Propriedades de integrais

$f, g : [a,b] \longmapsto \mathbb{R}$ integráveis, $-\infty < a \leq b < +\infty$

(A) $\displaystyle\int_a^b \{f(x) + g(x)\}\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(B) $\displaystyle\int_a^b c\, f(x)\,dx = c \int_a^b f(x)\,dx \qquad c \in \mathbb{R}$

(C) $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \qquad \forall\, c \in \mathbb{R}$

i) $\displaystyle\int_a^a f(x)\,dx = 0$

ii) $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

(D) $\displaystyle\int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(x-c)\,dx \qquad \forall\, c \in \mathbb{R}$

$\displaystyle 0 = \int_a^a f = \int_a^b f + \int_b^a f$

(E) Se $f \leq g$ em $[a,b] \Rightarrow \displaystyle\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

(F) $\displaystyle\int_a^b f(x)\,dx = \frac{1}{k} \int_{ak}^{bk} f\left(\frac{x}{k}\right)dx \qquad k \neq 0$

(*) $$\int_a^b \{f(x)+g(x)\}\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

dim. $$\underline{I}(f+g) = \overline{I}(f+g) = I(f)+I(g).$$

Sejam s_1 e $s_2 : [a,b] \longmapsto \mathbb{R}$ funções escadas $s_1 \leq f$ e $s_2 \leq g$.

$s_1 + s_2 \leq f+g$ f e g integráveis $\rightsquigarrow \begin{cases} \underline{I}(f) = I(f) \text{ e} \\ \underline{I}(g) = I(g) \end{cases}$

$$I(f) + I(g) = \sup\left\{ \int_a^b s_1\,dx : s_1 \leq f \right\} + \sup\left\{ \int_a^b s_2\,dx : s_2 \leq g \right\}$$

$$\underset{\text{Propriedade sup}}{=} \sup\left\{ \int_a^b s_1\,dx + \int_a^b s_2\,dx : s_1 \leq f \text{ e } s_2 \leq g \right\}$$

$$= \sup\left\{ \int_a^b (s_1+s_2)\,dx : s_1+s_2 \leq f+g \right\} \leq \underline{I}(f+g)$$

pois $\underline{I}(f+g) = \sup\left\{ \int_a^b s\,dx \;.\; s \leq f+g \right\}$

Sejam agora $t_1 \geq f$ e $t_2 \geq g$ $\quad f, g$ integráveis $\rightsquigarrow \begin{cases} \overline{I}(f) = I(f) \\ \overline{I}(g) = I(g) \end{cases}$

$$I(f) + I(g) \underset{\uparrow}{=} \inf\left\{ \int_a^b \{t_1 + t_2\}\,dx : t_1 + t_2 \geq f+g \right\} \geq \overline{I}(f+g)$$

Propriedade ínfimo

$$\therefore \quad \overline{I}(f+g) \leq I(f) + I(g) \leq \underline{I}(f+g) \leq \overline{I}(f+g)$$

$$\underline{I}(f+g) = I(f) + I(g) = \overline{I}(f+g).$$

$$\therefore \quad I(f+g) = I(f) + I(g) \qquad \blacksquare$$

(B) $\displaystyle\int_a^b c\, f(x)\,dx = c\int_a^b f(x)\,dx \qquad c\in\mathbb{R}$

Spg $\quad c>0 \qquad \underline{I}(cf) = \sup\left\{ \int_a^b s\,dx : \ s\leq cf \right\}$

$$= \sup\left\{ \int_a^b c\,s'\,dx : \ s'\leq f \right\} = \sup\left\{ c\int_a^b s'\,dx : \ s'\leq f \right\} = c\,\underline{I}(f) = c\,I(f)$$

Analogamente

$$\overline{I}(cf) = \inf\left\{ \int_a^b t\,dx \ . \ t\geq fc \right\} = \inf\left\{ \int_a^b c\,t'\,dx ; \ t'\geq f \right\}$$

$$= c\,\overline{I}(f) = c\,I(f)$$

$$\therefore \qquad \underline{I}(cf) = c\,I(f) = \overline{I}(cf) \qquad \blacksquare$$

OBS. (A) e (B) $\displaystyle\int_a^b \{\alpha f + \beta g\} = \alpha\int_a^b f + \beta\int_a^b g \qquad f,g : [a,b] \longmapsto \mathbb{R}$ int.

α,β const.

(c) $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad \forall c \in \mathbb{R}$

Spg. $a < b < c$. Nesse caso existem $\int_a^b f$ e $\int_b^c f$.

$$\int_a^c f(x)\,dx = \underline{I}(f) \overset{?}{=} \overline{I}(f) \overset{?}{=} \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

$$\underset{\text{escada}}{s} \leq f \longrightarrow \int_a^c s(x)\,dx = \int_a^b s(x)\,dx + \int_b^c s(x)\,dx$$

$$\underline{I}(f) = \sup\left\{ \int_a^c s(x)\,dx : s \leq f \right\} = \sup\left\{ \int_a^b s\,dx + \int_b^c s\,dx : s \leq f \right\}$$

$$\underset{\text{prop. sup}}{=} \sup\left\{ \int_a^b s\,dx : s \leq f \right\} + \sup\left\{ \int_b^c s\,dx : s \leq f \right\} = \underline{I}^{ab}(f) + \underline{I}^{bc}(f)$$

$$= \int_a^b f + \int_b^c f.$$

Analogamente se mostra $\overline{I}(f) = \int_a^b f + \int_b^c f$. $\blacksquare$

(F) $$\int_a^b f(x)\,dx = \frac{1}{k}\int_{ak}^{bk} f\left(\frac{x}{k}\right)dx \qquad k \neq 0$$

dem. $k>0$ $\quad g(x) = f\left(\frac{x}{k}\right) \quad a \le \frac{x}{k} \le b \longleftrightarrow ak \le x \le bk$

s' é função escada de f em $[a,b]$ com $s' \le f$ se só se

$s(x) = s'(x/k)$ é função escada de g em $[ak,bk]$ com $s \le g$.

$$\underline{I}(g) = \sup\left\{\int_{ak}^{bk} s(x)\,dx \;;\; s \le g\right\} = \sup\left\{k\int_a^b s'(x)\,dx \;;\; s' \le f\right\} = k\int_a^b f$$

já que $$\int_{ak}^{bk} s(x)\,dx = \frac{1}{k^{-1}}\int_{ak k^{-1}}^{bk k^{-1}} s\left(\frac{x}{k^{-1}}\right)dx = k\int_a^b s(kx)\,dx = k\int_a^b s'(x)\,dx.$$

Analogamente $\bar{I}(g) = k\int_a^b f$ $\quad\rightarrow\quad \underline{I}(g) = \bar{I}(g) = k\int_a^b f \quad \therefore \frac{1}{k}\int_{ak}^{bk} f\left(\frac{x}{k}\right)dx = \int_a^b f(x)\,dx$ ■