1. **3.16** Derive $P_3(x)$ da fórmula de Rodrigues e verifique que $P_3(\cos\theta)$ satisfaz a equação diferencial para a função angular $\Theta(\theta)$ para $\ell = 3$. Verifique que P_3 e P_1 são funções ortogonais por integração explícita.

solução

formula de Rodrigues $P_l(x) = \frac{1}{2^l\, l!} \frac{d^l}{dx^l}(x^2-1)^l$

$l = 3 \quad P_3(x) = \frac{1}{2^3 \times 3!} \frac{d^3}{dx^3}\left[(x^2-1)^3\right]$, $\frac{d}{dx}(x^2-1)^3 = 3(2x)(x^2-1)^2 = 6x(x^2-1)^2$

$\frac{d^2}{dx^2}\left[6x(x^2-1)^2\right] = 6(x^2-1)^2 + 2 \times 2x(x^2-1)6x = 6(x^2-1)^2 + 24x^2(x^2-1)$

$\frac{d^3}{dx^3}\left[6(x^2-1)^2 + 24x^2(x^2-1)\right] = 6 \times 2(2x)(x^2-1) + 24(2x)(x^2-1) + 24x^2(2x)$

$= 24x(x^2-1) + 48x(x^2-1) + 48x^3$

$\frac{1}{48}\frac{d^3}{dx^2}\left[(x^2-1)^3\right] = \frac{24x^3}{48} - \frac{24x}{48} + \frac{48x^3}{48} - \frac{48x}{48} + \frac{48}{48}x^3 =$

$= \frac{1}{2}x^3 - \frac{1}{2}x + x^3 - x + x^3 = 2x^3 + \frac{1}{2}x^3 - \frac{1}{2}x - x$

$= \frac{4x^3 + x^3}{2} - \frac{x + 2x}{3} = \frac{5}{2}x^3 - \frac{3x}{2}$

$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x \quad ; \quad P_3(\cos\theta) = \frac{5}{2}(\cos\theta)^3 - \frac{3}{2}\cos\theta = \frac{\cos\theta}{2}\left(5\cos\theta - 3\right)$$

$$P_3(x) = \frac{1}{2}\cos\theta\left(5\cos^2\theta - 3\right)$$

Verificamos que $P_3(\cos\theta)$ satifaz a função angular $\Theta(\theta)$ par $l = 3$

$$\frac{1}{\operatorname{sen}\theta}\frac{d}{d\theta}\left(\operatorname{sen}\theta\frac{dP_3}{d\theta}\right) \overset{?}{=} -l(l+1)P_3 \quad ;$$

$$\frac{dP_3}{d\theta} = \frac{1}{2}(-\operatorname{sen}\theta)\left(5\cos^2\theta - 3\right) + \frac{1}{2}\cos\theta\left(5\times 2\left(-\operatorname{sen}\theta\cos\theta\right)\right)$$

$$= \frac{1}{2}\left(-5\operatorname{sen}\theta\cos^2\theta + 3\operatorname{sen}\theta - 10\operatorname{sen}\theta\cos^2\theta\right) = \frac{1}{2}\left(-15\operatorname{sen}\theta\cos^2\theta + 3\operatorname{sen}\theta\right)$$

$$\frac{dP_3}{d\theta} = \frac{3}{2}\operatorname{sen}\theta\left(1 - 5\cos^2\theta\right)$$

$$\frac{1}{\operatorname{sen}\theta}\frac{d}{d\theta}\left(\operatorname{sen}\theta\frac{dP_3}{d\theta}\right) = 3\cos\theta\left(6 - 10\cos^2\theta\right) = -3\cos\theta\left(10\cos^2\theta - 6\right)$$

$$= -2\times 3\cos\theta\left(5\cos^2\theta - 3\right)\times\frac{1}{2}\times 2$$

$$= -4\times 3\times\frac{1}{2}\cos\theta\left(5\cos^2\theta - 3\right)$$

$$3(3+1) \Rightarrow l = 3$$

$$\Delta P_3(\cos\theta) = -(12)\,P_3(\cos\theta) \Rightarrow l(l+1) = 12$$

P_1 e P_3 são orthogonais se $\int_{-1}^{1} P_1(x) \cdot P_2(x)\, dx = ?$

$$P_1(x) = x \; ; \; P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$$

$$\int_{-1}^{1} P_1(x) \cdot P_3(x)\, dx = \int_{-1}^{1} \left(\frac{5}{2}x^3 - \frac{3}{2}x\right) x = \int_{-1}^{1} \left[\frac{5}{2}x^4 - \frac{3}{2}x^2\right] d\cancel{x} =$$

$$\int_{-1}^{1} P_1(x) \cdot P_3(x)\, dx = \left[\frac{\cancel{5}}{2} \times \frac{x^5}{\cancel{5}} - \frac{\cancel{3}}{2}\frac{x^3}{\cancel{3}}\right]_{-1}^{1} = \left(\frac{\cancel{5}}{2} \times (1)^5 - \frac{1}{2}(1) - \left(\frac{1}{2}\right) + \left(\frac{-1}{2}\right)\right)$$

$$\int_{-1}^{1} P_1(x) \cdot P_3(x) = 0 \Rightarrow P_1(x) \perp P_3(x)$$

2. **3.18** O potencial na superfície de uma esfera de raio R é dado por

$$V_0 = k\cos 3\theta,$$

onde k é uma constante. Encontre o potencial dentro e fora da esfera, bem como densidade superficial de carga $\sigma(\theta)$ na esfera. Suponha que inexiste carga dentro ou fora da esfera.

$$V_0 = K\cos 3\theta = K(4\cos^3\theta - 3\cos\theta) = K\left[\alpha P_3(\cos\theta) - B P_1(\cos\theta)\right]$$

$$K\cos 3\theta = 4(\cos\theta)^3 - 3\cos\theta$$

$$K\cos 3\theta = \alpha P_3(\cos\theta) - B P_1(\cos\theta) = \alpha\frac{1}{2}(5\cos^3\theta - 3\cos\theta) + B\cos\theta$$

$$K\cos 3\theta = \frac{5}{2}\alpha\cos^3\theta - \frac{3}{2}\alpha\cos\theta + B\cos\theta = \frac{5}{2}\alpha\cos^3\theta + \left(B - \frac{3}{2}\right)\cos\theta$$

$$\frac{5}{2}\alpha = 4 \Rightarrow \alpha = 4\times\frac{2}{5} = \frac{8}{5}$$

$$B - \frac{3}{2}\alpha = -3 \Rightarrow B = -\frac{15}{5} + \frac{3}{\cancel{2}}\times\frac{\cancel{8}\,4}{5} = -\frac{15}{5} + \frac{12}{5} = \frac{-3}{5}$$

$$V_\theta = K\left[\frac{8}{5}P_3(\cos\theta) - \frac{3}{5}P_1(\cos\theta)\right]$$

$$V(r,\theta) = \begin{cases} \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) & \text{para } r \leq R \\ \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) & \text{para } r \geq R \end{cases}$$

$$A_\ell = \frac{2\ell+1}{2R^\ell}\int_0^\pi V_0(\theta)\,P_\ell(\cos\theta)\,\text{sen}\,\theta\,d\theta \quad ; \quad V_0(\theta) = \left[\frac{8}{5}P_3(\cos\theta) - \frac{3}{5}P_1(\cos\theta)\right]k$$

$$A_\ell = \frac{2\ell+1}{2R^\ell}\int_0^\pi \frac{K}{5}\left[8\left(P_3(\cos\theta) - 3P_1(\cos\theta)\right]P_\ell(\cos\theta)\,\text{sen}\,\theta\,d\theta$$

$$A_\ell = \frac{2\ell+1}{2R^\ell}\int_0^\pi \left(\frac{K}{5}\times 8P_3(\cos\theta)P_\ell(\cos\theta)\,\text{sen}\,\theta\,d\theta - \frac{3K}{5}P_1(\cos\theta)P_\ell(\cos\theta)\,\text{sen}\,\theta\,d\theta\right)$$

$$A_\ell = \frac{2\ell+1}{2R^\ell}\left(\int_0^\pi \frac{K}{5}\times 8\,\underbrace{P_3(\cos\theta)P_\ell(\cos\theta)\,\text{sen}\,\theta\,d\theta}_{=\,0 \text{ se } \ell\neq 3;\ =\,2/7 \text{ se } \ell=3} - \frac{3K}{5}\int_0^\pi \underbrace{P_1(\cos\theta)P_\ell(\cos\theta)\,\text{sen}\,\theta\,d\theta}_{=\,0 \text{ se } \ell\neq 0;\ =\,2/3 \text{ se } \ell=1}\right)$$

$$A_\ell = \begin{cases} \dfrac{8K}{5R^3} & \text{se } \ell = 3 \\[2mm] -\dfrac{3K}{5R^1} & \text{si } \ell = 1 \\[2mm] 0 & \text{si } \ell\neq 1 \text{ e } \ell\neq 3 \end{cases} \quad ; \quad V(r,\theta) = \sum_{l=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) \quad \text{si } r\leq R$$

$$V(r,\theta) = \frac{8K}{5R^3}\times r^3 P_3(\cos\theta) + \frac{3k}{5R}\,r\,P_1(\cos\theta)$$

$$V(r,\theta) = \frac{8}{5}k\left(\frac{r}{R}\right)^3 P_3(\cos\theta) - \frac{3}{5}k\,\frac{r}{R}\,P_1(\cos\theta)$$

si $r \geq R$ $\quad V(r,\theta) = \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$

Continuidade : $V(R,\theta) = \frac{B_\ell}{R^{\ell+1}} \cancel{P_\ell(\cos\theta)} = A^\ell R^\ell \cancel{P_\ell(\cos\theta)}$

$$\frac{B_\ell}{R^{\ell+1}} = A^\ell R^\ell \Rightarrow B_\ell = A^\ell R^\ell \times R^{(\ell+1)} = A^\ell R^{(\ell+1+\ell)} = A^\ell R^{2\ell+1}$$

$$B_\ell = \begin{cases} \frac{8k}{5\cancel{R^3}} \times R^{\cancel{7}\,4} & \text{se } \ell = 3 \\ -\frac{3k}{5\cancel{R}} \times R^{\cancel{3}\,2} & \text{se } \ell = 1 \end{cases} \Rightarrow B_\ell = \begin{cases} \frac{8k}{5} R^4 & \text{se } \ell = 3 \\ -\frac{3k}{5} R^2 & \text{se } \ell = 1 \end{cases}$$

$$V(r,\theta) = \frac{8k}{5}\frac{R^4}{r^4} P_3(\cos\theta) - \frac{3k}{5}\frac{R^2}{r^2} P_1(\cos\theta) = \frac{8}{5}k\left(\frac{R}{r}\right)^4 P_3(\cos\theta) - \frac{3}{5}k\left(\frac{R}{r}\right)^2 P_1(\cos\theta)$$

$$\sigma_0 = \varepsilon_0 \sum_{\ell=0} (2\ell+1) A_\ell R^{\ell-1} P_\ell(\cos\theta) = \varepsilon_0 \left(3 A_1 P_1 + 7 A_3 R^2 P_3\right.$$

$$= \varepsilon_0 \left[3\left(\frac{3k}{5R}\right) P_1 + 7\,\frac{8k}{5R^3} \times R^2 P_3 = \frac{\varepsilon_0 k}{5R}\left(-9 P_1(\cos\theta) + 56 P_3(\cos\theta)\right)\right.$$

$$= \frac{\varepsilon_0 k}{5R}\left\{-9\cos\theta + \frac{56}{2}\left(5\cos^3\theta - 3\cos\theta\right) = \frac{\varepsilon_0 k}{5R}\cos\theta\left(-9 + 28 \times 5\cos^2\theta - 28\times 3\right) = \frac{\varepsilon_0 k}{5R}\left(140\cos^3\theta - 93\right)\right.$$

3. **3.19** Suponha que o potencial $V(\theta)$ na superfície de uma esfera seja especificado, e que não haja carga dentro ou fora da esfera. Mostre que a densidade de carga na esfera é dada por

$$\sigma(\theta) = \frac{\epsilon_0}{2R} \sum_{\ell=0}^{\infty} (2\ell+1)^2 C_\ell P_\ell(\cos\theta),$$

onde

$$C_\ell = \int_0^\pi V_0(\theta) P_\ell(\cos\theta) \sin\theta \, d\theta.$$

$$\begin{cases} V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) & r \leq R \\ V(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) & r \geq R \end{cases}$$

$$\left(\frac{dV}{dr}\text{fora} - \frac{dV}{dr}\text{dentro}\right)_{r=R} = -\frac{\sigma}{\varepsilon_0} \; ; \quad \frac{dV_d}{dr} = \sum_{\ell=0}^{\infty} \ell A_\ell r^{\ell-1} P_\ell(\cos\theta) \; ; \quad \frac{dV_f}{dr} = \frac{-(\ell+1)}{r^{\ell+2}} P_\ell(\cos\theta)$$

$$V(r=R,\theta)\text{dentro} = V(r=R,\theta)\text{fora} \Rightarrow A_\ell R^\ell P_\ell \cancel{(\cos\theta)} = \frac{B_\ell}{R^{\ell+1}} P_\ell \cancel{(\cos\theta)}$$

$$A_\ell R^\ell = \frac{B_\ell}{R^{\ell+1}} \Rightarrow B_\ell = A_\ell R^\ell \times R^{\ell+1} = A_\ell R^{2\ell+1}$$

$$+\sum_{\ell=0}^{\infty} \frac{(\ell+1)B_\ell P_\ell(\cos\theta)}{R^{\ell+2}} + \sum_{\ell=0}^{\infty} \ell A_\ell r^{\ell-1} P_\ell(\cos\theta) = +\frac{\sigma}{\varepsilon_0} \Rightarrow$$

$$\sum_{\ell=0}^{\infty} (\ell+1) \frac{A_\ell R^{2\ell+1}}{R^{\ell+2}} P_\ell(\cos\theta) + \sum_{\ell=0}^{\infty} \ell A_\ell R^{\ell-1} P_\ell(\cos\theta) = \frac{\sigma}{\varepsilon_0}$$

$$\sigma = \varepsilon_0 \sum_{l=0} (2l+1) A_l R^{l-1} P_l(\cos\theta) = D$$

$$A_l = \frac{2l+1}{2R^l}\int_0^{\pi} V_0 P_l(\cos\theta)\,\mathrm{sen}\,\theta\, d\theta \quad ; \quad \sigma(\theta) = \frac{\varepsilon_0}{2}\sum_{l=0} \frac{(2l+1)^2}{2R^l}\int_0^{\pi} R^{l-1} P_l(\cos\theta)\,\mathrm{sen}\,\theta\, d\theta$$

$$\sigma(\theta) = \frac{\varepsilon_0}{2R}\sum 2l+1 \int P_l^2(\cos\theta)\,\mathrm{sen}\,\theta\, d\theta$$

$$\sigma(\theta) = \frac{\varepsilon_0}{2R}\sum_{l=0}^{\infty}(2l+1)^2 C_l P_l(\cos\theta) \quad \text{com} \quad C_l = \int_0^{\pi} V_0(\theta) P_l(\cos\theta)\,\mathrm{sen}\,\theta\, d\theta$$