PSI 5794 – MATRIX ANALYSIS Prof. Cássio [signature]

FEV/18/2019
LEC #1

0. INTRODUCTION & MOTIVATION

What is a Matrix ?

- A rectangular array of ~~members~~ (NUMBERS)/scalars
  e.g., DIGITAL IMAGE: Array of pixels
- A (Graphic) representation of a LINEAR transformation between two finite dimension vector spaces
  (Coordi)
  the Coordinates of a L.T. given bases for the two vec spaces
  A Graphic interface for a L.T.
- A "vector" in a well defined vector space
  the vec space of L.T.'s

In engineering and in the hard sciences whenever a problem can be posed in matrix form, all the tools and theory of Matrices may be explored to unveil properties and structure that are not evident in the original problem

1) Image processing: Image enhancing, feature extraction image compression

$$B = UAV^*$$

tranf. image ↗ ↑ image

2D-DCT
2D-Haar
2D-FFT

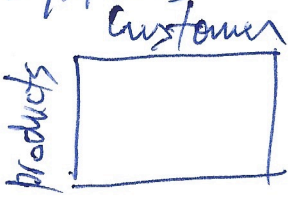
2) Control Systems

$$a_N \overset{(N)}{y} + \dots + a_1 \overset{(1)}{y} + a_0 y = f(u) \longrightarrow \begin{cases} \dot{x} = Ax + Bu \\ y = Hx + Eu \end{cases}$$

N^{th} order diff eq.
input/output description

1^{st} order Matrix diff eq
- internal stability
- Observability & Controlability

3) Recommendation Systems
(BIG DATA)



1. LINEAR VECTOR SYSTEMS

Fundamental problem in LINEAR ALGEBRA:
Solve simultaneous M eqs in N unknowns

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N = b_1$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N = b_M$$

(I)

a_{ij} — i: eq, row; j: unknown, col

Scalar form

$$\sum_{l=1}^{N} a_{kl}\, x_l = b_k \qquad k = 1, M$$

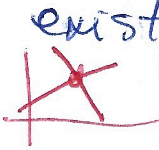
Matrix form

$$A_{M\times N}\, X_{N\times 1} = b_{M\times 1}$$

For a given b, the structure of matrix A will tell whether a solution vector X^o exists

SOLUTION

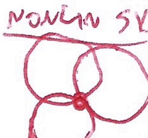
$X^o = \{X \mid Ax = b\}$

- Yes
  - UNIQUE  Crossing Lines
  - Infinitely many  Coincident Lines
- No  Parallel lines (non-coincident)

Systematic Solution

- Direct Methods
  e.g.: Gauss ELIMINATION / Gauss Jordan
- Iterative / Sequential Methods
  e.g.: Richardson's, Gauss-Seidel

NONLIN SYS



$(x - x_k)^2 + (y - y_k)^2 = r_k^2$ $k = 1, 3$

BACK

DIRECT METHODS

Transform the original sys (I) into an equivalent system that is ~~easyer~~ to solve

Arrive at ~~the~~ an exact solution after a finite number of arithmetic operations proportional to system size $M \cdot N$.

Computationally intensive for large scale sys (i.e, thousands of vars). Relevant for theoretical purposes.

Example: GAUSS ELIMINATION, GAUSS-JORDAN

ITERATIVE METHODS

Never transform the original system (I).

~~At the k^{th} iteration a sequ~~

A sequence of matrix-vector products as $A X_k$ (k^{th} iteration) are carried out

A good approximate solution is achieved within (typically) hundreds of iterations, even for large scale systems

Example: Richardson's Method

For an initial guess X_0, iterate

$$X_{k+1} = X_k + \mu \left(b - A X_k \right) \qquad \mu > 0 \qquad k = 1, 2, \dots$$

Convergence?

1.1. GAUSS ELIMINATION & GAUSS-JORDAN – □-SYS

Both GE & GJ methods perform elementary row ops to transform the original sys into a triangular (GE) or a diagonal (GJ) ~~systems~~ equivalent system:

- row scaling
- row exchange
- LIN COMB over rows

Pivots: nonzero elements in strategic positions in A → they define the multipliers

GAUSS ELIMINATION · GAUSS-JORDAN

EACH col is a stage VIA pivots

EACH pivot handles an entire column

Pivot: across main diagonal

Gauss Elimination: Upper triangularize & Back substitution

$$Ax = b \longrightarrow Ux = c$$

$$\Theta\left(\frac{N^3}{3}\right) \quad \times \div \quad + -$$

Gauss-Jordan: Diagonalize (Identity Matrix)

$$Ax = b \rightarrow Ix = x^0$$

$$\Theta\left(\frac{N^3}{2}\right) \quad \times \div \quad + -$$

Example: GE

Strang 12

$$2x_1 + x_2 + x_3 = 5$$
$$4x_1 - 6x_2 = -2$$
$$-2x_1 + 7x_2 + 2x_3 = 9$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$U \qquad c \quad$ B.S.

$$x_3 = 2$$

$$-8x_2 - 2x_3 = -12 \Rightarrow x_2 = 1$$

$$2x_1 + x_2 + x_3 = 5 \Rightarrow x_1 = 1$$

$$x^0 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Example: GJ

$$\left[\begin{array}{ccc|c} 2 & 2 & 6 & 4 \\ 2 & 1 & 7 & 6 \\ -2 & -6 & -7 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 2 & 1 & 7 & 6 \\ -2 & -6 & -7 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & -4 & -1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & -4 & -1 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 4 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & -5 & -5 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 4 & 4 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \rightarrow \underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{I} \Big| \underbrace{\begin{array}{c} 0 \\ -1 \\ 1 \end{array}}_{x^0} \rightarrow x^0 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

1.2. FINITE PRECISION ARITHMETICS

Lots of arithmetic ops are required in GE & GJ. As such they may suffer from numerical problems due to finite precision, since all ops turn out to be nonlinear due to roundoff errors.

A floating point number q with t digits and base β has the form $q = \pm\, 0.d_1 d_2 \cdots d_t \times \beta^L$ $\quad \beta, L, d_k$ integers

Arithmetic Ops in t digits can be modelled via a quantizing function $f(\cdot)$. For real a, b, c:

$f(a) \neq a$, $f(b) \neq b$, $f(c) \neq c$

$f(ab+c) = f\Big(f\big(f(a)f(b)\big) + f(c)\Big) \neq ab + c$ $\quad$ BACK

Example: $\frac{10}{3} \cdot 2 + \frac{7}{6}$ in 2-digits f.p.

$f(2) = 0{,}2 \cdot 10^1$, $f(7/6) = 0{,}17 \cdot 10^1$, $f(10/3) = 0{,}33 \cdot 10^1$

$f\big(f(0{,}33 \cdot 10^1 \times 0{,}2 \cdot 10^1) + 0.17 \cdot 10^1\big) = f(0{,}66 \cdot 10^1 + 0{,}17 \cdot 10^1) = 0{,}83 \cdot 10^1$

$= 8{,}3 \neq 7{,}833\ldots$ (6% error)

1.3. GAUSSIAN ELIMINATION WITH ROW PIVOTING

GJ is numerically poor, however useful for theoretical purposes, or to calculate the inverse of (small) matrices. GE, on the other hand can be made quite robust with a couple of modifications.

1) **Row pivoting**: avoid large multipliers in the elimination process.

- FOR EACH COL: select across the col the largest number (in mod) $(|\cdot|)$ and bring it to the pivotal position via row exchange

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~~Example~~ Example 1

$$-10^{-4}x + y = 1$$
$$x + y = 2$$

$$x^{\circ} = \begin{bmatrix} 1 \\ 1{,}0002 \end{bmatrix} \cdot \frac{1}{1{,}0001}$$

f.p. 3-digit solution:

multip $= 10^4$

$$\left[\begin{array}{cc|c} -10^{-4} & 1 & 1 \\ 1 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} -10^{-4} & 1 & 1 \\ \boxed{0} & \underbrace{1+10^4}_{=10^4} & \underbrace{2+10^4}_{=10^4} \end{array}\right] \xrightarrow{f(\cdot)} \left[\begin{array}{cc|c} -10^{-4} & 1 & 1 \\ 0 & 10^4 & 10^4 \end{array}\right] \to \hat{x}^{\circ} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

never actually computed

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row pivoting f.p. 3-digit sol:

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ -10^{-4} & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1+10^{-4} & 2+10^{-4} \end{array}\right] \xrightarrow{f(\cdot)} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right] \to \hat{x}^{\circ} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

more reasonable

2) **Row scaling**: scale the eqs so that the largest number in **each** row is 1. See example 1.5.2 Meyer22.

Row scaling & row pivoting make GE a quite robust method.

1.4. Echelon Forms & Rectangular Systems

The col structure of A in $A_{M\times N}\, X_{N\times 1} = b_{M\times 1}$ dictates whether or not a solution exists, and if it is unique or there are infinitely many. → why? Beck ↺

Applying the GE ~~&~~ or GJ elimination over a general rectangular matrix $A_{M\times N}$ unveils its col structure, however defective ▽ or defective \ may occur: the echelon forms (escalonada)

□-sys with unique sols always perfect ▽ or \

GE:

$$\begin{array}{|c|c|c|c|c|c|}\hline X & X & & & & \\ \hline 0 & 0 & X & X & X & \\ \hline 0 & 0 & 0 & 0 & 0 & X \\ \hline \end{array}$$

Row Echelon form: $A_{M\times N} \xrightarrow{GE} E_{M\times N}$

$A_{M\times N}$ — Compress Nonzero entries

Proceed with the elementary ops until

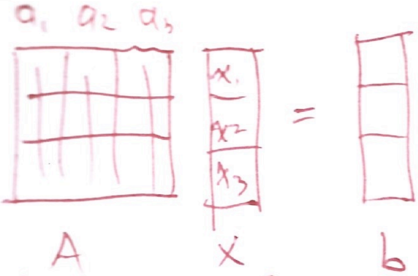
- Nonzero entries in E ~~are across~~ lie on or above a "broken" diagonal, a stair-step line (defective ▽), going down the rows, as far to the ~~left~~ right as possible
- Pivots are the first nonzero entries in each row ($\neq 0$) recall pivots are nonzero numbers always
- Rows of zeros (if any) are packed at the bottom

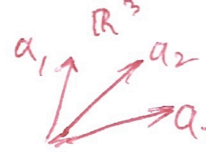
Example

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 2 & 4 & 0 & 4 & 4 \\ 1 & 2 & 3 & 5 & 5 \\ 2 & 4 & 0 & 4 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & -2 & -2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 3 & 3 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Cols containing pivots are L.I. (basic Cols) We'll use them to build b

the remaining cols are LIN COMBs of the pivotal Cols (NON-BASIC cols)





if a_1, a_2, a_3 are LI in $\mathbb{R}^3$, any b can be reached via Lin Comb of a_i's

$$\boxed{x_1} \cdot a_1 + \boxed{x_2} \cdot a_2 + \boxed{x_3} \cdot a_3 \stackrel{?}{=} b$$

scalar

cols

However, if the a_i's are coplanar (colinear), ~~some~~ b's are not reachable

row ops change the cols, but do <u>not</u> change the interdependence of cols in A (ie, its col structure)

$$A = \begin{array}{ccc} a_1 & a_2 & a_3 \\ \end{array}$$

$$A = \left[\begin{array}{c|c|c} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & -1 \end{array}\right] \xrightarrow{GE} \underbrace{\begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \end{array}}_{E}$$

$u_3 = 2u_1 - 3u_2$

same holds for A:

$a_3 = 2a_1 - 3a_2$

However it is much easier to see this in E

- Entries in E are <u>not</u> uniquely determined by A but the pivots positions / Col structure are

$$A \to E_1,\quad A \to E_2,\quad A \to E_3$$

try switching rows $A \to A'$ and find $E' \neq E$

- Same Col relations in E hold for A

<u>REDUCED ROW ECHELON FORM</u>: $A_{m \times n} \xrightarrow{GJ} E_{A_{m\times n}}$

We form a defective diagonal (identity) matrix. Besides the el row ops, make sure:
- Pivots are scaled to 1
- Anihilates entries above and below pivots

<u>Example</u>: same as before

$$\begin{matrix} 1 & 2 & 1 & 3 & 3 \\ 2 & 4 & 0 & 4 & 4 \\ 1 & 2 & 3 & 5 & 5 \\ 2 & 4 & 0 & 4 & 7 \end{matrix} \xrightarrow{GJ} \begin{matrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \triangleq E_A$$

- E_A is uniquely determined by A, but different matrices may have the same E_A

Example: $A \to \underbrace{A_1 \to A_2 \to A_3}_{\text{intermediate matrices in GJ}} \to E_A$

$$A_1 \to E_A,\quad A_2 \to E_A,\quad A_3 \to E_A$$

- As with GE, Col structure in E_A also holds for A.

1.5. Consistency of Lin Sys

GJ is allowed, but more complex

Systematic way to find out: GE on $[A|b]$

$$[A|b] \xrightarrow{GE} [E|c]$$

If in any stage of GE a row

$0\ 0\ 0 \cdots 0 | \alpha$ shows up, it means

$0^T x = \alpha$ → $\alpha = 0$: eqs redundant: multiple sols

→ $\alpha \neq 0$: Sys inconsistent, No Sol

Algebraic interpretation: CANNOT produce a nonzero α via Lin comb of zeros

Geometric interp.: back of page 6

1.6. Homogeneous Systems ($N(A)$)

Any system of the form $AX = 0$

- Always Consistent: admits trivial sol $X^0 = 0$
- Are there nontrivial sols? $\{X^0 \neq 0 \mid AX^0 = 0\}$

Systematic Approach

1) $A \xrightarrow{GE} E$: find out the pivotal cols — or $A \xrightarrow{GJ} E_A$

2) Back subs on E: Solve eqs for pivotal/ basic vars — those related to pivotal cols

Example

1) GE on $A_{3\times 4}$: E

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & -5 & -5 \end{bmatrix} \rightarrow \begin{array}{cccc} & \overbrace{x_2 \quad\quad\quad x_4}^{\text{free vars}} & & \\ \end{array}$$

$$\begin{array}{c} \\ \\ \end{array}\begin{bmatrix} \boxed{1} & 2 & \boxed{2} & 3 \\ \boxed{0} & 0 & \boxed{-3} & -3 \\ \boxed{0} & 0 & \boxed{0} & 0 \end{bmatrix}$$

$$\underbrace{x_1 \quad\quad x_3}_{\text{basic vars}}$$

Back

1.7. NON-HOMOGENEOUS SYSTEMS

$A_{M\times N} X_{N\times 1} = b_{M\times 1}$ $b_{M\times 1} \neq 0$: system may be INCONSISTENT

Systematic Solution

- Find echelon form on $[A|b]$ $\xrightarrow{GE}$ $[E|c]$ & B.S. ; $\xrightarrow{GJ}$ $[E_A|d]$
- Check for consistency: $0^T x \neq 0$ at any row?
- R pivot vars (basic vars) + N−R free vars
- Solution: $X^0 = \underbrace{X_p}_{\text{from } d \text{ in } [E_A|d]} + \underbrace{X_h}_{\text{previous method}}$

Example: via GJ

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 & 5 \\ 3 & 6 & 1 & 4 & 7 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 4 \\ 0 & 0 & -3 & -3 & -3 \\ 0 & 0 & -5 & -5 & -5 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

$$\to \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \begin{array}{ccccc} & x_2\text{-free} & & x_4 & \\ \boxed{1} & 2 & \boxed{0} & 1 & 2 \\ \boxed{0} & 0 & \boxed{1} & 1 & 1 \\ \boxed{0} & 0 & \boxed{0} & 0 & 0 \\ x_1 & & x_3 & & \end{array}$$

basic

$$\left.\begin{array}{r} x_1 + 2x_2 + x_4 = 2 \\ x_3 + x_4 = 1 \end{array}\right\} \quad \begin{array}{l} x_1 = 2 - 2x_2 - x_4 \\ x_2 \text{ free} \\ x_3 = 1 - x_4 \\ x_4 \text{ free} \end{array}$$

$$X^0 = \underbrace{\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{X_p} + \underbrace{x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{X_h} \quad \forall\ x_2, x_4$$

2) Back subst. on E: $Ex = 0$ solve for basic vars

$$\left.\begin{aligned} x_1 + 2x_2 + 2x_3 + 3x_4 &= 0 \\ -3x_3 - 3x_4 &= 0 \end{aligned}\right\}$$

x_2, x_4: free vars
(from step 1)
from col structure

$-3x_3 = 3x_4 \iff x_3 = -x_4$

$x_1 + 2x_2 + 2x_3 + 3x_4 = 0$

$x_1 = -2x_2 + 2x_4 - 3x_4$

$x_1 = -2x_2 - x_4$

$$x_h = \begin{bmatrix} -2x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

multiple sols. Why?

vars > eqs

$x_2; x_4$: free to choose pars

$$x_h = t_1 h_1 + t_2 h_2, \quad h_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad h_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$$

t_1, t_2: pars.

particular sols.

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$A_{M\times N} X_{N\times 1} = b_{M\times 1}$ $b_{M\times 1} \neq 0$: system may be INCONSISTENT

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- Find echelon form on $[A|b]$ $\xrightarrow{GE}$ $[E|c]$ & B.S.; $\xrightarrow{GJ}$ $[E_A|d]$
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$$\rightarrow \begin{array}{c} \\ \\ \\ \\ \end{array} \begin{array}{ccccc} & x_2\text{-free} & & x_4 & \\ \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right. & \begin{array}{c} 2 \\ 0 \\ 0 \end{array} & \begin{array}{c} 0 \\ 1 \\ 0 \end{array} & \begin{array}{c} 1 \\ 1 \\ 0 \end{array} & \left|\begin{array}{c} 2 \\ 1 \\ 0 \end{array}\right] \\ x_1 & & x_3 & & \end{array}$$

basic

$$\left.\begin{array}{r} x_1 + 2x_2 + x_4 = 2 \\ x_3 + x_4 = 1 \end{array}\right\} \begin{array}{l} x_1 = 2 - 2x_2 - x_4 \\ x_2 \text{ free} \\ x_3 = 1 - x_4 \\ x_4 \text{ free} \end{array}$$

$$X^0 = \underbrace{\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{X_p} + \underbrace{x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}}_{X_h} \quad \forall\ x_2, x_4$$

HomeWork

MEYER

1.5.4 Finite precision row scaling pivoting

2.1.1 Row echelon
2.1.2

2.3.1 Consistency
2.3.3 of Lin Sys

2.4.1 Hom Lin Sys
2.4.2

2.5.2 Non-hom
2.5.7 Lin Sys

Reading

Meyer ch1, ch2

~~Secs.~~ ch1: ~~all~~ 18 except 1.4 (ill-conditioning will be covered later)

ch2: everything