Solução Prova 2

1. (a) Como as quantidades de equilíbrio são independentes do tempo, então as derivadas temporais se anulam. Então:

$$\boxed{\begin{aligned}
&\boldsymbol{\nabla}\cdot(n_{e\,0}\boldsymbol{u}_{e\,0}) = 0\\
&\boldsymbol{\nabla}\cdot(n_{i\,0}\boldsymbol{u}_{i\,0}) = 0\\
&m_e\, n_{e\,0}\,(\boldsymbol{u}_{e\,0}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}_{e\,0} = -e\, n_{e\,0}\,(\boldsymbol{E}_0 + \boldsymbol{u}_{e\,0}\times\boldsymbol{B}_0) - \boldsymbol{\nabla}p_{e\,0}\\
&m_i\, n_{i\,0}\,(\boldsymbol{u}_{i\,0}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}_{i\,0} = e\, n_{i\,0}\,(\boldsymbol{E}_0 + \boldsymbol{u}_{i\,0}\times\boldsymbol{B}_0) - \boldsymbol{\nabla}p_{i\,0}\\
&\boldsymbol{\nabla}p_{e\,0} = k_B\, T_{e\,0}\boldsymbol{\nabla}n_{e\,0}\\
&\boldsymbol{\nabla}p_{i\,0} = \gamma_i k_B\, T_{i\,0}\boldsymbol{\nabla}n_{i\,0}\\
&\boldsymbol{\nabla}\cdot\boldsymbol{E}_0 = \frac{e}{\epsilon_0}\,(n_{i\,0} - n_{e\,0})\\
&\boldsymbol{\nabla}\times\boldsymbol{B}_0 = \mu_0\, e\,(n_{i\,0}\,\boldsymbol{u}_{i\,0} - n_{e\,0}\,\boldsymbol{u}_{e\,0})
\end{aligned}} \tag{1}$$

(b) Linearizando as equações: $f(\boldsymbol{r},t) = f_0(\boldsymbol{r}) + f_1(\boldsymbol{r},t)$, a da continuidade fica:

$$\underbrace{\frac{\partial n_{\alpha\,0}}{\partial t}}_{=0} + \frac{\partial n_{\alpha\,1}}{\partial t} + \boldsymbol{\nabla}\cdot\left(n_{\alpha\,0}\boldsymbol{u}_{\alpha\,0} + n_{\alpha\,0}\boldsymbol{u}_{\alpha\,1} + n_{\alpha\,1}\boldsymbol{u}_{\alpha\,0} + \underbrace{n_{\alpha\,1}\boldsymbol{u}_{\alpha\,1}}_{\text{(não linear)}}\right) = 0$$

em que $\alpha = e,\, i$ é o índice das espécies.

O divergente do primeiro termo é 0, pela relação de equilíbrio. Desprezando os termos não-lineares. Temos que:

$$\boxed{\begin{aligned}
&\frac{\partial n_{e\,1}}{\partial t} + \boldsymbol{\nabla}\cdot(n_{e\,0}\boldsymbol{u}_{e\,1} + n_{e\,1}\boldsymbol{u}_{e\,0}) = 0\\
&\frac{\partial n_{i\,1}}{\partial t} + \boldsymbol{\nabla}\cdot(n_{i\,0}\boldsymbol{u}_{i\,1} + n_{i\,1}\boldsymbol{u}_{i\,0}) = 0
\end{aligned}} \tag{2}$$

Para a equação de momento:

$$\begin{aligned}
&m_\alpha(n_{\alpha\,0} + n_{\alpha\,1})\left[\frac{\partial}{\partial t}\boldsymbol{u}_{\alpha\,0} + \frac{\partial}{\partial t}\boldsymbol{u}_{\alpha\,1} + (\boldsymbol{u}_{\alpha\,0}\cdot\boldsymbol{\nabla})(\boldsymbol{u}_{\alpha\,0} + \boldsymbol{u}_{\alpha\,1}) + (\boldsymbol{u}_{\alpha\,1}\cdot\boldsymbol{\nabla})(\boldsymbol{u}_{\alpha\,0} + \boldsymbol{u}_{\alpha\,1})\right]\\
&= q_\alpha(n_{\alpha\,0} + n_{\alpha\,1})\left[\boldsymbol{E}_0 + \boldsymbol{E}_1 + (\boldsymbol{u}_{\alpha\,0} + \boldsymbol{u}_{\alpha\,1})\times(\boldsymbol{B}_0 + \boldsymbol{B}_1)\right] - \boldsymbol{\nabla}p_{\alpha\,0} - \boldsymbol{\nabla}p_{\alpha\,1}
\end{aligned}$$

Novamente, as derivadas temporais de quantidades de equilíbrio se anulam e termos não-lineares como $(\boldsymbol{u}_{\alpha\,1}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{\alpha\,1}$ e $\boldsymbol{u}_{\alpha\,1}\times\boldsymbol{B}_1$ são desprezados. Usando as relações de equilíbrio do item (a), temos:

$$m_\alpha\, n_{\alpha\,0}\left[\frac{\partial}{\partial t}\boldsymbol{u}_{\alpha\,1} + (\boldsymbol{u}_{\alpha\,0}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{\alpha\,1} + (\boldsymbol{u}_{\alpha\,1}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{\alpha\,0}\right] + m_\alpha\, n_{\alpha\,1}(\boldsymbol{u}_{\alpha\,0}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{\alpha\,0} =$$
$$= q_\alpha\left[n_{\alpha\,0}\,\boldsymbol{E}_{\alpha\,1} + n_{\alpha\,1}\,\boldsymbol{E}_{\alpha\,0} + n_{\alpha\,0}(\boldsymbol{u}_{\alpha\,0}\times\boldsymbol{B}_1 + \boldsymbol{u}_{\alpha\,1}\times\boldsymbol{B}_0) + n_{\alpha\,1}\boldsymbol{u}_{\alpha\,0}\times\boldsymbol{B}_0\right] - \boldsymbol{\nabla}p_{\alpha\,1}$$

$$m_\alpha\, n_{\alpha\,0}\left[\frac{\partial}{\partial t}\boldsymbol{u}_{\alpha\,1} + (\boldsymbol{u}_{\alpha\,0}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{\alpha\,1} + (\boldsymbol{u}_{\alpha\,1}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{\alpha\,0}\right] = q_\alpha\, n_{\alpha\,0}\,(\boldsymbol{E}_1 + \boldsymbol{u}_{\alpha\,0}\times\boldsymbol{B}_1 + \boldsymbol{u}_{\alpha\,1}\times\boldsymbol{B}_0)$$
$$-\boldsymbol{\nabla}\,p_{\alpha\,1} - n_{\alpha,1}\Big[m_\alpha(\boldsymbol{u}_{\alpha\,0}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{\alpha\,0} - q_\alpha(\boldsymbol{E}_0 + \boldsymbol{u}_{\alpha\,0}\times\boldsymbol{B}_0)\Big]$$

O último termo, pela relação de equilíbrio, é $-\boldsymbol{\nabla}\,p_{\alpha\,0}/n_{\alpha_0}$, então:

$$\boxed{\begin{aligned}
&m_e\, n_{e\,0}\left[\frac{\partial}{\partial t}\boldsymbol{u}_{e\,1} + (\boldsymbol{u}_{e\,0}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{e\,1} + (\boldsymbol{u}_{e\,1}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{e\,0}\right] = -e\, n_{e\,0}\,(\boldsymbol{E}_1 + \boldsymbol{u}_{e\,0}\times\boldsymbol{B}_1 + \boldsymbol{u}_{e\,1}\times\boldsymbol{B}_0) - \\
&-\boldsymbol{\nabla}\,p_{e\,1} + \frac{n_{e\,1}}{n_{e\,0}}\boldsymbol{\nabla}\,p_{e\,0} \\
&m_i\, n_{i\,0}\left[\frac{\partial}{\partial t}\boldsymbol{u}_{i\,1} + (\boldsymbol{u}_{i\,0}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{i\,1} + (\boldsymbol{u}_{i\,1}\cdot\boldsymbol{\nabla})\boldsymbol{u}_{i\,0}\right] = e\, n_{i\,0}\,(\boldsymbol{E}_1 + \boldsymbol{u}_{i\,0}\times\boldsymbol{B}_1 + \boldsymbol{u}_{i\,1}\times\boldsymbol{B}_0) - \\
&-\boldsymbol{\nabla}\,p_{i\,1} + \frac{n_{i\,1}}{n_{i\,0}}\boldsymbol{\nabla}\,p_{i\,0}
\end{aligned}} \tag{3}$$

A equação de estado fica:

$$\boldsymbol{\nabla}p_{\alpha\,0} + \boldsymbol{\nabla}p_{\alpha\,1} = \gamma_\alpha\, k_B\, T_{\alpha\,0}(\boldsymbol{\nabla}n_{\alpha\,0} + \boldsymbol{\nabla}n_{\alpha\,1})$$

Logo:

$$\boxed{\begin{aligned}\boldsymbol{\nabla}p_{e\,1} &= k_B\, T_{e\,0}\boldsymbol{\nabla}n_{e\,1}\\ \boldsymbol{\nabla}p_{i\,1} &= \gamma_i\, k_B\, T_{i\,0}\boldsymbol{\nabla}n_{i\,1}\end{aligned}} \tag{4}$$

A Lei de Gauss tem o formato:

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}_0 + \boldsymbol{\nabla}\cdot\boldsymbol{E}_1 = \frac{e}{\epsilon_0}(n_{i\,0} + n_{i\,1} - n_{e\,0} - n_{e\,1})$$

Usando a relação do equilíbrio para a Lei de Gauss:

$$\boxed{\boldsymbol{\nabla}\cdot\boldsymbol{E}_1 = \frac{e}{\epsilon_0}(n_{i\,1} - n_{e\,1})} \tag{5}$$

Por último, linearizando a Lei de Ampère:

$$\boldsymbol{\nabla}\times\boldsymbol{B}_0+\boldsymbol{\nabla}\times\boldsymbol{B}_1=\mu_0\,e[n_{i\,0}\boldsymbol{u}_{i\,0}+n_{i\,0}\boldsymbol{u}_{i\,1}+n_{i\,1}\boldsymbol{u}_{i\,0}+n_{i\,1}\boldsymbol{u}_{i\,1}-(n_{e\,0}\boldsymbol{u}_{e\,0}+n_{e\,0}\boldsymbol{u}_{e\,1}+$$
$$+\,n_{e\,1}\boldsymbol{u}_{e\,0}+n_{e\,1}\boldsymbol{u}_{e\,1})]$$

Usando a relação de equilíbrio da Lei de Ampère e desprezando termos não-lineares:

$$\boxed{\boldsymbol{\nabla}\times\boldsymbol{B}_1=\mu_0\,e\,(n_{i\,0}\boldsymbol{u}_{i\,1}+n_{i\,1}\boldsymbol{u}_{i\,0}-n_{e\,0}\boldsymbol{u}_{e\,1}-n_{e\,1}\boldsymbol{u}_{e\,0})} \tag{6}$$

(c) Como as quantidades pertubativas são harmônicas, podemos fazer: $\boldsymbol{\nabla}\to i\boldsymbol{k}$; $\frac{\partial}{\partial t}\to -i\omega$. Então, para a equação de continuidade, como $n_{\alpha\,0}$ é constante e $\boldsymbol{u}_{\alpha\,0}$ é 0, então:

$$-\,i\omega\,\bar{n}_{e\,1}+i\,n_{e\,0}\,\boldsymbol{k}\cdot\bar{\boldsymbol{u}}_{e\,1}+i\,\bar{n}_{e,1}\underbrace{(\boldsymbol{k}\cdot\boldsymbol{u}_{e\,0})}_{=0}=0$$
$$-\,i\omega\,\bar{n}_{i\,1}+i\,n_{i\,0}\,\boldsymbol{k}\cdot\bar{\boldsymbol{u}}_{i\,1}+i\,\bar{n}_{i,1}\underbrace{(\boldsymbol{k}\cdot\boldsymbol{u}_{i\,0})}_{=0}=0$$

$$\boxed{\begin{aligned}\frac{\bar{n}_{e\,1}}{n_{e,0}}&=\frac{\boldsymbol{k}\cdot\bar{\boldsymbol{u}}_{e\,1}}{\omega}\\ \frac{\bar{n}_{i\,1}}{n_{i,0}}&=\frac{\boldsymbol{k}\cdot\bar{\boldsymbol{u}}_{i\,1}}{\omega}\end{aligned}} \tag{7}$$

Para a equação de energia:

$$i\boldsymbol{k}\,\bar{p}_{e\,1}=k_B\,T_{e\,0}(i\boldsymbol{k})\,\bar{n}_{e\,1}$$
$$i\boldsymbol{k}\,\bar{p}_{i\,1}=\gamma_i k_B\,T_{i\,0}(i\boldsymbol{k})\,\bar{n}_{i\,1}$$

$$\boxed{\begin{aligned}\bar{p}_{e\,1}&=k_B\,T_{e\,0}\,\bar{n}_{e\,1}\\ \bar{p}_{i\,1}&=k_B\,T_{i\,0}\,\bar{n}_{i\,1}\end{aligned}} \tag{8}$$

Já a Lei de Gauss:

$$\boxed{i(\boldsymbol{k}\cdot\bar{\boldsymbol{E}}_1)=\frac{e}{\epsilon_0}(\bar{n}_{i\,1}-\bar{n}_{e\,1})} \tag{9}$$

Para a Lei de Ampère:

$$i\boldsymbol{k}\times\bar{\boldsymbol{B}}_1=\mu_0\,e(\bar{n}_{i0}\,\bar{\boldsymbol{u}}_{i1}+\underbrace{\bar{n}_{i1}\,\boldsymbol{u}_{i0}}_{=0}-n_{e\,0}\,\bar{\boldsymbol{u}}_{e\,1}-\underbrace{\bar{n}_{e\,1}\,\boldsymbol{u}_{e\,0}}_{=0})$$

Usando a condição de quasineutralidade: $n_{i\,0} = n_{e\,0} = n_0$

$$\boxed{i\,\boldsymbol{k} \times \bar{\boldsymbol{B}}_1 = \mu_0\, e\, n_0(\bar{\boldsymbol{u}}_{i1} - \bar{\boldsymbol{u}}_{e\,1})} \tag{10}$$

Para a equação de momento:

$$\begin{aligned}
-\,i\omega\, m_e\, n_{e\,0}\, \bar{\boldsymbol{u}}_{e\,1} &= -e\, n_{e\,0}(\bar{\boldsymbol{E}}_1 + \bar{\boldsymbol{u}}_{e\,1} \times \boldsymbol{B}_0) - i\boldsymbol{k}\, \bar{p}_{e\,1} \\
-\,i\omega\, m_i\, n_{i\,0}\, \bar{\boldsymbol{u}}_{i\,1} &= e\, n_{i\,0}(\bar{\boldsymbol{E}}_1 + \bar{\boldsymbol{u}}_{i\,1} \times \boldsymbol{B}_0) - i\,\boldsymbol{k}\, \bar{p}_{i\,1}
\end{aligned}$$

$$\boxed{\begin{aligned}
\bar{\boldsymbol{u}}_{e\,1} &= -\frac{ie}{m_e\omega}\bar{\boldsymbol{E}}_1 + \frac{\bar{p}_{e\,1}}{m_e\,\omega\, n_{e\,0}}\boldsymbol{k} \\
\bar{\boldsymbol{u}}_{i\,1} &= \frac{ie}{m_i\,\omega}\bar{\boldsymbol{E}}_1 + \frac{\bar{p}_{e\,1}}{m_i\,\omega\, n_{i\,0}}\boldsymbol{k}
\end{aligned}} \tag{11}$$

(d) Multiplicando as 2 equações em (11) por $\boldsymbol{k}$, temos:

$$\begin{aligned}
\bar{\boldsymbol{u}}_{e\,1} \cdot \boldsymbol{k} &= -\frac{ie}{m_e\omega}\bar{\boldsymbol{E}}_1 \cdot \boldsymbol{k} + \frac{\bar{p}_{e\,1}}{m_e\,\omega\, n_{e\,0}}k^2 \\
\bar{\boldsymbol{u}}_{i\,1} \cdot \boldsymbol{k} &= -\frac{ie}{m_i\omega}\bar{\boldsymbol{E}}_1 \cdot \boldsymbol{k} + \frac{\bar{p}_{i\,1}}{m_i\,\omega\, n_{i\,0}}k^2
\end{aligned}$$

$$\begin{aligned}
\bar{\boldsymbol{u}}_{e\,1} \cdot \boldsymbol{k} &= -\frac{e^2}{\epsilon_0\, m_e\,\omega}(\bar{n}_{i\,1} - \bar{n}_{e\,1}) + \frac{\bar{p}_{e\,1}}{m_e\,\omega\, n_{e\,0}}k^2 \\
\bar{\boldsymbol{u}}_{e\,1} \cdot \boldsymbol{k} &= \frac{e^2}{\epsilon_0\, m_e\,\omega}(\bar{n}_{i\,1} - \bar{n}_{e\,1}) + \frac{\bar{p}_{e\,1}}{m_e\,\omega\, n_{e\,0}}k^2
\end{aligned}$$

Usando os resultados em (7) e multiplicando por $\omega\, n_0$:

$$\begin{aligned}
\omega^2\, \bar{n}_{e\,1} &= -\frac{e^2\, n_0}{\epsilon_0\, m_e}(\bar{n}_{i\,1} - \bar{n}_{e\,1}) + \frac{\bar{p}_{e\,1}}{m_e}k^2 \\
\omega^2\, \bar{n}_{i\,1} &= \frac{e^2\, n_0}{\epsilon_0\, m_i}(\bar{n}_{i\,1} - \bar{n}_{e\,1}) + \frac{\bar{p}_{i\,1}}{m_i}k^2
\end{aligned}$$

Como $\omega_p^2 = \frac{e^2\, n_0}{\epsilon_0\, m}$ e $\bar{p}_{\alpha\,1} = \gamma_\alpha\, k_B\, T_{\alpha\,0}\, \bar{n}_{\alpha\,1}$ e tirando as barras das quantidades:

$$\begin{aligned}
n_{e\,1}(\omega^2 - \omega_{pe}^2) &= -\omega_{pe}^2\, n_{i\,1} + \frac{k_B\, T_{e\,0}\, n_{e\,1}}{m_e}\, k^2 \\
n_{i\,1}(\omega^2 - \omega_{pi}^2) &= -\omega_{pi}^2\, n_{e\,1} + \frac{\gamma_i\, k_B\, T_{i\,0}\, n_{i\,1}}{m_i}\, k^2
\end{aligned}$$

Solução Prova 2

Escrevendo $v_{th,\alpha}^2 = \frac{\gamma_\alpha k_B T_{\alpha 0}}{m_\alpha}$, temos que:

$$n_{e\,1}(\omega^2 - \omega_{pe}^2 - k^2 v_{th,e}^2) = -\omega_{pe}^2 n_{i\,1}$$
$$n_{i\,1}(\omega^2 - \omega_{pi}^2 - k^2 v_{th,i}^2) = -\omega_{pi}^2 n_{e\,1}$$

$$\implies n_{e\,1} = -\frac{n_{1\,i}}{\omega_{pi}^2}(\omega^2 - \omega_{pi}^2 - k^2 v_{th,i}^2)$$

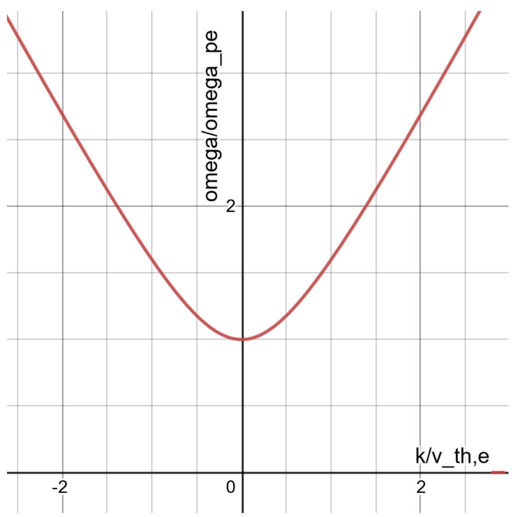
Logo, substituindo na primeira equação acima:

$$(\omega^2 - \omega_{pi}^2 - k^2 v_{th,i}^2)(\omega^2 - \omega_{pe}^2 - k^2 v_{th,e}^2)\, n_{1\,i} = \omega_{pe}^2 \omega_{pi}^2 n_{1\,i}$$

Enfim:

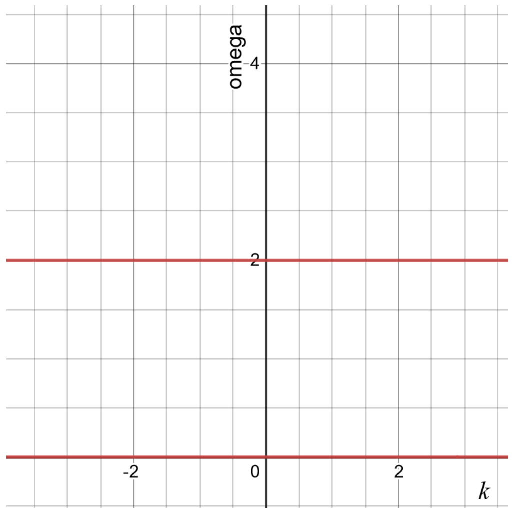
$$\boxed{(\omega^2 - \omega_{pi}^2 - k^2 v_{th,i}^2)(\omega^2 - \omega_{pe}^2 - k^2 v_{th,e}^2) = \omega_{pe}^2 \omega_{pi}^2} \tag{12}$$

(e) Para $T_{e\,0} \gg T_{i\,0}$, a relação de dispersão fica:



Essas oscilações gerarão as ondas de Langmuir. Em geral, percebemos que, devido a condição $m_e \ll m_i$, a parte dependente das quantidades dos íons na relação de dispersão é praticamente desprezável. Isso, novamente, indica que usar a aproximação de íons parados não é ruim e o erro cometido é pequeno.

Para $T_{e\,0} = T_{i\,0} = 0$, o esboço da relação de dispersão é:



Ou seja, as ondas serão oscilações de plasma-elétron, em que a frequência é a frequência ω_{pe}.