Integrais de seno e cosseno

Vamos precisar dos seguintes resultados:

inteiro positivo

(a) $$\frac{a}{n}\sum_{k=1}^{n}\cos\left(\frac{ka}{n}\right) < \operatorname{sen} a < \frac{a}{n}\sum_{k=0}^{n-1}\cos\left(\frac{ka}{n}\right), \quad n\geq 1 \quad a\in\left(0,\frac{\pi}{2}\right]$$

(b) Se $I\in\mathbb{R}$ satisfaz

$$\frac{b-a}{n}\sum_{k=1}^{n} f(x_k) < I < \frac{b-a}{n}\sum_{k=0}^{n-1} f(x_k) \qquad \forall n\geq 1$$

então $$\int_a^b f(x)\,dx = I.$$

$f: [a,b]\longmapsto\mathbb{R}$ função decrescente.

OBS: (i) $x_k = k\frac{a}{n}$

(ii) Como cosseno satisfaz (a) temos $\int_0^a \cos x\,dx = \operatorname{sen} a$ por (b).

Teo. Sejam $\operatorname{sen} x$ e $\cos x$ as funções satisfazendo as propriedades fundamentais 1-4. Então $\forall a \in \mathbb{R}$ temos

$$\int_0^a \cos x \, dx = \operatorname{sen} a \quad (*) \qquad \text{e} \qquad \int_0^a \operatorname{sen} x \, dx = 1 - \cos a \quad (*)(*).$$

dem. Inicialmente verificamos (*). O caso $a \in [0, \pi/2]$ segue da obs. anterior.

Suponha agora $a \in [-\frac{\pi}{2}, 0]$.

$$\int_0^a \cos x \, dx = (-1) \int_0^{-a} \cos(-x) dx = -\int_0^{-a} \cos x \, dx = -\operatorname{sen}(-a) = \operatorname{sen} a.$$

Note que cosseno é par e $-a \in [0, \pi/2]$.

Suponha agora $a \in [\pi/2, \frac{3\pi}{2}] \rightsquigarrow -\frac{\pi}{2} \leq a - \pi \leq \frac{\pi}{2}$

$$\int_0^a \cos x \, dx = \int_0^{\pi/2} \cos x \, dx + \int_{\pi/2}^a \cos x \, dx = \operatorname{sen} \frac{\pi}{2} + \int_{-\pi/2}^{a-\pi} \cos(x + \pi) dx$$

$$\int_0^a \cos x\,dx = 1 + \int_{-\frac{\pi}{2}}^{a-\pi} \cos(x+\pi)\,dx = 1 - \int_{-\frac{\pi}{2}}^{a-\pi} \cos x\,dx = 1 - \int_0^{a-\pi} \cos x\,dx - \int_{-\frac{\pi}{2}}^{0} \cos x\,dx$$

$$= \cancel{1} - \operatorname{sen}(a-\pi) - \cancel{\operatorname{sen}}\left(\frac{\pi}{2}\right) = \operatorname{sen} a$$

já que

$$\int_{-\frac{\pi}{2}}^{0} \cos x\,dx = (-1)\int_{\frac{\pi}{2}}^{0} \cos(-x)\,dx = \int_0^{\pi/2} \cos x\,dx \quad \text{e}$$

$$\operatorname{sen}(a-\pi) = \operatorname{sen} a \underbrace{\cos(\pi)}_{-1} - \cos a \underbrace{\operatorname{sen}(\pi)}_{0} = -\operatorname{sen} a.$$

Dessa forma $\displaystyle\int_0^a \cos x\,dx = \operatorname{sen} a \quad \forall\, a \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$

Agora usamos que cosseno e seno são periódicos de período 2π.

$a \in \left(\frac{3\pi}{2}, 2\pi\right)$ $\qquad$ $\displaystyle\int_0^a \cos x\,dx = \int_0^{\frac{3\pi}{2}} \cos x\,dx + \int_{\frac{3\pi}{2}}^{a} \cos x\,dx$

$$= \operatorname{sen}\left(\frac{3\pi}{2}\right) + \int_{-\pi/2}^{a-2\pi} \cos(x+2\pi)\,dx = -1 + 1 + \operatorname{sen} a = \operatorname{sen} a$$

já que

$a - 2\pi \in \left(-\frac{\pi}{2}, 0\right)$

$$\int_{-\pi/2}^{a-2\pi} \cos x\,dx = \int_{-\frac{\pi}{2}}^{0} \cos x\,dx + \int_0^{a-2\pi} \cos x\,dx$$

$$= \operatorname{sen}\frac{\pi}{2} + \operatorname{sen}(a-2\pi) = 1 + \operatorname{sen} a.$$

$$\therefore \int_0^a \cos x\,dx = \operatorname{sen} a \qquad \forall\, a \in \left[-\frac{\pi}{2}, 2\pi\right].$$

Mais, $\forall\, a \in \mathbb{R}$ seja k inteiro tq. $|a - 2k\pi| \leq 2\pi$. Spg. $a > 0$ e $a \geq 2k\pi$.

$$\int_0^a \cos x\,dx = \int_0^{2\pi k} \cos x\,dx + \int_{2\pi k}^{a} \cos x\,dx$$

$$= k\int_0^{2\pi} \cos x\,dx + \int_0^{a-2\pi k} \cos(x + 2\pi k)\,dx \qquad a - 2k\pi \in (0, 2\pi)$$

$$= k\,\mathrm{sen}(2\pi) + \int_0^{a-2\pi k} \cos(x)\,dx = k\,\underbrace{\cancel{\mathrm{sen}\,2\pi}}_{0} + \mathrm{sen}(a - 2k\pi)$$

$$= \mathrm{sen}\,a \quad \text{já que}$$

$$\mathrm{sen}(a - 2k\pi) = \mathrm{sen}\,a\,\underbrace{\cancel{\cos(2k\pi)}}_{1} - \cos a\,\underbrace{\cancel{\mathrm{sen}(2k\pi)}}_{0} = \mathrm{sen}\,a \quad \text{e}$$

$$\int_0^{2k\pi} \cos x\,dx = \left(\int_0^{2\pi} + \int_{2\pi}^{4\pi} + \ldots + \int_{2(k-1)\pi}^{2k\pi}\right) \cos x\,dx \overset{\text{Exercício}}{=} k\int_0^{2\pi} \cos x\,dx$$

Agora mostramos $\int_0^a \operatorname{sen} x\,dx = 1 - \cos a$. Para isso usamos $\operatorname{sen}\left(x+\frac{\pi}{2}\right) = \cos x$

$$\int_0^a \operatorname{sen} x\,dx = \int_{-\frac{\pi}{2}}^{a-\pi/2} \operatorname{sen}\left(x+\frac{\pi}{2}\right)dx = \int_{-\frac{\pi}{2}}^{a-\frac{\pi}{2}} \cos x\,dx = \int_{-\frac{\pi}{2}}^{0} \cos x\,dx + \int_0^{a-\pi/2} \cos x\,dx$$

$$= \operatorname{sen}\frac{\pi}{2} + \operatorname{sen}\left(a-\frac{\pi}{2}\right) = 1 - \cos a. \blacksquare$$

Exemplos: 1) $\int_a^b \cos x\,dx = \int_0^b \cos x\,dx + \int_a^0 \cos x\,dx = \operatorname{sen} b - \operatorname{sen} a$

já que $\int_a^0 \cos x\,dx = -\int_0^a \cos x\,dx = -\operatorname{sen} a$. Analogamente obtemos

$$\int_a^b \operatorname{sen} x\,dx = \int_a^0 \operatorname{sen} x\,dx + \int_0^b \operatorname{sen} x\,dx = -(1-\cos a) + 1 - \cos b = \cos a - \cos b.$$

2) $$\int_a^b \cos(cx)\,dx = \frac{1}{c}\int_{ac}^{bc} \cos x\,dx = \frac{1}{c}\,\mathrm{sen}\,x \Big|_{ac}^{bc} = \frac{1}{c}\left(\mathrm{sen}\,(bc) - \mathrm{sen}\,(ac)\right).$$

$$f(x)\Big|_a^b = f(b) - f(a)$$

$$\int_a^b \mathrm{sen}(cx)\,dx = -\frac{1}{c}\cos(x)\Big|_{ac}^{bc} = -\frac{1}{c}\left(\cos(bc) - \cos(ac)\right).$$

3) $$\int_0^a \mathrm{sen}^2 x\,dx = \frac{1}{2}\int_0^a (1-\cos 2x)\,dx = \frac{1}{2}\left(a - \frac{1}{2}\left(\mathrm{sen}\,2a - \cancelto{0}{\mathrm{sen}\,0}\right)\right) = \frac{a}{2} - \frac{\mathrm{sen}\,2a}{4}$$

pois

$$\mathrm{sen}^2 x = \frac{1}{2}\left(1 - \cos 2x\right)$$ já que $$\cos 2x = \cos(x+x) = \cos^2 x - \mathrm{sen}^2 x = 1 - 2\,\mathrm{sen}^2 x$$