Integral Indefinida

Em cada caso o problema é encontrar uma função F cuja derivada é uma função conhecida f. Se a função F existir, será denominada de antiderivada de f. Considere os exemplos.

Obs.: $\int f(x)\,dx = F(x) + C = [F(x)+C]' = f(x)$

a) $\int x^8\,dx = \dfrac{x^9}{9} + C$

b) $\int x^5\,dx$

c) $\int \cos x\,dx = \operatorname{sen} x + C$

d) $\int \operatorname{sen} x\,dx =$

e) $\int \dfrac{dt}{t}$

f) $\frac{1}{4}\int \operatorname{sen} 4x\, 4\,dx \longrightarrow$ função composta

$u = 4x$

$du = \underline{\underline{4}}\,dx$

$= \frac{1}{4}\int \underbrace{\operatorname{sen} 4x}_{\operatorname{sen} u}\, \underbrace{4\,dx}_{du} \Rightarrow \frac{1}{4}\int \operatorname{sen} u\,.du$

$= -\frac{1}{4}\cos 4x + C$

g) $\int \cos \dfrac{x}{6}\,dx$

h) $\int (x^3 + 5x + 8)\,dx$

$= \int x^3 dx + \int 5x\,dx + \int 8\,dx$

$= \int x^3 dx + 5\int x\,dx + 8\int dx$

$= \frac{x^4}{4} + \underline{\underline{C}} + 5\frac{x^2}{2} + \underline{\underline{C}} + 8x + \underline{\underline{C}}$

$= \frac{x^4}{4} + \frac{5}{2}x^2 + 8x + C$

i) $\int \frac{dx}{(x+1)^2} = \int (x+1)^{-2} dx$

$= \frac{(x+1)^{-1}}{-1} + C$

$= \frac{-1}{x+1} + C$

j) $\int \frac{dx}{\underbrace{x^2-6x+9}_{(x-3)^2}} = \int (x-3)^{-2} dx$

$= \frac{(x-3)^{-1}}{-1} + C = \frac{-1}{x-3} + C$

k) $\int \frac{x\,dx}{x^2+1}$

l) $\int \underbrace{(x^2+3x+7)^{31}}_{u} \underbrace{(2x+3)dx}_{du}$

$u = x^2 + 3x + 7$

$\frac{du}{dx} = 2x + 3$

$du = (2x+3)dx$

$\int u^n du = \frac{u^{n+1}}{n+1} + C$

$= \frac{(x^2+3x+7)^{32}}{32} + C$

m) $\displaystyle\int \frac{dx}{x^2+1}$ $\Delta<0$

$$\int \frac{du}{u^2+1} = \text{arctg}\, u + C \quad \text{ou} \quad \int \frac{du}{a^2+u^2} = \frac{1}{a}\,\text{arctg}\,\frac{u}{a} + C$$

$$= \text{arctg}\, x + C$$

n) $$\int \frac{dx}{x^2+5} = \frac{1}{5}\int \frac{dx}{\frac{x^2}{5}+1}$$

$$\rightarrow \int \frac{du}{a^2+u^2} = \frac{1}{a}\,\text{arctg}\,\frac{u}{a} + C$$

ou

$$= \frac{1}{5}\int \frac{dx}{\underbrace{\left(\frac{x}{\sqrt{5}}\right)^2+1}_{u^2+1}} \rightarrow \int \frac{du}{u^2+1} = \text{arctg}\, u + C$$

$$u = \frac{x}{\sqrt{5}}$$

$$u' = \frac{dx}{\sqrt{5}}$$

$$= \int \frac{\frac{1}{\sqrt{5}}\,dx}{\left(\frac{x}{\sqrt{5}}\right)^2+1} = \frac{\sqrt{5}}{5}\,\text{arctg}\,\frac{x}{\sqrt{5}} + C$$

o) $$\int \frac{dx}{x^2+10x+28}$$

p) $$\int \frac{x\,dx}{\underbrace{x^2-6x+15}_{u}}$$

$$x^2-6x+15=0$$
$$\Delta = b^2-4ac$$
$$\Delta = 36-4(1)(15)$$
$$\Delta = -24 < 0$$

$$u = x^2-6x+15$$

$$\frac{du}{dx} = 2x-6$$

$$du = 2x-6\,dx$$

$$= \frac{1}{2}\int \frac{2x\,dx}{x^2-6x+15} = \frac{1}{2}\int \frac{(2x-6+6)\,dx}{x^2-6x+15}$$

$$= \frac{1}{2}\int \frac{(2x-6+6)\,dx}{x^2-6x+15}$$

$$= \frac{1}{2}\int \frac{2x-6\;dx}{x^2-6x+15} + \frac{1}{2}\underbrace{\int \frac{6\,dx}{x^2-6x+15}}_{I_1} \qquad (1)$$

$$I_1 = \int \frac{6\,dx}{x^2-6x+15} \qquad x^2-6x+15 = (x-3)^2+6$$

$$= \int \frac{6\,dx}{(x-3)^2+\underbrace{6}_{\to\, =1}} = \frac{1}{6}\int \frac{6\,dx}{\frac{(x-3)^2}{6}+\frac{6}{6}} = \frac{1}{6}\int \frac{6\,dx}{\Big(\underbrace{\frac{x-3}{\sqrt{6}}}_{u}\Big)^2+1}$$

$$u = \frac{x-3}{\sqrt{6}}$$

$$\frac{du}{dx} = \frac{1}{\sqrt{6}} \Rightarrow du = \frac{dx}{\sqrt{6}} \qquad \boxed{\int \frac{du}{u^2+1} = \text{arc tg}\, u + C}$$

$$= \frac{1}{\cancel{6}}\cdot\cancel{6}\cdot\sqrt{6}\int \frac{\frac{1}{\sqrt{6}}\,dx}{\left(\frac{x-3}{\sqrt{6}}\right)^2+1} = \frac{1}{2}\cdot\sqrt{6}\;\text{arc tg}\,\frac{x-3}{\sqrt{6}}$$

Retornando na equação (1):

$$= \frac{1}{2}\ln(x^2-6x+15) + \frac{\sqrt{6}}{2}\,\text{arc tg}\,\frac{x-3}{\sqrt{6}} + C$$

q) $\frac{1}{2}\int \text{sen}\,(\underbrace{x^2+3}_{u})\,2x\,dx \qquad u = x^2+3,\quad du = 2x\,dx \qquad \boxed{\int \text{sen}\,u\,du = -\cos u + C}$

$$= -\frac{1}{2}\cos(x^2+3) + C$$

r) $\int \text{sen}^7 7x\, \cos 7x\, dx$

s) $\displaystyle\int \frac{\text{sen}\,7x}{(\cos 7x+3)^7}\,dx = -\frac{1}{7}\int \underbrace{(\cos 7x+3)}_{u}^{-7}\underbrace{(-7)\text{sen}\,7x\,dx}_{du}$

$u = \cos 7x + 3$

$du = -\text{sen}\,7x \,.\, 7\,dx$

$du = -7\,\text{sen}\,7x\,dx$

$$= -\frac{1}{7}\,\frac{(\cos 7x+3)^{-6}}{-6} + C$$

$$= \frac{1}{42\,(\cos 7x+3)^6} + C$$

$$\boxed{\int u^n\,du = \frac{u^{n+1}}{n+1} + C}$$

t) $\displaystyle\frac{1}{9}\int \sec^2 \underbrace{(3x^3+7)}_{u}\,\underbrace{9x^2dx}_{du}$

$u = 3x^3+7$

$du = 9x^2dx$

$$\frac{1}{9}\int \sec^2 u\,du$$

$$\int \sec^2 u\,du = \text{tg}\,u + C$$

$$= \frac{1}{9}\,\text{tg}\,(3x^3+7) + C$$

u) $\displaystyle\frac{1}{8}\int \text{cossec}^2 \underbrace{(4x^2+2)}_{u}\,\underbrace{8x\,dx}_{du}$

$u = 4x^2+2$

$du = 8x\,dx$

$$= \frac{1}{8}\cdot\int \text{cossec}^2 u\,du$$

$\to \int \text{cossec}^2 u\,du = -\text{cotg}\,u + C$

Obs. $d(\text{cotg}\,u) = -\text{cossec}^2 u\,du$

$$= -\frac{1}{8}\,\text{cotg}\,(4x^2+2) + C$$

v) $\displaystyle\int \frac{\sec^2 5x\,dx}{\sqrt[3]{a+b.\text{tg}\,5x}} = \frac{1}{5b}\int \underbrace{(a+b.\text{tg}\,5x)}_{u}^{-1/3}.\,5b\sec^2 5x\,dx$

$u = a + b.\,\text{tg}\,5x$

$du = b.\sec^2 5x.\,5\,dx$

$$= \frac{1}{5b}\,\frac{(a+b\,\text{tg}\,5x)^{\frac{2}{3}}}{\frac{2}{3}} + C$$

$$= \frac{3}{10b}\,(a+b\,\text{tg}\,5x)^{\frac{2}{3}} + C$$

w) $\int \frac{x+7}{x-2}\,dx$

Obs: O polinômio tem que ter um grau maior ou igual ao denominador

$$
\begin{array}{l|l}
\cancel{x}+7 & x-2 \\
\hline
-\cancel{x}+2 & 1 \\
\hline
9 &
\end{array}
$$

$$= \int \left(1 + \frac{9}{x-2}\right)dx$$

$$= \int dx + 9\int \frac{\overbrace{dx}^{du}}{\underbrace{x-2}_{u}}$$

$u = x-2$
$du = dx$

$$= x + 9\ln|x-2| + C$$

x) $\int \frac{x^3-7x+4}{x^2-2x+1}\,dx$

$$
\begin{array}{l|l}
\cancel{x^3}+0x^2-7x+4 & x^2-2x+1 \\
\hline
-\cancel{x^3}+2x^2-x & x+2 \\
\hline
\cancel{2x^2}-8x+4 & \\
-\cancel{2x^2}+4x-2 & \\
\hline
-4x+2 &
\end{array}
$$

$$= \int \left[x+2 + \left(\frac{-4x+2}{x^2-2x+1}\right)\right]dx$$

$$= \int x\,dx + \int 2\,dx + \int \frac{-4x+2}{x^2-2x+1}\,dx \quad (1)$$

I_1

$x^2-2x+1=0$
$\Delta = b^2-4ac$
$\Delta = 4-4$
$\Delta = 0$

$$I_1 = -2\int \frac{2x-1}{x^2-2x+1}\,dx$$

$x^2-2x+1=0$
$\Delta = b^2-4ac$
$\Delta = 4-4$
$\Delta = 0$

$$= -2\int \frac{2x-2+2-1}{x^2-2x+1}\,dx$$

$$= -2\int \frac{2x-2\,dx}{\underbrace{x^2-2x+1}_{u}} - 2\int \frac{dx}{x^2-2x+1}$$

$$-2\int \underbrace{\frac{2x-2}{x^2-2x+1}}\,dx - 2\int \frac{dx}{\underbrace{x^2-2x+1}_{\hookrightarrow (x-1)^2}} \Rightarrow -2\underbrace{\int (x-1)^{-2}\,dx}_{\frac{-2(x-1)^{-1}}{-1}}$$

$u = x^2 - 2x + 1$

$du = 2x - 2\,dx$

$$= -2\ln|x^2-2x+1| + \frac{2}{x-1}$$

Retornando a equação (1):

$$= \frac{x^2}{2} + 2x - 2\ln|x^2-2x+1| + \frac{2}{x-1} + C$$

z) $\displaystyle\int \frac{\overbrace{\ln 4x}^{u}}{x}\,\overbrace{dx}^{du}$ $\qquad u = \ln 4x$

$du = \frac{\cancel{4}}{\cancel{4}x}\,dx$

$$\boxed{\int u^n\,du = \frac{u^{n+1}}{n+1} + C}$$

$$= \frac{\ln^2 4x}{2} + C$$

Definição

Se $F(x)$ é uma primitiva de $f(x)$, a expressão $F(x)+C$ é denominada integral indefinida da função $f(x)$, sendo representada por $\int f(x)\,dx = F(x) + C$.

$f(x) \to$ função integrando

$f(x)\,dx \to$ integrando

Da definição da integral indefinida decorre que:

a) $\int f(x)\,dx = F(x) + C \Leftrightarrow [F(x)+C]' = F'(x) + 0 = f(x)$

b) $\int f(x)dx$ representa uma família de funções (a família de todas as primitivas da função integrando)

Proposições

a) $\int k f(x)\, dx = k \int f(x) dx$, $k \in \mathbb{R}$

Seja $F(x)$ uma primitiva de $f(x)$. Logo $k.F(x)$ é uma primitiva da função $k f(x)$ pois, $[k.F(x)]' = k.F'(x) = k.f(x)$. Portanto,

$$\int k\, f(x)\, dx = k[F(x) + C]$$
$$= kF(x) + kC_1$$
$$k\,\underbrace{[F(x) + C_1]}_{f(x)}$$
$$= k f(x) dx$$

b) $\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$

Sejam $F(x)$ e $G(x)$ primitivas das funções $f(x)$ e $g(x)$, respectivamente. Logo $F(x)$ e $G(x)$ é uma primitiva de $f(x)$ e $g(x)$, pois $[F(x) + G(x)]' = F'(x) + G'(x) = f(x) + g(x)$.

Integração por Substituição de Variáveis

De acordo com as fórmulas de primitivação, não é apresentado como calcular as integrais do tipo:

$$\int \underbrace{(3x^2-1)}_{u}^{1/3} \cdot 4x\,dx \qquad u = 3x^2-1 \quad du = 6x\,dx$$

$$= \frac{1}{6}\, 4 \int (3x^2-1)^{1/3} \cdot \underbrace{6x\,dx}_{du}$$

$$\frac{4}{6}\int u^{1/3} \cdot du$$

Neste caso considerou-se u como uma função de x e du como a diferencial de u de modo que:

$$u = f(x) \qquad du = f'(x)\,dx$$

Então tem-se:

$$= \frac{4}{6}\,\frac{u^{4/3}}{\frac{4}{3}}$$

$$= \frac{\cancel{4}}{\cancel{6}_{2}} \cdot \frac{\cancel{3}}{\cancel{4}} \cdot (3x^2-1)^{4/3} + C$$

$$= \frac{1}{2}(3x^2-1)^{4/3} + C$$

Porém, pode ser verificado a resposta correta com a utilização da regra da Cadeia para derivar a função:

$$\frac{1}{\cancel{2}} \cdot \frac{\cancel{4}^{2}}{\cancel{3}}(3x^2-1)^{1/3} \cdot \cancel{6}^{2}x = (3x^2-1)^{1/3} \cdot 4x$$

Considere os exemplos:

a) $\int x^2 \underbrace{(1-4x^3)}_{u}^{1/5} dx$

$u = 1-4x^3$
$du = -12x^2 dx$

$= -\frac{1}{12}\int \underbrace{(1-4x^3)}_{u}^{1/5} \underbrace{(-12)x^2 dx}_{du}$

$= -\frac{1}{12}\int u^{1/5} du$

$= \frac{-1}{12} \frac{u^{\frac{6}{5}}}{\frac{6}{5}}$

$= -\frac{1}{12} \frac{5}{6} (1-4x^3)^{\frac{6}{5}} + C$

$= \frac{-5}{72} (1-4x^3)^{\frac{6}{5}} + C$

b) $\int \frac{x^{2/3}}{(2-x^{5/3})^5} dx = -\frac{3}{5}\int (\underbrace{2-x^{5/3}}_{u})^{-5} \left(\frac{-5}{3}\right) x^{2/3} dx$

$u = 2 - x^{5/3}$
$du = -\frac{5}{3} x^{2/3} dx$

$= \frac{-3}{5} u^{-5}$

$= -\frac{3}{5} \frac{u^{-4}}{-4}$

$= -\frac{3}{5} \frac{(2-x^{5/3})^{-4}}{-4} + C$

$= \frac{3}{20(2-x^{5/3})^4} + C$

c) $\int \frac{x\,dx}{\sqrt{5-4x^2}}$

d) $\int \cot g^7 3x \ \operatorname{cossec}^2 3x\, dx$