Exo 1

1. 3.1 Encontre o potencial médio em uma superfície esférica de raio R devido a uma carga q localizada dentro da superfície. Mostre que

$$V_{\text{med}} = V_{\text{centro}} + \frac{Q_{int}}{4\pi\epsilon_0 R},$$

onde V_{centro} é o potencial no centro devido a todas a cargas externas, e Q_{int} é a carga interna.

$$V_{med} = \frac{1}{4\pi R^2}\int V \cdot ds = \frac{1}{4\pi R}\int \frac{q}{4\pi\varepsilon_0 r} R^2 \operatorname{sen}\theta \, d\theta \, d\phi \ ; \quad r^2 = R^2 + z^2 - 2Rz\cos\theta \quad = du$$

$$V_{med} = \frac{1}{4\pi \cancel{R^2}} \times \frac{q}{4\pi\varepsilon_0}\int \frac{\cancel{R^2}\operatorname{sen}\theta \, d\theta \, d\phi}{(R^2 + z^2 - 2Rz\cos\theta)^{1/2}} = \frac{q \times \cancel{2\pi}}{\cancel{4\pi}_{2} \times 4\pi\varepsilon_0}\int_0^{\pi} \frac{\operatorname{sen}\theta \, d\theta}{\sqrt{R^2 + z^2 - 2Rz\cos\theta}}$$

$$V_{med} = \frac{q}{2 \times 4\pi\varepsilon_0 R z}\left[\sqrt{R^2 + z^2 - 2Rz\cos\theta}\right]_0^{\pi}$$

$$V_{med} = \frac{q}{2 \times 4\pi\varepsilon_0}\frac{1}{Rz}\left(\sqrt{R^2 + z^2 + 2Rz} - \sqrt{R^2 + z^2 - 2Rz}\right)$$

$$V_{med} = \frac{q}{4\pi\varepsilon_0 \times 2Rz}\left(z + R - |z - R|\right)$$

como $z < R \Rightarrow z - R < 0$

$|z - R| = -z + R$

$$V_{med} = \frac{q}{4\pi\varepsilon_0 \times 2Rz}\left(z + \cancel{R} + z - \cancel{R}\right)$$

$$V_{med} = \frac{q \times \cancel{2}\cancel{z}}{4\pi\varepsilon_0 \times \cancel{2} R \cancel{z}} = \frac{q}{4\pi\varepsilon_0 R} \Rightarrow V_{med} = \frac{q}{4\pi\varepsilon_0 R}$$

$q = Q_{int}$

Mostre que $V_{med} = V_{centro} + \dfrac{Q_{int}}{4\pi\varepsilon_0 R}$

Se $q = Q_{int} \Rightarrow V_{med\,int} = \dfrac{Q_{int}}{4\pi\varepsilon_0 R}$

$V_{med\,ext} = V_{cent} \Rightarrow V_{med} = V_{med\,int} + V_{med\,ext}$

$$V_{med} = V_{cent} + \frac{Q_{int}}{4\pi\varepsilon_0 R}$$

2. 3.7

(a) Com ajuda da lei dos cossenos, mostre que o potencial na superfície de uma esfera condutora com raio R, aterrada, devido a uma carga q a uma distância $a > R$ do centro, é

$$V(r,\theta) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{1}{\sqrt{R^2+(ra/R)^2-2ra\cos\theta}}\right),$$

onde r e θ são as coordenadas esféricas quando o eixo z passa pela carga q.

(b) Encontre a carga superficial induzida na esfera, em função de θ. Integre para encontrar a carga induzida.

Exercicio 2

$$V = \frac{q}{4\pi\varepsilon_0 \mathscr{r}} + \frac{q'}{4\pi\varepsilon_0 \mathscr{r}'}$$

$$\mathscr{r}^2 = r^2 + a^2 - 2ra\cos\theta \;;$$

$$\mathscr{r}'^2 = r^2 + b^2 - 2rb\cos\theta$$

$$V_m = V_n = 0 \quad ($$

$$V_m = \frac{q}{\cancel{4\pi\varepsilon_0}(a+R)} + \frac{q'}{\cancel{4\pi\varepsilon_0}(R+b)} = 0,$$

$$V_n = \frac{q}{\cancel{4\pi\varepsilon_0}(a-R)} + \frac{q'}{\cancel{4\pi\varepsilon_0}(R-b)} = 0,$$

$$\frac{q}{a+R} + \frac{q'}{R+b} = 0 \Rightarrow q(R+b) + (a+R)q' = 0$$

$$\frac{q}{a-R} + \frac{q'}{R-a} = 0 \qquad q(R-b) + (a-R)q' = 0$$

$$2qR + 2aq' = 0$$

$$\cancel{2}qR = -\cancel{2}aq' \Rightarrow q' = \frac{-qR}{a}$$

$$\frac{\cancel{q}}{a+R} - \frac{\cancel{q}R}{a(R+b)} = 0 \Rightarrow \frac{1}{a+R} - \frac{R}{a(R+b)} = 0$$

$$a\cancel{R} + ab - \cancel{Ra} - R^2 = 0$$

$$R^2 - ab = 0 \Rightarrow \boxed{b = \frac{R^2}{a}} ; \; r' = \left(r^2 + b^2 - 2rb\cos\theta\right)^{1/2} = r^2 + \frac{R^4}{a^2} - 2r\frac{R^2}{a}\cos\theta$$

$$V = \frac{q}{4\pi\varepsilon_0 r} + \frac{q'}{4\pi\varepsilon_0 r'} = \frac{q}{4\pi\varepsilon_0\sqrt{r^2 + a^2 - 2ra\cos\theta}} + \frac{q'}{4\pi\varepsilon_0\sqrt{r^2 + \frac{R^4}{a^2} - 2r\frac{R^2}{a}\cos\theta}}$$

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q'

$$q = -\frac{q R}{a} ; \quad V = \frac{q}{4\pi\varepsilon_0\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{qR}{a\sqrt{r^2 + \frac{R^4}{a^2} - \frac{2rR^2\cos\theta}{a}}}$$

$$V = \frac{q}{4\pi\varepsilon_0\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{q}{4\pi\varepsilon_0\left(\frac{a}{R}\right)\sqrt{r^2 + \frac{R^4}{a^2} - 2r\frac{R^2}{a}\cos\theta}}$$

$$V = \frac{q}{4\pi\varepsilon_0\left(r^2 + a^2 - 2ra\cos\theta\right)^{1/2}} - \frac{q}{4\pi\varepsilon_0\sqrt{\frac{a^2r^2}{R^2} + \frac{\cancel{a^2}R^{\cancel{4}2}}{\cancel{R^2}\cancel{a^2}} - 2r\cancel{R^2} \times \frac{a\cancel{R}}{\cancel{R^2}\cancel{a}}\cos\theta}}$$

$$V = \frac{q}{4\pi\varepsilon_0\left(r^2 + a^2 - 2ra\cos\theta\right)^{1/2}} - \frac{q}{4\pi\varepsilon_0\left(R^2 + \left(\frac{ar}{R}\right)^2 - 2ra\cos\theta\right)}$$

b) $\sigma = -\varepsilon_0 \frac{\partial V}{\partial n} = -\varepsilon_0 \frac{\partial V}{\partial r}$; $V = \frac{q}{4\pi\varepsilon_0}\left[\left(r^2+a^2-2ra\cos\theta\right)^{-1/2} - \left(R^2+\left(\frac{ar}{R}\right)^2 - 2ra\cos\theta\right)^{-1/2}\right]$

$$\frac{\partial V}{\partial r} = \frac{q}{4\pi\varepsilon_0}\left(-\cancel{\frac{1}{2}}\left(\cancel{2}r-\cancel{2}a\cos\theta\right)\left(r^2+a^2-2ar\cos\theta\right)^{-3/2} + \cancel{\frac{1}{2}}\left(\cancel{2}\frac{a^2}{R^2}r-\cancel{2}a\cos\theta\right)\left(R^2+\left(\frac{ar}{R}\right)^2-2ra\cos\theta\right)^{-3/2}\right)$$

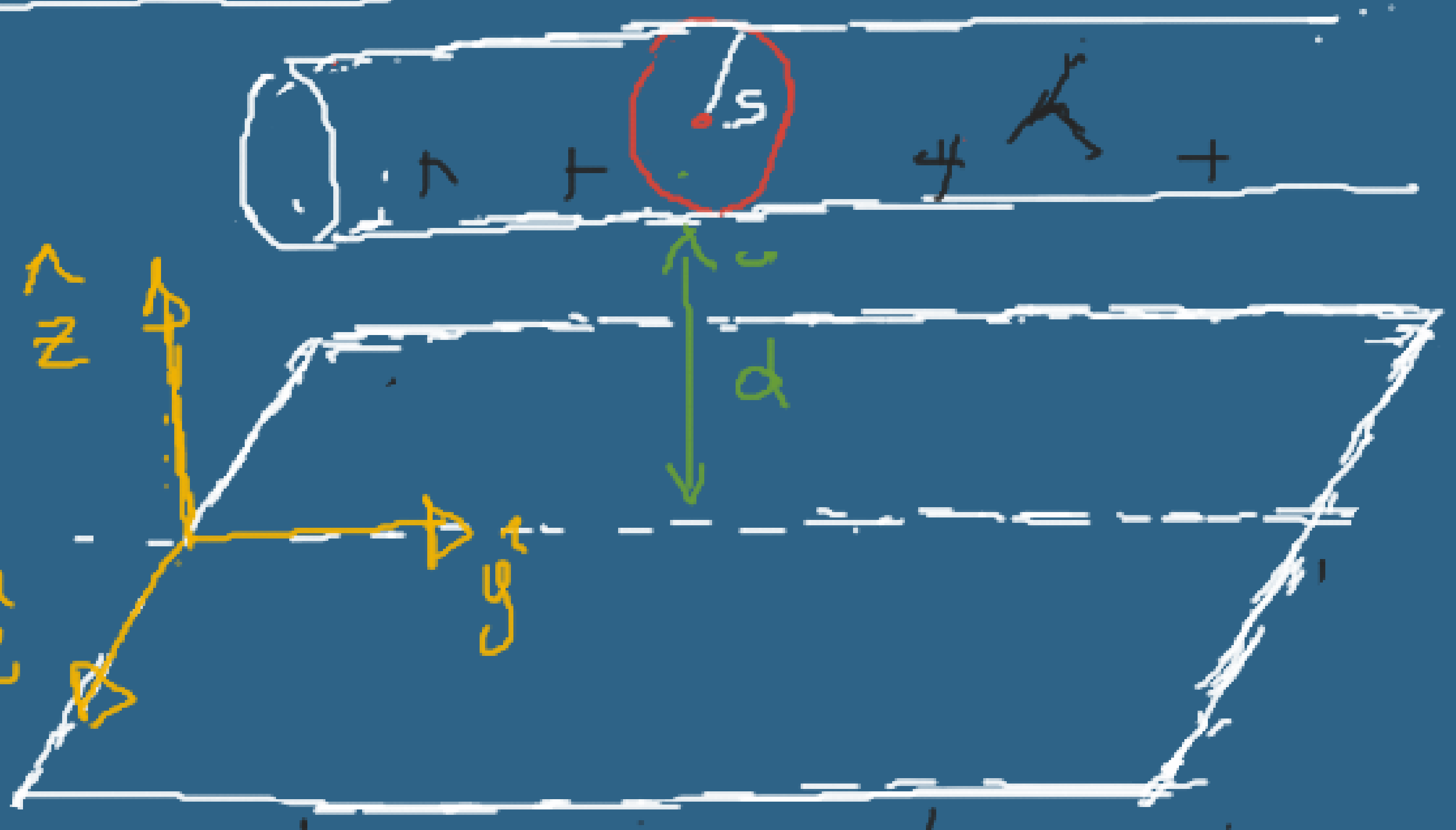
$$\frac{\partial V}{\partial r} = \frac{q}{4\pi\varepsilon_0}\left[-\left(r-a\cos\theta\right)\left(r^2+a^2-2ar\cos\theta\right)^{-3/2} + \left(\frac{a^2}{R^2}r-a\cos\theta\right)\left(R^2+\left(\frac{a}{R}\right)^2r^2-2ra\cos\theta\right)^{-3/2}\right]$$

$$\frac{\partial V}{\partial r} = \frac{q}{4\pi\varepsilon_0}\left[-\left(R-\cos\theta\right)\left(R^2+a^2-2aR\cos\theta\right)^{-3/2} + \left(\frac{a^2}{\cancel{R^2}}\times\cancel{R}-a\cos\theta\right)\left(R^2+\frac{a^2R^2}{R^2}-2Ra\cos\theta\right)^{-3/2}\right]$$

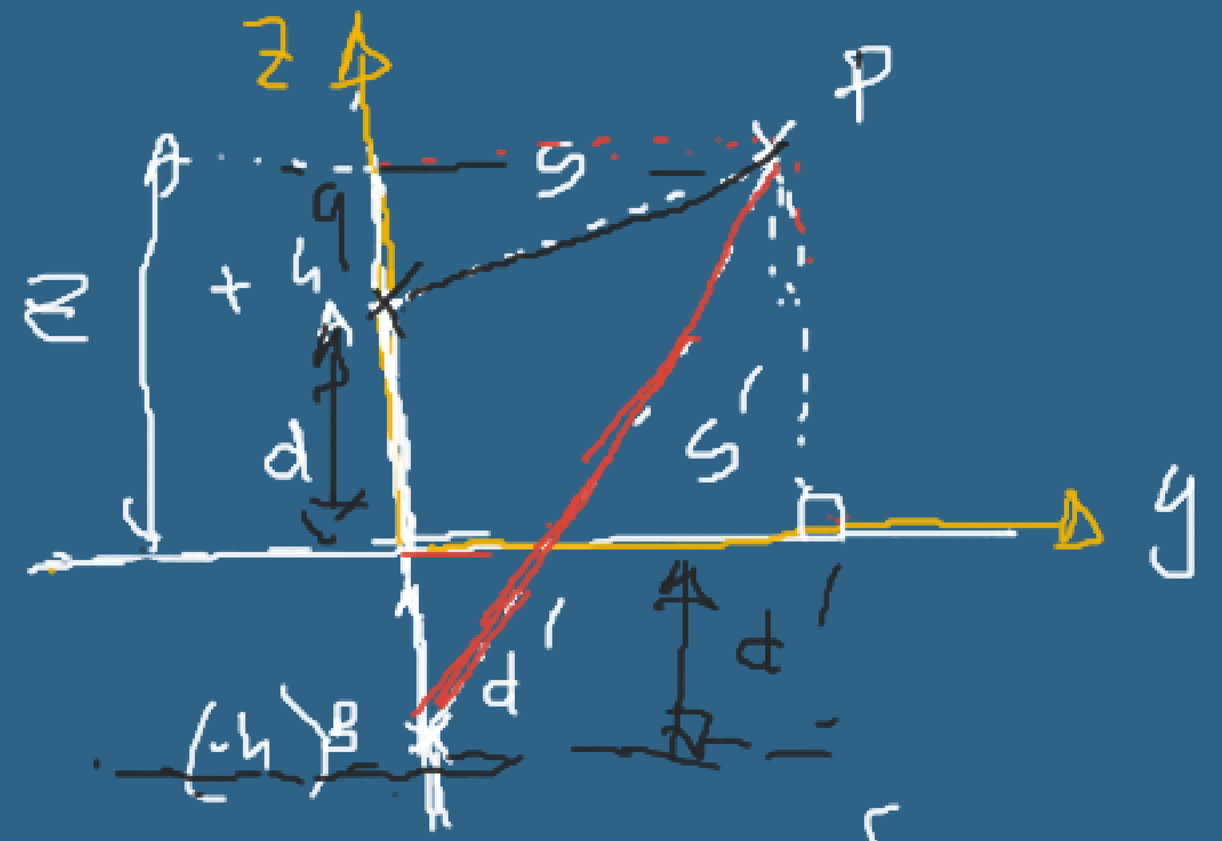
$$\sigma = \frac{q\,\cancel{\varepsilon_0}}{4\pi\cancel{\varepsilon_0}R}\,\frac{-R+\cancel{a\cos\theta}+\frac{a^2}{R}-\cancel{a\cos\theta}}{\left(R^2+a^2-2aR\cos\theta\right)^{3/2}} = \frac{q}{4\pi R}\,\frac{-\left(R^2-a^2\right)}{\left(R^2+a^2-2Ra\cos\theta\right)^{3/2}}$$

$$\boxed{\frac{q}{4\pi R}(R^2-a^2)(R^2+a^2-2Ra\cos\theta)^{-3/2}.}$$

Exercício 3



$\Longrightarrow$



$V_{\text{fio infinito}} = -\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{s}{a}$

$V = V(+\lambda) + V(-\lambda)$

$V = -\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{s}{a} - \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{s'}{a}$

$V = -\frac{\lambda}{2\pi\varepsilon_0}\left(\ln s - \cancel{\ln a} - \ln s' + \cancel{\ln a}\right)$

$V = -\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{s}{s'} = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{s'}{s}$

$\vec{s}' = (z+d')\hat{z} + y\hat{y} \Rightarrow s' = \left[y^2 + (z+d)^2\right]^{1/2}$

$\vec{s} = (z-d)\hat{z} + y\hat{y} \Rightarrow s = \left((z-d)^2 + y^2\right)^{1/2}$

$V = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{\left(y^2+(z+d)^2\right)^{1/2}}{\left(y^2+(z-d')^2\right)^{1/2}}$

$V(z=0,y) = 0 \Rightarrow \frac{y^2+(0+d')^2}{y^2+(0-d)^2} = 0$

$d = d'$

$V(y,z) = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{y^2+(z+d)^2}{y^2+(z-d)^2}\right)^{1/2}$

5. **3.12** Encontre o potencial no sistema da Fig. 3.17, supondo que a fronteira em $x = 0$ consista de duas lâminas metálicas: uma, que vai de $y = 0$ a $y = a/2$, está no potencial V_0, enquanto a outra, que vai de $y = a/2$ a $y = a$, está no potencial $-V_0$.

$$\begin{cases} y = 0 \rightarrow V = 0 \\ y = a \quad V = 0 \end{cases}$$

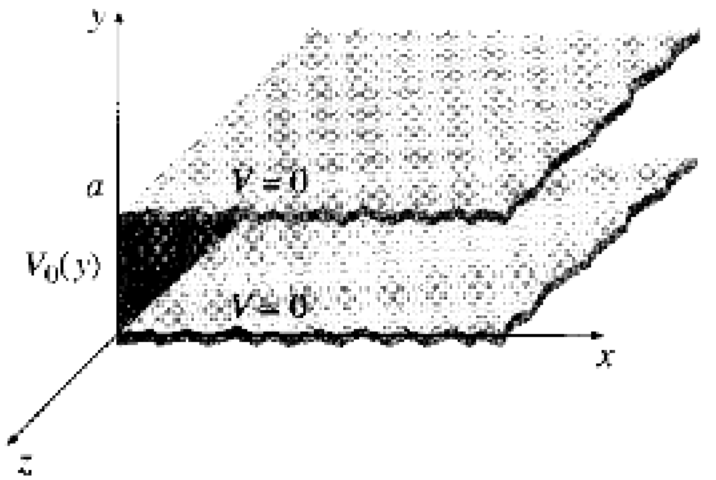


Figure 3.17

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \operatorname{sen} \frac{n\pi}{a} y$$

$$V_0(y) = \begin{cases} +V_0, & \text{for } 0 < y < a/2 \\ -V_0, & \text{for } a/2 < y < a \end{cases}$$

$$C_n = \frac{2}{a} \int_0^a V_0(y) \operatorname{sen} \frac{n\pi}{a} y \, dy$$

$$\Delta V = 0$$

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a) \quad \text{(Eq. 3.30)},$$

$$C_n = \frac{2}{a} \int_0^{\frac{a}{2}} V_0 \operatorname{sen} \frac{n\pi}{a} y \, dy - \int_{a/2}^{a} V_0 \operatorname{sen} \frac{n\pi}{a} y \, dy$$

$$C_n = \frac{2V_0}{a} \int_0^{a/2} \operatorname{sen} \frac{n\pi}{a} y \, dy - \frac{2V_0}{a} \int_{a/2}^{a} \operatorname{sen} \frac{n\pi}{a} y \, dy$$

$$C_n = \frac{2V_0}{a} \left(\left[-\frac{a}{n\pi} \cos \frac{n\pi}{a} y \right]_0^{a/2} - \frac{a}{n\pi} \left[-\cos \frac{n\pi}{a} y \right]_{a/2}^{a} \right)$$

$$C_n = \frac{2V_0}{a} \times \frac{a}{n\pi} \left[\left(-\cos\left(\frac{n\pi}{\cancel{a}} \times \frac{\cancel{a}}{2}\right) + \overset{1}{\cos 0} \right) - \left(-\cos \frac{n\pi}{\cancel{a}} \times \cancel{a} + \cos \frac{n\pi}{\cancel{a}} \frac{\cancel{a}}{2} \right) \right]$$

$$C_n = \frac{2V_0}{\cancel{a}} \times \frac{\cancel{a}}{n\pi} \left(\left(1 - \cos\frac{\pi}{2}n\right) - \left(-\cos n\pi + \cos\frac{n\pi}{2}\right) \right)$$

$$C_n = \frac{2V_0}{n\pi} \left(1 - \cos\frac{\pi}{2}n + \cos n\pi - \cos\frac{n\pi}{2} \right)$$

$$C_n = \frac{2V_0}{n\pi} \left(1 + (-1)^n - 2\cos\frac{n\pi}{2} \right)$$

$$C_n = \begin{cases} \dfrac{8V_0}{n\pi} & \text{se } n \text{ par} \\ 0 & \text{e outro} \end{cases} \qquad n = 4j+2 \text{ com } j = 0, 1, 2, 3, 4$$

$$V(x, y) = \frac{8V_0}{\pi} \sum_{n=2,4,6} \frac{e^{-\frac{n\pi}{a}x} \operatorname{sen} \frac{n\pi}{a} x}{n}$$

6. **3.13** Suponha agora que, no sistema da Fig. 3.17, a lâmina em $x = 0$ seja um condutor no potencial V_0. Determine a densidade de carga nessa lâmina.

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5} \frac{1}{n} e^{-\frac{n\pi}{a}x} \operatorname{sen} \frac{n\pi}{a} y$$

$$\sigma = -\varepsilon_0 \frac{dV}{dn} = -\varepsilon_0 \frac{dV}{dx}$$

$$\frac{dV}{dx} = \frac{-4V_0}{\cancel{\pi}} \sum \frac{1}{\cancel{n}} \times -\frac{\cancel{n}\cancel{\pi}}{a} e^{-\frac{n\pi}{a}x} \operatorname{sen} \frac{n\pi}{a} y$$

$$\frac{dV}{dx} = \frac{-4V_0}{\cancel{\pi}} \sum \frac{\cancel{\pi}}{a} e^{-\frac{n\pi}{a}x} \operatorname{sen} \frac{n\pi}{a} y$$

$$\left.\frac{dV}{dx}\right|_{x=0} = \frac{-4V_0}{a} \sum_{n=1,2,3} e^{-\frac{n\pi}{a}0} \operatorname{sen} \frac{n\pi}{a} y$$

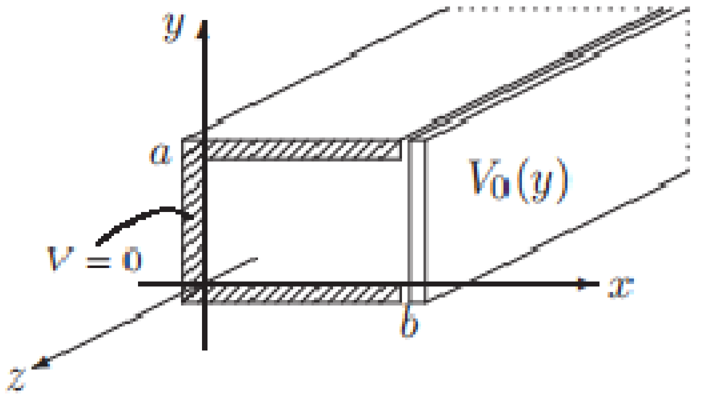
$$\left.\frac{dV}{dx}\right|_{x=0} = -\frac{4V_0}{a} \sum_{n=1} \operatorname{sen} \frac{n\pi}{a} y$$

$$\sigma = -\varepsilon_0 \frac{dV}{dn}$$

$$\sigma = +\frac{4\varepsilon_0 V_0}{a} \sum_{n=1,3,5} \operatorname{sen} \frac{n\pi}{a} y$$

$$\sigma = \frac{4\varepsilon_0 V_0}{a} \sum_{n=1,3,5} \operatorname{sen} \frac{n\pi}{a} y$$

7. **3.14** Uma calha metálica retangular corre paralelamente ao eixo z, de $-\infty$ a ∞. Três de seus lados, em $y = 0$, $y = a$ e $x = 0$, estão aterrados. O quarto lado, em $x = b$, está num potencial $V_0(y)$. Encontre uma fórmula geral para o potencial dentro da calha.



$$\left\{\begin{array}{ll} \text{(i)} & V(x,0)=0, \\ \text{(ii)} & V(x,a)=0, \\ \text{(iii)} & V(0,y)=0, \\ \text{(iv)} & V(b,y)=V_0(y). \end{array}\right\}$$

Metodos de separação de variaveis

$$V(x,y) = X(x)Y(y) \; ; \; \Delta V = \frac{\partial^2 V(x,y)}{\partial x^2} + \frac{\partial^2 V(x,y)}{\partial y^2} = \frac{\partial^2}{\partial x^2}\left(X(x)Y(y)\right) + \frac{\partial^2}{\partial y^2}\left(X(x)Y(y)\right) = 0$$

$$\Delta V(x,y) = Y(y)\frac{\partial^2 X(x)}{\partial x^2} + X(x)\frac{\partial^2 Y(y)}{\partial y^2} = 0$$

$$\frac{1}{XY}\Delta V(x,y) = \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0 \Rightarrow \begin{cases} \dfrac{1}{X}\dfrac{\partial^2 X}{\partial x^2} = cte = C_1 = +k^2 \\ \dfrac{1}{Y}\dfrac{\partial^2 Y}{\partial y^2} = cte = C_2 = -k^2 \end{cases}$$

$$\begin{cases} X(x) = Ae^{-kx} + Be^{kx} \\ Y(y) = C\,\mathrm{sen}\,ky + D\cos ky \end{cases} \Rightarrow V(x,y) = \left(Ae^{-kx} + Be^{kx}\right)\left(C\,\mathrm{sen}\,kx + D\cos kx\right)$$

i) $$V(x,0) = 0 = \left(Ae^{-kx} + Be^{kx}\right)\left(C\,\mathrm{sen}\,k\,y + D\cos k\,y\right) = 0$$

$\to 0$ $\quad 0 \quad + D$

$$\left(Ae^{-kx} + Be^{kx}\right)(\Delta) = 0 \Rightarrow \Delta = 0$$

$$V(x) = \left(Ae^{-kx} + Be^{kx}\right) C \operatorname{sen} ky$$

ii) $V(x,a) = 0 = \left(Ae^{-kx} + Be^{kx}\right) \operatorname{sen} ka = 0 \Rightarrow \operatorname{sen} ka = 0 \Rightarrow ka = n\pi$

$$V(x,y) = \left(Ae^{-kx} + Be^{kx}\right) C \operatorname{sen} \frac{n\pi}{a} x$$

iii) $V(0,y) = 0 = (A + B) C \operatorname{sen} \frac{n\pi}{a} x = 0 \Rightarrow A + B = 0 \Rightarrow A = -B$

$$V(x,y) = \left(Ae^{-\frac{n\pi}{a}x} - Ae^{\frac{n\pi}{a}}\right) C \operatorname{sen} \frac{n\pi}{a} y = 2AC \operatorname{senh} \frac{n\pi}{a} x \operatorname{sen} \frac{n\pi}{a} y$$

$$V(x,y) = 2AC \operatorname{senh} \frac{n\pi}{a} x \operatorname{sen} \frac{n\pi}{a} y = \sum C_n \operatorname{senh} \frac{n\pi}{a} x \operatorname{sen} \frac{n\pi}{a} y$$

$$V(b,y) = \sum_{n=1}^{\infty} C_n \operatorname{senh} \frac{n\pi}{a} x b \operatorname{sen} \frac{n\pi}{a} y = V_0(y)$$

$n \neq n'$

0

Orthogonal.

$$\int_0^a V_0 \operatorname{sen} \frac{\pi n}{a} y \, dy = \sum_{n=1}^{\infty} C_n \operatorname{senh} \frac{n\pi}{a} b \int_0^a \underbrace{\operatorname{sen} \frac{n\pi}{a} y \operatorname{sen} \frac{n\pi}{a} dy}_{= \frac{a}{2} \ \ \sin n = n'}$$

$$\int_0^a V_0 \operatorname{sen} \frac{\pi n}{a} y \, dy = C_n \operatorname{sen} \frac{n\pi b}{a} \times \frac{a}{2}$$

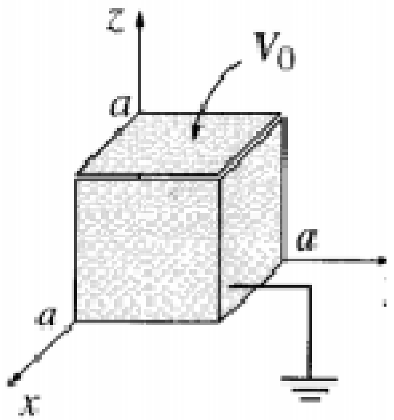
$$\int_0^{\pi} V_0 \,\mathrm{sen}\, \frac{\pi}{a} n y \, dy = \frac{a}{2} C_n \,\mathrm{sen}\, \frac{n\pi}{a} b \Rightarrow$$

$$C_n = \frac{2}{\mathrm{senh}\, \frac{\pi b}{a}} \int_0^{\pi} V_0 \,\mathrm{sen}\, \frac{\pi}{a} n y \, dy$$

$$C_n = \frac{2}{\mathrm{senh}\, \frac{\pi}{a} b} V_0 \left[-\frac{a}{n\pi} \cos\theta \right]_0^a = \frac{-2 V_0}{\mathrm{senh}\, \frac{\pi}{a} b} \frac{a}{n\pi} \left[\cos \frac{\pi}{a} n y \right]_0^a$$

$$C_n = \frac{2}{a\,\mathrm{senh}\left(\frac{n\pi}{a}\right)} V_0 \int_0^a \mathrm{sen}\left(\frac{n\pi y}{a}\right) dy = \frac{2V_0}{a\,\mathrm{senh}\left(\frac{n\pi}{a}\right)} \times \begin{cases} 0 & \text{se } n \text{ for par} \\ \frac{2a}{n\pi} & n \text{ não par} \end{cases}$$

8. **3.15** Uma caixa cúbica, com arestas a, consiste de cinco placas de metal, soldadas umas às outras e aterradas, como na Fig. 3.23. O topo é uma placa metálica separada, isolada das outras e mantida no potencial V_0. Encontre o potencial no interior da caixa.



$$\left\{\begin{array}{llll} \text{(i)} & V=0 & \text{when} & x=0, \\ \text{(ii)} & V=0 & \text{when} & x=a, \\ \text{(iii)} & V=0 & \text{when} & y=0, \\ \text{(iv)} & V=0 & \text{when} & y=a, \\ \text{(v)} & V=0 & \text{when} & z=0, \\ \text{(vi)} & V=V_0 & \text{when} & z=a. \end{array}\right\}$$

$$V(x,y,z) = X(x)\,Y(y)\,Z(z)$$

$$\Delta V(x,y,z) = \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y(y)}{dy^2} + \frac{1}{Z}\frac{d^2Z(z)}{dz^2} = 0 \Rightarrow C_1 + C_2 + C_3 = 0$$

$$C_3 = -C_1 - C_2$$

$$C_1 = -K^2 \quad C_2 = -L^2 \quad C_3 = k^2 + L^2$$

$$X(x) = A \operatorname{sen} kx + B \cos kx \; ; \; Y(y) = C \operatorname{sen} Ly + D \cos Ly \; ;$$

$$Z(z) = E e^{\sqrt{k^2+L^2}\,z} + G e^{-\sqrt{k^2+L^2}\,z}$$

$$V(x=0,y,z) = 0 \Rightarrow A \underbrace{\operatorname{sen}(0)}_{0} + B = 0 \Rightarrow B = 0$$

$$V(x=a) = 0 \Rightarrow A \operatorname{sen} ka = 0 \Rightarrow ka = n\pi \Rightarrow k = \frac{n\pi}{a}$$

$$V(x,y=0,z) = 0 \Rightarrow C \underbrace{\operatorname{sen}(0)}_{0} + D = 0 \Rightarrow D = 0$$

$$V(x,y=a,z) = 0 \Rightarrow C \operatorname{sen} L \times a = 0 \Rightarrow La = m\pi \Rightarrow L = \frac{m\pi}{a}$$

$$V(x,y,z) = A \operatorname{sen} \frac{n\pi}{a} x \; C \operatorname{sen} \frac{m\pi}{a} y \left(e^{\sqrt{k^2+L^2}\,z} + e^{-\sqrt{k^2+L^2}\,z} \right)$$

$$V(x, y, z=0) = 0 \Rightarrow E e^{0} + G e^{0} = 0 \Rightarrow E + G = 0 \Rightarrow G = -E$$

$$V(x, y, z) = A \operatorname{sen} \frac{n\pi}{a} x \; C \operatorname{sen} \frac{m\pi}{a} y \, 2 \operatorname{senh}\left(\sqrt{k^2 + l^2}\right) z$$

$$k^2 + l^2 = \frac{n^2 \pi^2}{a^2} + \frac{m^2 \pi^2}{a^2} = \frac{\pi^2}{a^2}\left(n^2 + m^2\right)$$

$$V(x, y, z) = 2AC \operatorname{sen} \frac{n\pi}{a} x \operatorname{sen} \frac{m\pi}{a} y \operatorname{senh}\left(\frac{\pi}{a}\sqrt{m^2 + n^2}\right) z$$

$$V(x, y, z) = \sum_n \sum_m \operatorname{sen} \frac{n\pi}{a} x \operatorname{sen} \frac{m\pi}{a} y \operatorname{senh} \frac{\pi}{a} \sqrt{m^2 + n^2} \, x$$

$$V_0 = \sum \sum \operatorname{sen} \frac{n\pi}{a} x \operatorname{sen} \frac{m\pi}{a} y \operatorname{senh} \frac{\pi}{a} \sqrt{m^2 + n^2} \, a$$

$$\iint V_0 \operatorname{sen} \frac{\pi n'}{a} x \, dx \operatorname{sen} \frac{m'\pi}{a} dy = \sum \sum \operatorname{senh} \frac{\pi}{\cancel{a}} \sqrt{m^2 + n^2} \, \cancel{a} \underbrace{\int_0^a \operatorname{sen} \frac{n\pi}{a} x \operatorname{sen} \frac{n'\pi}{a} x \, dx}_{\frac{a}{2} \text{ se } n = n'} \int_0^a \operatorname{sen} \frac{m'\pi}{a} \;\text{''}\; dy$$

$$\text{''} \quad \frac{a}{2} \text{ se } m = m'$$

$$C_{n,m} \operatorname{senh} \frac{\pi}{a} \sqrt{m^2 + n^2} \times \frac{a^2}{4} = \int_0^a \int_0^a V_0 \operatorname{sen} \frac{\pi n}{a} x \operatorname{sen} \frac{m\pi}{a} \, dx \, dy$$

$$C_{n,m}\,\mathrm{senh}\left(\pi\sqrt{n^2+m^2}\right) = \left(\frac{2}{a}\right)^2 V_0 \int_0^a\int_0^a \mathrm{sen}(n\pi x/a)\,\mathrm{sen}(n\pi y/a)\,dx\,dy$$

$$C_{n,m}\,\mathrm{senh}\left(\pi\sqrt{n^2+m^2}\right) = \begin{cases} 0 & \text{se } m \text{ é par} \\ \dfrac{16V_0}{\pi^2 nm} & \text{se } m \text{ nao é par} \end{cases}$$

$$V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{n=1}\sum_{m=1,2,3}\frac{1}{nm}\,\mathrm{sen}(n\pi x/a)\,\mathrm{sen}(m\pi y/a)\,\frac{\mathrm{sen}\left(\pi\sqrt{n^2+m^2}\,z/a\right)}{\mathrm{senh}\,\pi\sqrt{n^2+m^2}}$$

$$V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{n=1,2}\sum_{m=1,2,3}\frac{1}{nm}\,\mathrm{sen}(n\pi x/a)\,\mathrm{sen}(m\pi y/a)\,\frac{\mathrm{sen}\left(\pi\sqrt{n^2+\frac{m^2 z}{a}}\right)}{\mathrm{senh}\,\pi\sqrt{n^2+m^2}}$$

9. 3.17 Suponha que o potencial na superfície de uma esfera seja V_0. Encontre o potencial dentro e fora, pelo método de separação de variáveis.

$\nabla^2 V = 0$ cordenadas esféricas

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \operatorname{sen}\theta}\frac{\partial}{\partial\theta}\left(\operatorname{sen}\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2 \operatorname{sen}\theta}\frac{\partial^2 V}{\partial\phi^2} = \Delta V = 0$$

simetria azimutal

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \operatorname{sen}\theta}\frac{\partial}{\partial\theta}\left(\operatorname{sen}\theta\frac{\partial V}{\partial\theta}\right) = 0$$

$$V(r,\theta) = R(r)\,\Theta(\theta) \Rightarrow 0 = \frac{1}{\cancel{r^2}}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}(R(r)\Theta(\theta))\right) + \frac{1}{\cancel{r^2}\operatorname{sen}\theta}\frac{\partial}{\partial\theta}\left(\operatorname{sen}\theta\frac{\partial}{\partial\theta}(R(r)\Theta(\theta))\right)$$

$$\Theta(\theta)\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right) + R(r)\frac{1}{\operatorname{sen}\theta}\frac{\partial}{\partial\theta}\left(\operatorname{sen}\theta\frac{\partial V}{\partial\theta}\right) = 0$$

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta(\theta)}\frac{1}{\operatorname{sen}\theta}\frac{\partial}{\partial\theta}\left(\Theta(\theta)\right) = 0 \Rightarrow C_1 + C_2 = 0 \Rightarrow C_1 = -C_2$$

$$C_1 = \ell(\ell+1) \qquad C_2 = -\ell(\ell+1) \qquad \frac{\partial}{\partial r}\left[r^2\frac{\partial R(r)}{\partial r}\right] = \ell(\ell+1)R(r) \quad (1)$$

$$\frac{1}{\Theta \operatorname{sen}\theta}\frac{d}{d\theta}\left(\operatorname{sen}\theta \frac{dV}{d\theta}(\theta)\right) = C_2 = -l(l+1)$$

$$\frac{d}{d\theta}\left(\operatorname{sen}\theta \frac{dV}{d\theta}\right) = -l(l+1)\operatorname{sen}\theta\,\Theta \qquad (2)$$

Soluções (1) $R(r) = Ar^l + \frac{B}{r^{l+1}}$; (2) $\Theta(\theta) = P_L(\cos\theta)$

$$V(r,\theta) = R(r)\,\Theta(\theta) = \left(Ar^l + Br^{l+1}\right)P_L(\cos\theta)$$

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(Ar^l + \frac{B}{r^{l+1}}\right)P_l(\cos\theta)$$

Dentro $r < R \Rightarrow V(r,\theta) = \sum_{l=0}^{\infty} Ar^l P_L(\cos\theta)$

se $l \to 0 \Rightarrow \boxed{\frac{B}{r^{l+1}}} \to \infty$

Fora se $l \to \infty$ $\boxed{Ar^l} \to$ $\Rightarrow V(r,\theta) = \sum_{l=0}^{\infty} \frac{B}{r^{l+1}} P_L(\cos\theta)$

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_L(\cos\theta)$$

$$V(r,\theta) = \sum_{l=0} A\, r^{l}\, P_l(\cos\theta)$$

$$A_l = \begin{cases} 0 \text{ se } L \neq 0 \\ V_0 \text{ se } l = 0 \end{cases}$$

Quando $r = R \Rightarrow V(r,\theta) = V_0$

$$V_0 = \sum_{l=0}^{\infty} A_l R^{l} P_l(\cos\theta)$$

$$V(r,\theta) = A_0 r^0 P_0(\cos\theta)$$

$$V(r,\theta) = V_0$$

Truque de Fourier

$$\int_0^{\pi} V_0 P_L'(\cos\theta)\,\text{sen}\,\theta\, d\theta = \sum_{l=0}^{\infty} A_l R^{l} \int_0^{\pi} P_L(\cos\theta)\, P_L'(\cos\theta)\,\text{sen}\,\theta\, d\theta$$

...

$$\boxed{V(r,\theta) = V_0}$$

Dentro

$$\int_{-1}^{1} P_l(x)\, P_l'(x)\, dx = \int_0^{\pi} P_l(\cos\theta)\, P_l'(\cos\theta)\,\text{sen}\,\theta\, d\theta = \begin{cases} 0 \text{ se } l \neq l' \\ \dfrac{2}{2l+1} \text{ se } l = l \end{cases}$$

$$\int_0^{\pi} V_0 P_l(\cos\theta)\,\text{sen}\,\theta\, d\theta = A_l R^{l} \times \frac{2}{2l+1}$$

$$A_l = \frac{2l+1}{2R^{l}} V_0 \int_0^{\pi} P_L(\cos\theta)\,\text{sen}\,\theta\, d\theta = \frac{2l+1}{2R^{L}} V_0 \int_0^{\pi} P_l(\cos\theta)\, P_0(\cos\theta)\,\text{sen}\,\theta\, d\theta$$

com $P_0(\cos\theta) = 1$

$$\int_0^{\pi} P_L(\cos\theta)\, P_0(\cos\theta)\,\text{sen}\,\theta\, d\theta = \begin{cases} 0 \text{ se } L \neq 0 \\ 2 \text{ se } l = 0 \end{cases}$$

Fora

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

Qd $r = R$ $\Rightarrow$ $\frac{B_l}{R^{l+1}} P_l(\cos\theta) = V_0$

Orthogonalidade

$$\int_0^{\pi} V_0 P_{l'}(\cos\theta)\, \text{sen}\theta\, d\theta = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} \int_0^{\pi} \underbrace{P_l(\cos\theta) P_{l'}(\cos\theta)\, \text{sen}\theta\, d\theta}_{\begin{cases} 0 \text{ se } l \neq l' \\ \frac{2}{2l+1} \text{ se } l' = l \end{cases}}$$

$$\int_0^{\pi} V_0 P_l(\cos\theta)\, \text{sen}\theta\, d\theta = \frac{B_l}{R^{l+1}} \times \frac{2}{2l+1}$$

$$B_l = \frac{(2l+1) R^{l+1}}{2} V_0 \int_0^{\pi} P_{l'}(\cos\theta)\, \text{sen}\theta\, d\theta$$

$$\int_0^{\pi} P_l(\cos\theta)\, \text{sen}\theta\, d\theta = \int_0^{\pi} P_l(\cos\theta) P_0(\cos\theta)\, \text{sen}\theta\, d\theta = \begin{cases} 0 \text{ se } l \neq 0 \\ 2 \text{ se } l = 0 \end{cases}$$

$$B_l = \begin{cases} 0 \text{ se } l \neq 0 \\ V_0 R \text{ se } l = 0 \end{cases} \Rightarrow V(r,\theta) = \frac{B_0}{r^1} \underbrace{P_0(\cos\theta)}_{1} = \frac{V_0 R}{r} \Rightarrow \begin{cases} V(r,\theta) = \frac{V_0 R}{r} \text{ fora} \\ V(r,\theta) = V_0 \text{ dentro} \end{cases}$$