PGF5003: Classical Electrodynamics I
Problem Set 5

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(Due to June 22, 2021)

Guidelines: *write down the most relevant passages in your calculations, not only the final results. Do not forget to write the mathematical expressions that you are using in order to solve the questions. We strongly recommended the use of the International System of Units.*

1 Question (0.5 point)

Use the electric stress tensor formalism to prove that no isolated charge distribution $\rho(r)$ can exert a net force on itself. Distinguish the cases when $\rho(r)$ has a net charge and when it does not.

2 Question (1 point)

The charge and current densities for a single point charge q can be written formally as

$$\rho(\mathbf{x}', t') = q\delta\left[\mathbf{x}' - \mathbf{r}(t')\right] \tag{1}$$

$$\mathbf{J}(\mathbf{x}', t') = q\mathbf{v}(t')\delta\left[\mathbf{x}' - \mathbf{r}(t')\right] \tag{2}$$

where $\mathbf{r}(t')$ is the charge's position at time t' and $\mathbf{v}(t')$ is its velocity. In evaluating expressions involving the retarded time, one must put $t' = t - R(t')/c$, where $\mathbf{R} = \mathbf{x} - \mathbf{r}(t')$ (but $\mathbf{R} = \mathbf{x} - \mathbf{x}'(t')$ inside the delta functions).

a) *As a preliminary to deriving the Heaviside-Feynman expression for the electric and magnetic fields of a point charge, show that*

$$\int d^3x'\delta\left[\mathbf{x}' - \mathbf{r}(t_{ret})\right] = \frac{1}{\kappa}, \tag{3}$$

where $\kappa = 1 - \mathbf{v} \cdot \hat{R}/c$. Note that κ is evaluated at the retarded time.

b) *Starting with the Jefimenko generalizations of the Coulomb and Biot-Savart laws, use the expressions for the charge and current densities for a point charge and the result of part a to obtain the Heaviside-Feynman expressions for the electric and magnetic fields of a point charge,*

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left\{\left[\frac{\hat{R}}{\kappa R^2}\right]_{ret} + \frac{\partial}{c\partial t}\left[\frac{\hat{R}}{\kappa R}\right]_{ret} - \frac{\partial}{c^2\partial t}\left[\frac{\mathbf{v}}{\kappa R}\right]_{ret}\right\} \tag{4}$$

$$\mathbf{B} = \frac{\mu_0 q}{4\pi}\left\{\left[\frac{\mathbf{v}\times\hat{R}}{\kappa R^2}\right]_{ret} + \frac{\partial}{c\partial t}\left[\frac{\mathbf{v}\times\hat{R}}{\kappa R}\right]_{ret}\right\}. \tag{5}$$

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3 Question (1 point)

a) *When the current density $\mathbf{J}$ is independent of the time, the charge density is given by*

$$\rho(\mathbf{r}, t) = \rho(\mathbf{r}, 0) + \dot{\rho}(\mathbf{r}, 0)t. \tag{6}$$

In this case, show that the electric field is given by

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int d^3r' \rho(\mathbf{r}, t)\frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \tag{7}$$

where ρ is computed at the time t and not in the retarded time t_{ret}, which is identical to the electrostatic situation. Could you give an example where this situation happens?

b) *Show that the Biot-Savart law*

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int d^3r \frac{\mathbf{J}(\mathbf{r}', t) \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \tag{8}$$

is still valid even in the case the the density current $\mathbf{J}$ changes with time, being the time variation sufficiently small to use the first approximation order

$$\mathbf{J}(t_{ret}) = \mathbf{J}(t) + (t_{ret} - t)\dot{\mathbf{J}}(t) \tag{9}$$

what leads to this quantity be calculated in the time t and not in the retarded one.

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4 Question (1.5 point)

Two halves of a spherical metallic shell of radius R and infinite conductivity are separated by a very small insulating gap. An alternating potentials is applied between the two halves of the sphere, so that the potentials are $\pm V\cos(\omega t)$. Find:

a) *the electrical potential (inside and outside the sphere) when the voltages are in their peak. It is just other way to ask you to solve the static version of this problem using Legendre polynomials;*

b) *the momentum of dipole (extending the previous result for the time dependent potential and doing the comparison with the potential due to an electric dipole);*

c) *the first and second derivative in time of the moment of dipole;*

d) *the potential vector $\mathbf{A}$, the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ using dipole approximation;*
Hint: *you can use:*

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\dot{\mathbf{p}}(t_{ret})}{r}, \quad \mathbf{B} = \vec{\nabla} \times \mathbf{A} = \frac{\mu_0}{4\pi c}\frac{\ddot{\mathbf{p}} \times \hat{r}}{r} \text{ and } \mathbf{E} = c\mathbf{B} \times \hat{r}. \tag{10}$$

e) *Find the Poynting vector and the radiated power from the sphere.*

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5 Question (0.5 point)

A spherical shell of radius R uniformly charged has sinusoidal oscillations purely in the radial axis. What is the radiated power?

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6 Question (1 point)

A thin linear antenna of length d is excited in such a way that the sinusoidal current makes a full wavelength of oscillation as shown in the Figure.

a) *Calculate exactly the power radiated per unit of solid angle.* **Hint:** *compute the potential vector, find the electrical and magnetic field, compute the Poyinting vector and then, the radiated power.*

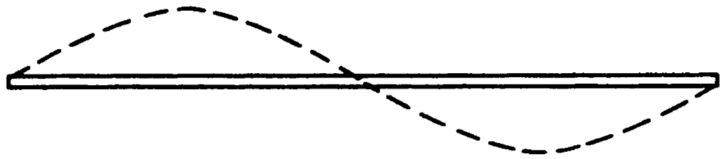

Figure 1: Figure for the exercise.

b) *Plot the angular distribution of radiation from item (a). Show your code to me!*

c) *Determine the total power radiated and find a numerical value for the radiation resistance.* **Hint:** *you may need the following result:* $I = \int_0^{\pi} dx \frac{\sin^2(\pi \cos x)}{\sin x} = 1.55718$.

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7 Question (1 point)

Consider the electric dipole and magnetic dipole radiation fields. **Hint:** *see, e.g., Jackson, Ch. 9.2 and 9.3.*

a) *What are these dipole fields in terms of the Transverse Electric and Transverse Magnetic radiation fields?*

b) *Assume that an oscillating electric dipole is aligned along the z direction. Express the electric dipole radiation field in terms of the scalar and vector spherical harmonics.*

c) *Assume that an oscillating magnetic dipole is aligned along the z direction. Express that dipole radiation field in terms of the scalar and vector spherical harmonics.*

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8 Question (1 point)

A scalar spherical wave ψ is emitted from a source at the origin in such a way that the Fourier transform are given by:

$$\tilde{\psi}(t, k, \hat{k}) = A \, \exp\left(-i\omega t\right) \, \exp\left(-ak\right) \, \cos\theta_k$$

a) *Show that this is a pure dipole pattern in real and in Fourier space;*

b) *Show that this corresponds to a pulse of width $\Delta r = a$ that propagates from the origin outwards. Plot the spatial dependence of that pulse as it propagates in space and in time.*

c) *Show that this pulse has a finite (and fixed) energy at any time. You can interpret the energy density as $|\psi|^2$.*

Hint: *For this problem you will need the following integral:*

$$\int_0^\infty dk\, k^2\, j_1(kx)\, e^{-ak} = 2x/(x^2+a^2)^2$$

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9 Question (1 point)

A particle with mass m and charge e moves in a uniform, static, electric field $\mathbf{E}_0$.

a) *Solve the velocity and position of the particle as explicit functions of time. Assume the initial velocity $\mathbf{v}_0$ is perpendicular to the electric field.*

b) *Eliminate the time to obtain the trajectory of the particle in space. Discuss the shape of the path for short and long times (define what you call as "short" and as "long" times!).*

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10 Question (1.5 point)

The magnetic field of the earth can be represented approximately by a magnetic dipole of magnetic moment $M = 8.1 \cdot 10^{25} Gauss \cdot cm^3$. Consider the motion of energetic electrons in the neighborhood of the earth under the action of this dipole. **Note:** $\mathbf{M}$ *points south.*

a) *Show that the equation for a line of magnetic force is $r = r_0 \sin^2\theta$, where θ is the usual polar angle (colatitude) measured from the axis of the dipole and find an expression for the magnitude of B along any line of force as a function of θ.*

b) *A positively charged particle circles around a line of force in the equatorial plane with a gyration radius a and a mean radius R ($a \ll R$). Show that the particle's azimuthal position (east longitude) changes approximately linearly in time according to*

$$\phi(t) = \phi_0 - \frac{3}{2}\left(\frac{a}{R}\right)^2 \omega_B (t - t_0) \tag{11}$$

where ω_B is the frequency of gyration at the radius R.

c) *If, in addition to its circular motion of part (b), the particle has a small component of velocity parallel to the lines of force, show that it undergoes small oscillations in θ around $\theta = \pi/2$ with a frequency $\Omega = \left(3/\sqrt{2}\right)(a/R)\,\omega_B$. Find the change in longitude per cycle of oscillation in latitude.*

d) *For an electron of $10MeV$ kinetic energy at a mean radius $R = 3 \cdot 10^7 m$, find ω_B and a, and so determine how long it takes to drift once around the earth and how long it takes to execute one cycle of oscillation in latitude.*

Hint: *look at the section 12.4 in Jackson's book.*