Aula 17 – Quantização do Campo eletromagnético

Começando pelas eq. de Maxwell:

No espaço livre de cargas ou correntes

$$\vec\nabla\cdot\vec D=\rho;\quad \vec D=\epsilon_0\vec E$$
$$\vec\nabla\cdot\vec B=0$$
$$\vec\nabla\times\vec E=-\partial\vec B/\partial t$$
$$\vec\nabla\times\vec H=\mu_0\vec J+\mu_0\epsilon_0\,\partial\vec E/\partial t$$

$\Rightarrow$
$$\boxed{\begin{aligned}&\nabla\cdot\vec E=0\ ; && \nabla\times\vec E=-\partial\vec B/\partial t\\ &\nabla\cdot\vec B=0\ ; && \nabla\times\vec B=\mu_0\epsilon_0\,\partial\vec E/\partial t\\ & && \hookrightarrow \mu_0\epsilon_0=\frac{1}{c^2}\end{aligned}}$$

$\hookrightarrow$ $\nabla\times(\nabla\times\vec B)$ $\rightsquigarrow$ Eq. de onda p/ $\vec B$ & $\vec E$

$\rightsquigarrow$ É conveniente expressar em termos dos potenciais $\phi(\vec r,t)$ e $\vec A(\vec r,t)$:

$$\left\{\begin{aligned}&\vec B=\vec\nabla\times\vec A\\ &\vec E=-\nabla\phi-\frac{\partial\vec A}{\partial t}\end{aligned}\right\}$$
$\leftarrow$ potenciais são $\oplus$ importantes na Mec. quântica!

$\Rightarrow$ Porém $\vec A$ e ϕ não são únicos $\rightsquigarrow$ Invariância de calibre ("gauge invariance")

$$\rightsquigarrow\left\{\begin{aligned}&\vec A'=\vec A+\nabla\chi\\ &\phi'=\phi-\frac{\partial\chi}{\partial t}\end{aligned}\right.$$

$\hookrightarrow$ Gauge de Coulomb: $\{\vec\nabla\cdot\vec A=0\ ,\ \phi=0\}$

$$\left\{\begin{aligned}&\vec B=\nabla\times\vec A\\ &\vec E=-\frac{\partial\vec A}{\partial t}\end{aligned}\right.$$

$\nabla\times\nabla\times\vec E$ $\rightsquigarrow$ Eq onda p/ $\vec E$

$\hookrightarrow$ Reescrevendo eq. de Maxwell p/ $\vec A$

$$\nabla\cdot(\nabla\times\vec A)=0\ ;\qquad \boxed{\nabla\times(\nabla\times\vec A)=-\frac{1}{c^2}\frac{\partial^2\vec A}{\partial t^2}}$$
$$\nabla\cdot(-\partial\vec A/\partial t)=0\ ;$$

$$\boxed{\nabla^2\vec A=\frac{1}{c^2}\frac{\partial^2\vec A}{\partial t^2}}$$

$\rightsquigarrow$ Eq. onda
$$\frac{\partial^2 f}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

Fazendo a escolha ("ansatz")

$$\vec A(\vec r,t)=\sum_m A_m\cdot\left[a_m(t)\vec u_m(\vec r)+a_m^*(t)\vec u_m^*(\vec r)\right]$$

$$\hookrightarrow\ \vec A(\vec r,t)=\sum_m\sqrt{\frac{\hbar}{2\omega_m\epsilon_0}}\cdot\left[a_m(t)\vec u_m(\vec r)+a_m^\dagger(t)\vec u_m^*(\vec r)\right]$$

Subst. na eq. de onda

$$\left\{\begin{aligned}&\nabla^2\vec u_m(\vec r)+\frac{\omega_m^2}{c^2}\vec u_m(\vec r)=0\\ &\frac{\partial^2 a_m(t)}{\partial t^2}+\omega_m^2 a_m=0\end{aligned}\right\}$$

Como solução ao funções:

$$\begin{cases} a_m(t) = a_m e^{-i\omega_m t} \\ a_m^*(t) = a_m^\dagger(t) = a_m^\dagger e^{i\omega_m t} \end{cases}$$

$$\vec{u}_m(\vec{r}) = \hat{e}_m \cdot e^{i\vec{k}_m\cdot\vec{r}} \cdot \frac{1}{\sqrt{V}}$$ volume do modo

$$V = L^3$$

polarização do modo

$$\int u_m^*(\vec{r})\, u_n(\vec{r})\, d\vec{r} = \delta_{mn}$$

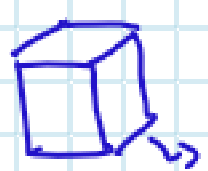
$$\boxed{k_m = \frac{2\pi}{\lambda_m} = \frac{\omega_m}{c}} \rightsquigarrow \omega_m = c|\vec{k}_m|$$

$$\lambda \cdot \nu = v$$

$\rightsquigarrow$ $\hat{e}_m$: polarização do modo m $\Rightarrow$ $\hat{e}_m \cdot \hat{e}_n = \delta_{mn}$

↳ deve obedecer $\hat{e}_m \cdot \boxed{\hat{k}_m} = 0$

P/ cada freq. ω_m $\rightarrow$ $\begin{cases} \text{2 polarizações ortogonais no plano } \hat{e}_m\cdot\hat{k}_m = 0 \\ \text{2 modos p/ cada } \omega_m \end{cases}$



$\rightsquigarrow L$

$$\vec{A}(\vec{r}+L\hat{x}) = \vec{A}(\vec{r}+L\hat{y}) = \vec{A}(\vec{r}+L\hat{z}) = A(\vec{r})$$

$$\hookrightarrow \vec{k}_m = \frac{2\pi}{L}(m_x\hat{x}, m_y\hat{y}, m_z\hat{z}) \; ; \qquad m_i = \{0, \pm1, \pm2, \dots\}$$

$$i = \{\hat{x}, \hat{y}, \hat{z}\}$$

$$\omega_m = c|\vec{k}_m|$$

A forma final de $\vec{A}(\vec{r},t)$

$$\vec{A}(\vec{r},t) = \sum_m \sqrt{\frac{\hbar}{2\omega_m\epsilon_0 V}} \cdot \hat{e}_m \left\{ a_m e^{i[\vec{k}_m\cdot\vec{r}-\omega_m t]} + a_m^\dagger e^{-i[\vec{k}_m\cdot\vec{r}-\omega_m t]} \right\}$$

alternativamente

$$\rightarrow \vec{A}(\vec{r},t) = \sum_m \left(\frac{\hbar}{2\omega_m\epsilon_0}\right)^{1/2} \left\{ a_m \vec{u}_m(\vec{r}) e^{-i\omega_m t} + \underbrace{a_m^\dagger u_m^*(\vec{r})\, e^{i\omega_m t}}_{c.c} \right\}$$

$\vec{u}_m(\vec{r}) \equiv$ depende das condições de contorno $\Rightarrow$ $\begin{cases} \rightsquigarrow \text{senoidal (cavidades)} \\ \rightsquigarrow \text{exponencial (espaço livre)} \\ \text{(ondas planas)} \end{cases}$

Finalmente

$$\vec{A}(\vec{r},t) = \sum_m \left(\frac{\hbar}{2\omega_m \epsilon_0}\right)^{1/2} \left\{ a_m \vec{u}_m(\vec{r})\, e^{-i\omega_m t} + a_m^{+} \vec{u}_m^{*}(\vec{r})\, e^{i\omega_m t} \right\}$$

$$\vec{E}(\vec{r},t) = i \sum_m \left(\frac{\hbar \omega_m}{2\epsilon_0 V}\right)^{1/2} \hat{e}_m \left\{ a_m e^{-i(\vec{k}_m\cdot\vec{r} - \omega_m t)} - a_m^{+} e^{i(\vec{k}_m\cdot\vec{r} - \omega_m t)} \right\}$$

$$\vec{B}(\vec{r},t) = \frac{1}{c}\, \hat{k}_m \times \vec{E} = -\frac{i}{c} \sum \sqrt{\frac{\hbar\omega_m}{2\epsilon_0 V}}\; \hat{e}_m \times \vec{k}_m \left\{ a_m e^{i(\vec{k}_m\cdot\vec{r} - \omega_m t)} - a_m^{+} e^{i(\vec{k}_m\cdot\vec{r} - \omega_m t)} \right\}$$

$$\mathcal{H} = U_{EM} = \frac{1}{2} \int \underbrace{\epsilon_0 \vec{E}^2 + \frac{1}{\mu_0} \vec{B}^2}_{\text{dens. energia do Campo}}\, \underbrace{d^3\vec{r}}_{dV \text{ (volume)}}$$

$\vec{E}^* \cdot \vec{E}$

$$= \frac{1}{2} \int_V \epsilon_0 \left(\frac{\partial^2 \vec{A}^2}{\partial t^2}\right) + \frac{1}{\mu_0} (\nabla \times \vec{A})^2\, d^3\vec{r}$$

$$\boxed{\mathcal{H} = \sum_m \hbar\omega_m \left(a_m a_m^{+} + a_m^{+} a_m \right)} \equiv \sum_m H_m \rightarrow \mathcal{H}_m = \hbar\omega_m \left(a_m^{+} a + a_m a_m^{+}\right)$$

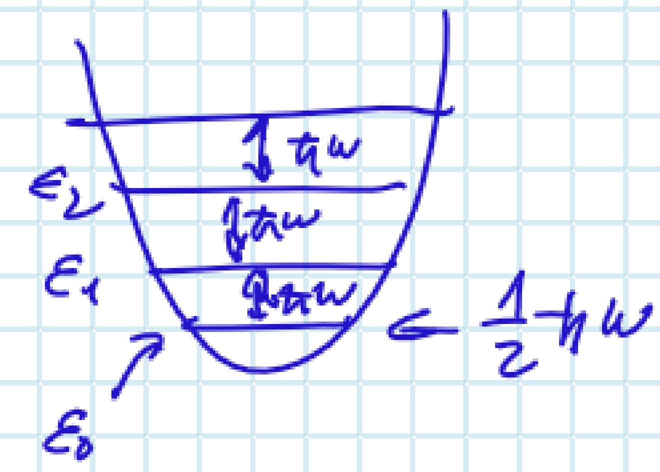
$\Downarrow$ Quantização do Campo

Faz-se a associação:

$$\left.\begin{array}{l} a_m \rightarrow \hat{a}_m \\ a_m^{*} = a_m^{+} \rightarrow \hat{a}_m^{+} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} [\hat{a}_m, \hat{a}_n^{+}] = \delta_{mn} \\ [a_m, a_n] = 0 \\ [a_m^{+}, a_n^{+}] = 0 \end{array}\right.$$

Solução do osc. harmônico quântico

osc. harm. 1D



$$\boxed{\hat{\mathcal{H}}_{EM} = \sum_m^{\infty} \hbar\omega_m \left(a_m^{+} a_m + 1/2 \right)}$$

$\sum_m \frac{1}{2}\hbar\omega_m$

$\mathcal{E} = \hbar\omega = h\nu$

$\hbar = \frac{h}{2\pi} \rightarrow \omega = 2\pi\nu$

$$\hookrightarrow \langle 0 | \hat{\mathcal{H}}_{EM} | 0 \rangle = \sum_m^{\infty} \hbar\omega_m$$

$$\left\{\begin{array}{l} \lambda\nu = v = c \\ \lambda = \frac{c}{\nu} = \frac{c}{\omega}\cdot 2\pi \rightarrow \omega = c\,\frac{2\pi}{\lambda} = ck \end{array}\right.$$