orthogonality ($\langle \cdot, \cdot \rangle = 0$).

2) A norm is a function of one vector argument; an inner product is a function of two vector arguments.

Thm: Cauchy – Schwarz Inequality proof HoFm 261

if $\langle x, y\rangle$ is an inner product over $(V, \mathbb{F})$

then
$$\boxed{|\langle x, y\rangle|^2 \leq \langle x, x\rangle \langle y, y\rangle} \quad \forall x, y \in V$$

with equality iff x and y are L.D., i.e.,

$\exists\, \alpha \in \mathbb{F} \mid y = \alpha x$. $(\mathbb{F} = \mathbb{C}$ or $\mathbb{R})$

Corollary: if $\langle \cdot, \cdot\rangle$ is a vector inner product in V, then
$$\boxed{\|x\| = (\langle x, x\rangle)^{1/2}}$$

is a norm induced by the inner product.

Proof: check all the properties for a norm:

(1) and (2) follow from the definition of inner prod.

(3) $\|cx\| = |c|\,\|x\| \quad \forall c \in \mathbb{F}$

$$\|x\| = (\langle x, x\rangle)^{1/2} \Rightarrow \|cx\| = (\langle cx, cx\rangle)^{1/2} = (c\langle x, cx\rangle)^{1/2} =$$

$$= (c\overline{\langle cx, x\rangle})^{1/2} = (c\overline{c\langle x, x\rangle})^{1/2} = (|c|^2\langle x, x\rangle)^{1/2} = |c|\,\|x\|$$

(4) $\|x + y\| \leq \|x\| + \|y\|$ (calculate $\|x+y\|^2$ then apply Cauchy-Schwarz)

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x+y\rangle + \langle y, x+y\rangle = \overline{\langle x+y, x\rangle} + \overline{\langle x+y, y\rangle}$$

$$= \overline{\langle x, x\rangle + \langle y, x\rangle} + \overline{\langle x, y\rangle + \langle y, y\rangle} = \langle x, x\rangle + \langle x, y\rangle + \overline{\langle x, y\rangle} + \langle y, y\rangle$$

$$= \langle x, x\rangle + 2\,\mathrm{Re}\{\langle x, y\rangle\} + \langle y, y\rangle = \|x\|^2 + 2\,\mathrm{Re}\{\cdot\} + \|y\|^2.$$

$$\left|\|x+y\|^2\right| = \left|\|x\|^2 + 2\,\mathrm{Re}(\cdot) + \|y\|^2\right| \leq \|x\|^2 + \|y\|^2 + |2\,\mathrm{Re}(\cdot)|$$

$$\leq \|x\|^2 + \|y\|^2 + 2|\langle x, y\rangle| \leq \|x\|^2 + \|y\|^2 + 2\langle x, x\rangle\langle y, y\rangle$$

(4) $\|x+y\| \leq \|x\| + \|y\|$ (Calculate $\|x+y\|^2$ then Cauchy-Schwarz)

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x+y\rangle + \langle y, x+y\rangle =$$

$$= \overline{\langle x+y, x\rangle} + \overline{\langle x+y, y\rangle} = \overline{\langle x,x\rangle + \langle y,x\rangle} + \overline{\langle x,y\rangle + \langle y,y\rangle}$$

$$= \|x\|^2 + \overline{\langle y,x\rangle} + \overline{\langle x,y\rangle} + \|y\|^2 = \|x\|^2 + \langle x,y\rangle + \overline{\langle x,y\rangle} + \|y\|^2$$

$$\cancel{= \|x\|^2 + |2\,\mathrm{Re}(\langle x,y\rangle)| + \|y\|^2} \quad = \|x\|^2 + 2\,\mathrm{Re}(\langle x,y\rangle) + \|y\|^2.$$

$$\left| \|x+y\|^2 \right| \leq \|x\|^2 + |2\,\mathrm{Re}(\cdot)| + \|y\|^2 .$$

but the result of $\langle x,y\rangle$ is a complex member

$\langle x,y\rangle = a + jb \Rightarrow |\langle x,y\rangle| = (a^2+b^2)^{1/2}$; and

$|\mathrm{Re}(\langle x,y\rangle)| = |\mathrm{Re}(a+jb)| = |a| = (a^2)^{1/2}$, thus

$|\langle x,y\rangle| \geq |\mathrm{Re}(\langle x,y\rangle)|$, so it follows that

$$\|x\|^2 + |2\,\mathrm{Re}(\cdot)| + \|y\|^2 \leq \|x\|^2 + 2|\langle x,y\rangle| + \|y\|^2$$

$$\leq \|x\|^2 + 2(\langle x,x\rangle\langle y,y\rangle)^{1/2} + \|y\|^2 = \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$

$$= (\|x\| + \|y\|)^2. \text{ then}$$

$$\left| \|x+y\|^2 \right| = \|x+y\|^2 \leq (\|x\| + \|y\|)^2 \Rightarrow \left(\|x+y\|^2\right)^{1/2} \leq \left[(\|x\|+\|y\|)^2\right]^{1/2}$$

$$\|x+y\| \leq \Big|\underbrace{\|x\|}_{\geq 0} + \underbrace{\|y\|}_{\geq 0}\Big| = \|x\| + \|y\|$$

$\therefore$ $\|x\| = \langle x,x\rangle^{1/2}$ is a vector norm.

$\therefore$ $\boxed{\|x+y\| \leq \|x\| + \|y\|}$

Examples

1) $\|X\| = (X^T X)^{1/2}$ $\mathbb{R}$ euclidean product

2) $\|X\| = (X^* X)^{1/2}$ $\mathbb{C}$ euclidean product

3) $\|X\|_Q \triangleq (X^* Q X)^{1/2}$, $Q = Q^*$, $Q > 0$ Weighted Norm

a) $Q = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$

ok, obeys the properties for norms

b) $Q = \begin{bmatrix} 2 & 0 \\ 0 & -1/2 \end{bmatrix}$

if $p = \begin{bmatrix} a \\ b \end{bmatrix}$ then

$\|p\|_Q < 0$ for $|a| < \frac{|b|}{2}$

Not a norm for this Q (Q is indefinite)

c) $Q = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

Q is rank deffic.

$N(Q)$ nontrivial

$\exists\, X \neq 0 \mid QX = 0$

For $X \in N(Q)$ and $X \neq 0$,

$\|X\| = 0$ for $X \neq 0$

Not a norm.

7.3 Vector p-norms (l_p norms)

they are a family of norms that calculate the value of the norm directly from the vector entries, in terms of a free to choose parameter p that is application dependent

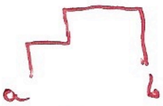
$$\boxed{\|X\|_p \triangleq \left(\sum_{k=1}^{N} |x(k)|^p \right)^{1/p}}$$

$p \sim$ weighting

$X \in \mathbb{C}$

Typical cases

1) $\|x\|_1 = \sum_{k=1}^{N} |x(k)|$ — Manhattan norm (l_1) used in compressive sensing



2) $\|x\|_2 = \left(\sum_{k=1}^{N} |x(k)|^2 \right)^{1/2}$ — Euclidean norm (l_2) minimizes energy (math tractability)

3) $\|x\|_\infty = \lim_{p \to \infty} \|x\|_p = \lim_{p \to \infty} \left(\sum_k |x(k)|^p \right)^{1/p} = \max_k |x(k)|$

l_∞ norm: Outliers removal, robust estimation / Control
Chebyshev criterion

Proof of (3): a) reorder $x \to \bar{x}$, $\bar{x}_1 = \max_k |x(k)|$ $\bar{x}_N = \min_k |x(k)|$

Note that $\|x\|_p = \|\bar{x}\|_p$

b) $$\|\bar{x}\|_p = \Big(\underbrace{|\bar{x}_1|^p + |\bar{x}_2|^p + \cdots + |\bar{x}_l|^p}_{\text{Assume } l \text{ largest are equal}} + \cdots + |\bar{x}_N|^p \Big)^{1/p}$$

$$= \left(|\bar{x}_1|^p \right)^{1/p} \left(l + \left| \frac{\bar{x}_{k+1}}{\bar{x}_1} \right|^p + \cdots + \left| \frac{\bar{x}_N}{\bar{x}_1} \right|^p \right)^{1/p}$$

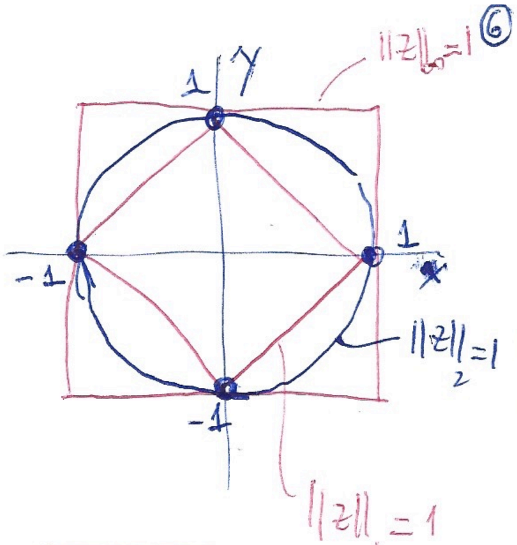
$= |\bar{x}_1| \, (\cdot)^{1/p}$. Now

$$\lim_{p \to \infty} \|x\|_p = |\bar{x}_1| \lim_{p \to \infty} \left(l + \underbrace{\left| \frac{\bar{x}_{k+1}}{\bar{x}_1} \right|^p}_{<1} + \cdots + \underbrace{\left| \frac{\bar{x}_N}{\bar{x}_1} \right|^p}_{<1} \right)^{1/p}$$

$$= |\bar{x}_1| \lim_{p \to \infty} (l)^{1/p} = |\bar{x}_1| = \max_k |x(k)| \quad //. \qquad l^{1/p} \to 1$$

Geometric Interpretation

$z = \begin{bmatrix} x \\ y \end{bmatrix}$



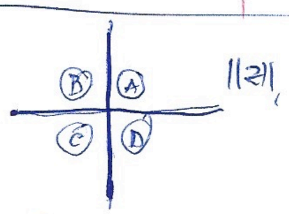
Note $\|z\|_p$ increases monotonically with $|x|$ and $|y|$

root loci

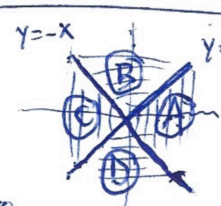
$\|z\|_1 = 1 \iff |x| + |y| = 1$

$\|z\|_2 = 1 \iff (|x|^2 + |y|^2)^{1/2} = 1$

$\|z\|_\infty = 1 \iff \max\{|x|, |y|\} = 1$



Ⓐ: $\cancel{x}\ x + y = 1$
Ⓑ: $-x + y = 1$
Ⓒ: $-x - y = 1$
Ⓓ: $x - y = 1$



Ⓐ: $\max\{|x|, |y|\} = |x| = x = 1$
Ⓑ: $\max\{|x|, |y|\} = |y| = y = 1$
Ⓒ: " " $= |x| = -x = 1$
Ⓓ: $= |y| = -y = 1$

Hölder's Inequality

Let $x, y \in \mathbb{C}^n (\mathbb{R}^n)$ then

$$|\langle x, y \rangle| \le \|x\|_p \|\cancel{x}\|_q \qquad \frac{1}{p} + \frac{1}{q} = 1$$

Ex: Signal scaling in (fixed point) cascading digital filters (bi-quads)

Minkowski's Inequality

For $p \ge 1$, $x, y \in \mathbb{C}^n (\mathbb{R}^n)$

Generalization of Δ-ineq.

$$\left[\sum_{i=1}^{n} |x(i) + y(i)|^p \right]^{1/p} \le \left(\sum_i |x(i)|^p \right)^{1/p} + \left(\sum_i |y(i)|^p \right)^{1/p}$$

$$\|x + y\|_p \le \|x\|_p + \|y\|_p$$

7.4. CONVERGENCE OF SEQUENCES

Vectors (and matrix) norms are useful to study iterative procedures which generate a sequence of vectors $\{x_i\}$ as

$$x_i = x_{i-1} + z_i$$

Example: iterative lin sys solutions, Adaptive filters, etc.

Def: A sequence $\{x_i\}$ is said to converge to a vector $x^0 \in V$ with respect to a norm $\|\cdot\|$ iff $\|x_i - x^0\| \to 0$ as $i \to \infty$.

Question: How to choose / construct a proper norm (among so many)?

Thm: All norms on $\mathbb{C}^N$ ($\mathbb{R}^N$) are equivalent, that is, for any two arbitrary norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ there exist constants C_m and C_M (possibly N-dependent) such that

$$\boxed{C_m \|x\|_\alpha \leq \|x\|_\beta \leq C_M \|x\|_\alpha} \quad \forall x \in \mathbb{C}^N$$

If we choose ll.ll_a and ll.ll_a diverges for the sequence of vecs at hand, this inequality says that picking another norm ll.ll_b does not lead to a different conclusion: Cm*ll.ll_a will push ll.ll_b from below to infinity, together with CM*ll.ll_a, i.e., ll.ll_b is sandwiched by norm ll.ll_a. Another point: if ll.ll_b converges, then the ineq says that Cm*ll.ll_a also converges since it is smaller; therefore, CM*ll.ll_a will converge together with Cm*ll.ll_a with norm ll.ll_b in between (again the sandwhich). We can find inequalities for ll.ll_a in between a sandwhich made with other constants and norm ll.ll_b.

<u>Examples</u>

1) $\|X\|_2 \leq \|X\|_1 \leq \sqrt{N}\,\|X\|_2$

a) $\|X\|_2^2 = \sum_\ell |X_\ell|^2 = |X_1|^2 + \cdots + |X_N|^2$

$\|X\|_1^2 = \left(\sum_\ell |X_\ell|\right)^2 = \left(|X_1| + \cdots + |X_N|\right)^2$

$\Big\rangle$ $\|X\|_2^2 \leq \|X\|_1^2$ or $\|X\|_2 \leq \|X\|_1$

b) Let $z_\ell = \begin{cases} 1 & \text{if } X_\ell \geq 0 \\ -1 & \text{if } X_\ell < 0 \end{cases}$, then $\|X\|_1 = \sum_\ell |X_\ell| = \sum_\ell z_\ell X_\ell$

$= z^* X$

Hölder's ineq: $|z^* X| \leq \|z\|_2 \, \|X\|_2$.

$\|z\|_2^2 = \sum_{\ell=1}^{N} (\pm 1)^2 = N \therefore \|z\|_2 = \sqrt{N}$. thus

$|\,\|X\|_1| = |z^* X| \leq \|z\|_2 \, \|X\|_2 = \sqrt{N}\,\|X\|_2$

or $\|X_1\| \leq \sqrt{N}\,\|X\|_2$.

From (a) and (b) follows (1).

2) $\|X\|_\infty \leq \|X\|_1 \leq N\,\|X\|_\infty$

a) For some ℓ we have $|X(\ell)| \geqslant |X(k)| \;\; \forall\, \ell \neq k$

then $\|X\|_1 = \cancel{\quad} |X_1| + |X_2| + \cdots + |X_\ell| + \cdots + |X_N|$

$$= \underbrace{|X_\ell|}_{\geqslant 0} + \underbrace{|X_1| + \cdots + |X_N|}_{\geqslant 0} \geqslant |X_\ell| = \max_{\ell} |X_\ell| = \|X\|_\infty$$

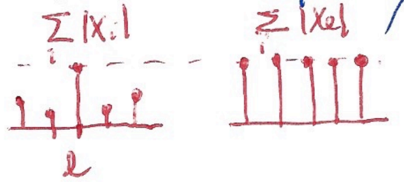
$\therefore \|X\|_1 \geqslant \|X\|_\infty$

b) $\|X\|_1 = |X_1| + \cdots + |X_\ell| + \cdots + |X_N|$ ($|X_\ell| =$ max entry in X)

$$\leq |X_\ell| + \cdots + |X_\ell| + \cdots + |X_\ell|$$

$$= N \cdot |X_\ell| = N \max_i |X_i|$$

$$= N\,\|X\|_\infty$$



$\therefore \|X\|_1 \leq N\,\|X\|_\infty$

from (a) and (b) follows (2).

in general, for ℓ_p vector norms

$$\boxed{\|X\|_p \leq \|X\|_r \leq N^{\left(\frac{1}{r} - \frac{1}{p}\right)} \|X\|_p}$$

7.5. Matrix Norms

~~We can try view a matrix~~

One possibility to construct a matrix norm $|||\cdot|||$ is by using what is already known: vector norms. For instance, by seeing a matrix A_{MN} as an abstract vector in the $M \cdot N \times 1$ vector space via $\underset{MN\times 1}{a} = \text{vec}(\underset{MN}{A})$, then applying any of the previous vec norms for, i.e. $|||A||| = \|a\|$. One such an example is the Frobenius norm $|||A|||_F = \|a\|_2$, or the l_2 norm for matrices. In this approach we see a matrix, in a sense, as a "fat vector".

However, matrix products are common place so it is instrumental to include a rule specifically conceived to account for the "size" of AB in terms of the individual "sizes" of A and B (the sub multiplicative axiom).

Def: A function $\|\cdot\| : \mathbb{F}^{m \times n} \to \mathbb{R}$ is a matrix norm if it satisfies the following properties

1. $\|A\| \geq 0$
2. $\|A\| = 0$ iff $A = 0$
3. $\|cA\| = |c| \, \|A\|$
4. $\|A+B\| \leq \|A\| + \|B\|$
5. $\|AB\| \leq \|A\| \, \|B\|$

Generalized matrix norms (1–4)

Consistent matrix norms (1–5)

Remarks: 1) Some authors consider only (1–4) as a matrix norm, and include (5) as an "extra" feature that makes the extended norm a consistent norm; consistent w.r.t. the matrix product;

2) $\|A^2\| = \|A\,A\| \leq \|A\| \, \|A\| = \|A\|^2$

3) $\|I\| = \|A A^{-1}\| \leq \|A\| \, \|A^{-1}\|$ or $\|A^{-1}\| \geq \dfrac{\|I\|}{\|A\|}$

4) || I || = || I x I || <= || I || || I ||, then || I ||>= 1

E.g., || I_2 ||_F = || [1 0; 0 1] ||_F = || a ||_F = sqrt(2)>1
where a = vec(I_2) = [1 0 0 1]^T

Frobenius Norm (matrix Euclidean norm)

$$\|A\|_F \triangleq \left(\sum_{i,j} |a_{ij}|^2\right)^{1/2} = \left[\mathrm{Tr}(A^*A)\right]^{1/2} = \left[\mathrm{Tr}(AA^*)\right]^{1/2}$$

or, as before, $\|A\|_F = \|a\|_2$, with $a = \mathrm{vec}(A)$.

a) Important for theoretical and practical reasons: it is easy to compute (entry wise)

b) Unitarily invariant. Let Q : $Q^*Q = I$ ($Q^TQ = I$)

$$\|QA\|_F^2 = \mathrm{Tr}\left((QA)^*(QA)\right) = \mathrm{Tr}(A^*Q^*QA) =$$
$$= \mathrm{Tr}(A^*A) = \|A\|_F^2$$

c) Proof that $\|A\|_F$ is a (consistent) matrix norm: check properties (1)–(5) from the definition

7.6: INDUCED MATRIX NORMS (Also called "Operator Norms")

Def: A matrix norm $|||\cdot|||$ induced by a vec norm $||\cdot||$ is defined as

$$\boxed{|||A||| = \max_{x \neq 0} \frac{||Ax||}{||x||}} \qquad (I)$$

Or, similarly

$$\boxed{|||A||| = \max_{||x||=1} ||Ax||} \qquad (II)$$

thm: the function (I) (or (II)) is a matrix norm in $\mathbb{F}^{N \times N}$ satisfying

a) $||Ax|| \leq |||A||| \, ||x||$ (subordinated to vec norm $||\cdot||$)

b) $|||I||| = 1$

Remark: some authors say that norms following (a) above are compatible, or vec norm $||\cdot||$ is compatible with matrix norm $|||\cdot|||$

<u>Proof</u>: check properties (1)–(5).

(1–3): exercise

(4): $\|A+B\| \le \|A\| + \|B\|$

$$\|A+B\| = \max_{x\neq 0} \frac{\|(A+B)x\|}{\|x\|} = \max_{x\neq 0} \frac{\|Ax + Bx\|}{\|x\|}$$

$$= \max_{x\neq 0}\left(\frac{\|Ax\|}{\|x\|} + \frac{\|Bx\|}{\|x\|}\right) \le \underbrace{\max_{x\neq 0} \frac{\|Ax\|}{\|x\|}}_{\text{separate maximizations}} + \underbrace{\max_{x\neq 0} \frac{\|Bx\|}{\|x\|}}$$

$$= \|A\| + \|B\|.$$

(5) $\|AB\| \le \|A\|\,\|B\|$

$$\max_{x\neq 0} \frac{\|ABx\|}{\|x\|} = \max_{\substack{x\neq 0\\ Bx\neq 0}} \frac{\|ABx\|}{\|x\|} = \max_{\substack{x\neq 0\\ Bx\neq 0}} \frac{\|ABx\|}{\|x\|}\,\frac{\|Bx\|}{\|Bx\|}$$

$$= \max_{\substack{x\neq 0\\ Bx\neq 0}} \frac{\|ABx\|}{\|Bx\|}\,\frac{\|Bx\|}{\|x\|} \le \max_{\underbrace{Bx}_{z}\neq 0} \frac{\|A\underbrace{Bx}_{z}\|}{\|\underbrace{Bx}_{z}\|}\ \max_{x\neq 0} \frac{\|Bx\|}{\|x\|}$$

$$= \|A\|\,\|B\|.$$

a) $|||A||| = \max_{x\neq 0} \frac{\|Ax\|}{\|x\|} = \frac{\|Ax^0\|}{\|x^0\|}$, for some x^0 that maximizes $\frac{\|Ax\|}{\|x\|}$.

then, for all other x, $|||A||| \le \frac{\|Ax\|}{\|x\|}$, or $\|Ax\| \le |||A|||\,\|x\|$

b) $|||I||| = 1$. $|||I||| = \max_{x\neq 0} \frac{\|Ix\|}{\|x\|} = \max_{x\neq 0} \frac{\|x\|}{\|x\|} = 1.$

7.7: INDUCED Matrix p-norms

there are a family of <u>matrix norms</u> induced by the $\|\cdot\|_p$ vector norms,

$$|||A|||_p \triangleq \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} .$$

Depending on the selected p, the resulting norm $|||A|||$ may be easily calculated directly from the entries of A.

<u>Examples</u>

1) $|||A|||_1 = \max_{j=1,N} \left(\sum_{i=1}^{M} |a_{ij}| \right)$ max col sum

2) $|||A|||_\infty = \max_{i=1,M} \left(\sum_{j=1}^{N} |a_{ij}| \right)$ max row sum

3) $|||A|||_2 = \max_k \left(\lambda_k (A^* A) \right)^{1/2} = \max_k \left(\lambda_k (A A^*) \right)^{1/2}$

$\triangleq \sigma_{MAX}(A)$, the maximum singular value of A $(\neq \rho(A))$

a) it is the "spectral norm", and it is anomalous, in the sense it is not directly extracted from entries of A, as $|||A|||_1$ and $|||A|||_\infty$. Not so easy to calculate

b) Unitarily invariant: $|||QA|||_2 = \max_k [\lambda_k ((QA)^*(QA))]$
$= \max_k [\lambda_k (A^* Q^* Q A)] = |||A|||_2$. $Q^*Q = I$

Example

$$A = \begin{array}{ccc|c} \begin{bmatrix} 1 & 2 & 3 \\ -9 & 1 & 8 \\ 0 & 0 & 6 \end{bmatrix} & & & \begin{array}{c} 6 \\ 18 \\ 6 \end{array} \end{array} \Bigg\} \text{ max row sum } |||A|||_\infty = 18$$

$$\underbrace{10 \quad 3 \quad 17}_{\text{max col sum } |||A|||_1 = 17.}$$

7.8. Matrix Convergence

As with vectors, we can think of a matrix sequence $\{A_k\}$ and explore matrix norms to study the sequence convergence. Recall that the spectral radius of a matrix A is given by

$$\rho(A) = \max_k \{ |\lambda_k|, \lambda_k \in \lambda(A) \}.$$

thm: If $|||\cdot|||$ is any matrix norm, then

$$\rho(A) \le |||A||| \quad , \text{ for all } A \in \mathbb{F}^{N\times N}.$$

proof: if $Ax_i = \lambda_i x_i$, then form $X = [x_i \, x_i \cdots x_i]$, so that $|\lambda_i| \, |||X||| =$ (next page)

$$|\lambda_i| \, ||| X ||| = ||| \, |\lambda_i| X ||| = ||| \lambda_i X |||$$

$$= ||| \lambda_i [x_i \; x_i \cdots x_i] ||| = ||| [\lambda_i x_i \; \lambda_i x_i \cdots \lambda_i x_i] |||$$

$$= ||| [A x_i \; A x_i \cdots A x_i] ||| = ||| A X ||| \leq ||| A ||| \, ||| X |||,$$

$$|\lambda_i| \, \cancel{||| X |||} \leq ||| A ||| \, \cancel{||| X |||}$$, since X groups the ~~eig.~~ i^{th} eivec of A, which is never zero. ($||| X ||| \neq 0$)

thus

$$|\lambda_i| \leq ||| A ||| \quad \forall i$$

~~and~~ so that it must hold for ~~that~~ the i that returns the $\max_i |\lambda_i|$, that is $\rho(A)$:

$$\boxed{\rho(A) \leq ||| A |||}$$

<u>thm:</u> Let $A \in \mathbb{F}^{n \times n}$. If there is a norm $|||\cdot|||$ such that $|||A||| < 1$, then

$$\lim_{k \to \infty} A^k = 0$$

that is, all entries of A^k go to zero as $k \to \infty$.

<u>proof:</u> $|||A^k||| = |||A^{k-1}A||| \le |||A^{k-1}|||\,|||A|||$

$$\le |||A^{k-2}|||\,|||A|||^2 \le |||A^{k-3}|||\,|||A|||^3$$

$$\vdots$$

$$\le |||A|||^k, \quad \text{or} \quad |||A^k||| \le |||A|||^k.$$

Since $|||A||| < 1$, $|||A|||^k \to 0$ as $k \to \infty$.

~~From the definition of matrix norms, this implies that~~. If $|||A|||^k \to 0$, then $|||A^k||| \to 0$.

from the definition of matrix norms, if $|||A^k||| \to 0$ as $k \to \infty$, then

$$A^k \to 0 \text{ as } k \to \infty.$$

~~Example:~~

or

$$f(A) = \sum_{k=0}^{\infty} c_k A^k,$$ for a scalar analytic function $f(\cdot)$.

We can quickly test if this matrix series converges by checking how

$|c_k| \, |||A^k|||$ evolves with k.

that is, $|||A||| < 1$ provides a necessary condition for convergence.

~~Example~~ ~~$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$ is well defined only if $|||A||| < 1$. that is,~~

$|||A^k||| \leq |||A|||^k \rightarrow$

For instance, if we can find $A = P \Lambda P^{-1}$, then it is easy to check for convergence.

What if A is defective? We can resort to matrix norms to study convergence, or use the Jordan form (previous Lecture).

7.7. CONDITION NUMBER $K(A)$ (SQUARE MATRICES)

Consider a consistent lin sys $Ax=b$ and assume $\exists\, A^{-1}$. Then consider a perturbed version of b (finite precision, noise) $b \to b+\delta b$

Question: How is the ~~exac~~ exact (UNKNOWN) solution $x = A^{-1}b$ affected by δb?

Since $b \to b + \underbrace{\delta b}_{\text{error vector}}$ then $x \to x + \underbrace{\delta x}_{\text{another error vector}} \triangleq \hat{x}$

then, the perturbed lin sys is

$$A\hat{x} = b + \delta b \iff A(x+\delta x) = b + \delta b$$

$$\underbrace{\cancel{Ax}}_{= \text{ due to consistency}} + A\delta x = \cancel{b} + \delta b \quad \therefore \quad A\delta x = \delta b$$

$$\boxed{\delta x = A^{-1}\delta b}$$

Similar procedures for $A \to A + \delta A$ are available

So that $\boxed{\|\delta x\| \le \|A^{-1}\|\,\|\delta b\|}$ (1)

From the original system $\|b\| \le \|A\|\,\|x\|$

or $\boxed{\dfrac{1}{\|x\|} \le \dfrac{\|A\|}{\|b\|}}$ (2) $\overset{(2)\times\|\delta x\|}{\Longrightarrow}$ $\dfrac{\|\delta x\|}{\|x\|} \le \dfrac{\|\delta x\|\,\|A\|}{\|b\|}$ (1)

$$\frac{\|\delta x\|}{\|x\|} \le \frac{\|A^{-1}\|\,\|\delta b\|\,\|A\|}{\|b\|} \quad \text{or} \quad \boxed{\underbrace{\frac{\|\delta x\|}{\|x\|}}_{\text{relative error on } x} \le \underbrace{\|A\|\,\|A^{-1}\|}_{\triangleq K(A)} \underbrace{\frac{\|\delta b\|}{\|b\|}}_{\text{relative error on } b}}.$$

$\boxed{K(A) \triangleq \|A\|\,\|A^{-1}\|}$ is the condition number.

$K(A) \geq 1$ and it quantifies how sensitive a lin sys is with respect to perturbations (in b, in A, both).

if $K(A)$ is small, then the lin sys can be solved reliably. Large $K(A)$: problems (ill-conditioning)