Seno e Cosseno - Definição Axiomática

Def: $f: \mathbb{R} \mapsto \mathbb{R}$ é periódica de período $p \in \mathbb{R}$ se satisfaz

$$f(x+p) = f(x) \quad \forall x \in \mathbb{R}.$$

Propriedades fundamentais: sen e cos

1) sen e cos estão definidas em $\mathbb{R}$.

2) $\cos 0 = \operatorname{sen} \pi/2 = 1$ e $\cos \pi = -1$. (valores especiais)

3) $\cos(x-y) = \cos x \cos y + \operatorname{sen} x \operatorname{sen} y$ (cosseno da diferença)

4) Desigualdades fundamentais:

$$0 < \cos x < \frac{\operatorname{sen} x}{x} < \frac{1}{\cos x} \quad \forall x \in (0, \pi/2).$$

Recordação: $\tan x = \dfrac{\operatorname{sen} x}{\cos x}$, $\cot x = \dfrac{\cos x}{\operatorname{sen} x}$, $\sec x = \dfrac{1}{\cos x}$, $\operatorname{cossec} x = \dfrac{1}{\operatorname{sen} x}$.

Teo Sejam sen e cos as funções acima satisfazem 1-4. Então

(a) $\operatorname{sen}^2 x + \cos^2 x = 1$ (identidade pitagórica)

(b) $\operatorname{sen} 0 = \cos\left(\frac{\pi}{2}\right) = \operatorname{sen} \pi = 0$

(c) $\cos(-x) = \cos x$ e $\operatorname{sen}(-x) = -\operatorname{sen} x$ $\forall x \in \mathbb{R}$
(função par) (função ímpar)

(d) $\operatorname{sen}\left(x + \frac{\pi}{2}\right) = \cos x$ e $\cos\left(x + \frac{\pi}{2}\right) = -\operatorname{sen} x$ $\forall x \in \mathbb{R}$

(e) $\cos(x + 2\pi) = \cos x$ e $\operatorname{sen}(x + 2\pi) = \operatorname{sen} x$ $\forall x \in \mathbb{R}$

(f) $\cos(x+y) = \cos x \cos y - \operatorname{sen} x \operatorname{sen} y$
$\operatorname{sen}(x+y) = \operatorname{sen} x \cos y + \operatorname{sen} y \cos x$ } exercício

(g) $\operatorname{sen} a - \operatorname{sen} b = 2 \operatorname{sen}\left(\frac{a-b}{2}\right) \cos\left(\frac{a+b}{2}\right)$ e $\cos a - \cos b = -2 \operatorname{sen}\left(\frac{a-b}{2}\right) \operatorname{sen}\left(\frac{a+b}{2}\right)$.

(h) Monotonicidade. Em $[0, \pi/2]$ o seno é estritamente crescente e o cosseno estritamente decrescente dem (a) Pelas propriedades 2 e 3: $1 = \cos(0) = \cos(x-x) = \cos^2 x + \operatorname{sen}^2 x$.

(b) Por (a) $1 = \operatorname{sen}^2 0 + \cos^2 0 = \operatorname{sen}^2 0 + 1^2 \rightsquigarrow \operatorname{sen}^2 0 = 0 \therefore \operatorname{sen} 0 = 0$.

Analogamente se obtém $\cos\left(\frac{\pi}{2}\right) = 0 = \operatorname{sen}(\pi)$.

(c) $\cos(-x) = \overset{1}{\cancel{\cos 0}} \cos x + \overset{0}{\cancel{\operatorname{sen} 0}} \operatorname{sen} x \rightsquigarrow \cos(-x) = \cos x$

(*) $\cos\left(\frac{\pi}{2} - y\right) = \operatorname{sen} y$ pois $\cos \frac{\pi}{2} = 0$ e $\operatorname{sen} \frac{\pi}{2} = 1$.

Por outro lado, $\operatorname{sen}(-y) = \cos\left(\frac{\pi}{2} + y\right) = \cos\left(\pi - \left(\frac{\pi}{2} - y\right)\right)$

$$= \overset{-1}{\cancel{\cos \pi}} \cos\left(\frac{\pi}{2} - y\right) + \overset{0}{\cancel{\operatorname{sen} \pi}} \operatorname{sen}\left(\frac{\pi}{2} - y\right)$$

$$= -\operatorname{sen} y.$$

(d) $-\operatorname{sen} x = \operatorname{sen}(-x) = \cos\left(\frac{\pi}{2} + x\right)$

$\operatorname{sen}\left(x+\frac{\pi}{2}\right)=\cos x$ segue de

$$y = y + \frac{\pi}{2}$$

$$\cos\left(\frac{\pi}{2} - y - \frac{\pi}{2}\right) = \cos(-y)$$

$$= \cos y = \operatorname{sen}\left(y+\frac{\pi}{2}\right) \text{ em } (*)$$

(e)

$$\cos(x+2\pi) = \cos\left(x+\frac{3\pi}{2}+\frac{\pi}{2}\right) = -\operatorname{sen}\left(x+\frac{3\pi}{2}\right)$$

$$= -\operatorname{sen}\left(x+\pi+\frac{\pi}{2}\right) = -\cos(x+\pi) = -\cos\left(x+\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$= -\left(-\operatorname{sen}\left(x+\frac{\pi}{2}\right)\right) = \cos x \quad \forall x \in \mathbb{R}.$$

Analogamente se prova $\operatorname{sen}(x+2\pi) = \operatorname{sen} x$.

(g)

$$
\begin{aligned}
- \quad \operatorname{sen}(x+y) &= \operatorname{sen}x \cos y + \operatorname{sen}y \cos x \\
\operatorname{sen}(x-y) &= \operatorname{sen}x \cos y - \operatorname{sen}y \cos x
\end{aligned}
$$

$$\operatorname{sen}(x+y) - \operatorname{sen}(x-y) = 2 \operatorname{sen}y \cos x$$

$$
\begin{aligned}
- \quad \cos(x+y) &= \cos x \cos y - \operatorname{sen}x \operatorname{sen}y \\
\cos(x-y) &= \cos x \cos y + \operatorname{sen}x \operatorname{sen}y
\end{aligned}
$$

$$\cos(x+y) - \cos(x-y) = -2 \operatorname{sen}x \operatorname{sen}y$$

$$y = \frac{a-b}{2} \qquad x = \frac{a+b}{2} \qquad \begin{aligned} x+y &= a \\ x-y &= b \end{aligned}$$

$$\operatorname{sen}a - \operatorname{sen}b = 2 \operatorname{sen}\left(\frac{a-b}{2}\right) \cos\left(\frac{a+b}{2}\right) \quad e \quad \cos a - \cos b = -2 \operatorname{sen}\left(\frac{a-b}{2}\right) \operatorname{sen}\left(\frac{a+b}{2}\right)$$

h) Em $(0, \frac{\pi}{2})$ temos por 4 que $\operatorname{sen} x$ e $\cos x$ são positivos.

Sejam $0 < b < a < \frac{\pi}{2}$ $\rightsquigarrow$ $\frac{a-b}{2}$ e $\frac{a+b}{2} \in (0, \frac{\pi}{2})$

Segue de g) que $\operatorname{sen} a - \operatorname{sen} b > 0$ e $\cos a - \cos b < 0$

a e b arbitrários em $(0, \pi/2)$ $\Rightarrow$ sen é estritamente crescente

e cos estritamente decrescente

$\blacksquare$

OBS:

(i) Seno e cosseno são monótonas por partes. Basta combinar as propriedade (d) e (h).

(ii) Outras propriedades: $\operatorname{sen} 2x = 2 \operatorname{sen} x \cos x$ e $\cos 2x = \cos^2 x - \operatorname{sen}^2 x$.

(Exercício)

(iii) É claro a unicidade de seno e cosseno?

Deve ser provado

$$(\operatorname{sen} x - \widehat{\operatorname{sen}}\, x)^2 + (\cos x - \widehat{\cos}\, x)^2 = \operatorname{sen}^2 x + \cos^2 x + \widehat{\operatorname{sen}}^2 x + \widehat{\cos}^2 x$$
$$- 2 \operatorname{sen} x\, \widehat{\operatorname{sen}}\, x - 2 \widehat{\cos}\, x \cos x$$

$$= 2\left(1 - \operatorname{sen} x\, \widehat{\operatorname{sen}}\, x - \widehat{\cos}\, x \cos x\right) = 2\left(1 - \cos\left(x + \frac{\pi}{2}\right) \widehat{\cos}\left(x + \frac{\pi}{2}\right) - \widehat{\cos}\, x \cos x\right)$$

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