MAE 314 1ª Prova 2001

1) a) $(X_1, X_4) \sim N_2\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \right)$

b) $X_1 | X_4 = x \sim N_1(\mu, \sigma^2)$

$$\mu = \mu_{X_1} + Cov(X_1, X_4)\left[Var(X_4)\right]^{-1}(x-2)$$

$$= 3 + 1 \cdot \frac{1}{2}(x-2) = 3 + \frac{x}{2} - 1 = 2 + \frac{x}{2}$$

$$\sigma^2 = Var(X_1) - Cov(X_1, X_4)\frac{1}{2}Cov(X_1, X_4)$$

$$= 3 - 1 \cdot \frac{1}{2} 1 = 3 - \frac{1}{2} = \frac{5}{2}$$

c) $Y = [2 \ 3 \ 1 \ 1] X \sim N_1(\mu, \sigma^2)$

$$\sigma^2 = [2 \ 3 \ 1 \ 1] \begin{bmatrix} 3 & 0 & 2 & 1 \\ 0 & 4 & 0 & 0 \\ 2 & 0 & 4 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \end{bmatrix} =$$

$$= [9 \ 12 \ 9 \ 5] \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \end{bmatrix} = 68$$

$$\mu = [2 \ 3 \ 1 \ 1] \begin{bmatrix} 3 \\ 1 \\ 4 \\ 2 \end{bmatrix} = 15$$

d) $X_2 \sim N(1,4)$ $\left(\dfrac{X_2-1}{2}\right) \sim N(0,1)$

$W = \left(\dfrac{X_2-1}{2}\right)^2 \sim \chi^2_1$ $P(W>c) = 0{,}10$ $c = 2{,}706$

2) a) Σ será p.d. se $\begin{vmatrix} 1 & k \\ k & 1 \end{vmatrix} = 1-k^2 > 0$ e

$k^2 < 1$ $-1 < k < 1$ e

$\begin{vmatrix} 1 & k & 0 \\ k & 1 & k \\ 0 & k & 1 \end{vmatrix} = -2k^2+1 > 0$ $k^2 < \dfrac{1}{2}$ $\dfrac{-1}{\sqrt{2}} < k < \dfrac{1}{\sqrt{2}}$

$\Rightarrow \dfrac{-\sqrt{2}}{2} < k < \dfrac{\sqrt{2}}{2}$

b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} \sim N_2(\mu, \Sigma)$

$\Sigma = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & k & 0 \\ k & 1 & k \\ 0 & k & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 4k+3 & -2k-1 \\ -2k-1 & 3 \end{bmatrix}$

$-2k-1 = 0$ $k = -\dfrac{1}{2}$ $Y_1 - Y_2 - Y_3$ e $Y_1+Y_2+Y_3$ têm dist

N_2, com covariância 0 $\Rightarrow$ independentes

$k = -\dfrac{1}{2} \in \left]-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right[$

3) $Y = CX + d \sim N(C\mu_x + d,\ C\Sigma C')$

$H_0: \mu_x = \mu_0$ $\quad T_x^2 = n(\bar{X} - \mu_0)' S^{-1} (\bar{X} - \mu_0)$

$H_a: \mu_x \neq \mu_0$

$H_0: \mu_Y = C\mu_0 + d$

$H_a: \mu_Y \neq C\mu_0 + d$

$$T_Y^2 = n(\bar{Y} - C\mu_0 - d)'(CSC')^{-1}(\bar{Y} - C\mu_0 - d)$$

$$= n(C\bar{X} + d - C\mu_0 - d)' C'^{-1} S^{-1} C^{-1} (C\bar{X} + d - C\mu_0 - d)$$

$$= n[C(\bar{X} - \mu_0)]' C'^{-1} S^{-1} C^{-1} [C(\bar{X} - \mu_0)]$$

$$= n(\bar{X} - \mu_0)' \underbrace{C' C'^{-1}}_{I} S^{-1} \underbrace{C^{-1} C}_{I} (\bar{X} - \mu_0) = n(\bar{X} - \mu_0)' S^{-1} (\bar{X} - \mu_0)$$

4) a) $\begin{vmatrix} 4-\lambda & 3 \\ 3 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (4-\lambda)^2 = 9$ RC:

$4-\lambda = 3 \Rightarrow \lambda = 1$

$4-\lambda = -3 \Rightarrow \lambda = 7$

$$\begin{bmatrix} 4-7 & 3 \\ 3 & 4-7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3x_1 + 3x_2 \\ 3x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad x_1 - x_2 = 0 \quad x_1 = x_2 = 1$$

$$L_1 = \begin{bmatrix} 1/\sqrt{2} = \sqrt{2}/2 \\ 1/\sqrt{2} = \sqrt{2}/2 \end{bmatrix}$$

$x_1 = 1 \quad x_2 = -1$

$$\begin{bmatrix} 4-1 & 3 \\ 3 & 4-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 + 3x_2 \\ 3x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad L_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\sqrt{2}/2 \end{bmatrix}$$

b) $Var(Y_1) = \left(\frac{\sqrt{2}}{2}\right)^2 \cdot 4 + \left(\frac{\sqrt{2}}{2}\right)^2 \cdot 4 + 2\,\frac{\sqrt{2}}{2}\,\frac{\sqrt{2}}{2} \cdot 3 =$

$= \frac{2}{4} \cdot 4 + \frac{2}{4} \cdot 4 + 3 = 7 = \lambda_1$

$Var(Y_2) = \frac{2}{4} \cdot 4 + \frac{2}{4} \cdot 4 - 3 = 1 = \lambda_2$

$Var(Y_1) + Var(Y_2) = 8 = \underbrace{Var(X_1)}_{4} + \underbrace{Var(X_2)}_{4}$

c) $Cov(Y_1, Y_2) = \underset{A}{\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}} \underset{Var(Y)}{\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}} \underset{B'}{\begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}} =$

$= \begin{bmatrix} 7\sqrt{2}/2 & 7\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix} = 7\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} - 7\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} = 0$

5) a) $Var(AX^{(1)}) = A\,Var(X^{(1)})\,A' =$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 1 \\ 1 & 5 \end{bmatrix}$$

b) $Cov(AX^{(1)}, BX^{(2)}) = A\,Cov(X^{(1)}, X^{(2)})\,B'$

$$= A \begin{bmatrix} +1/2 & -1/2 & 0 \\ 1 & -1 & 0 \end{bmatrix} B' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} +1/2 & -1/2 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 & 1/2 & 0 \\ 3/2 & -3/2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$