Retas

Retas // planos coord.

$$\begin{cases} Oxy : & c = 0 \\ Oxz : & b = 0 \\ Oyz : & a = 0 \end{cases}$$

$\vec{v} = (a, b, c)$

$\vec{v} \parallel r$

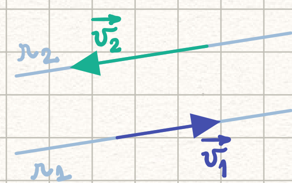
Retas // eixos coord.

$$\begin{cases} Ox : & b = c = 0 \\ Oy : & a = c = 0 \\ Oz : & a = b = 0 \end{cases}$$

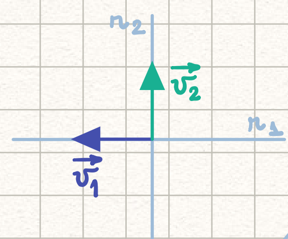
Sejam: $r_1 \parallel \vec{v_1}$ e $r_2 \parallel \vec{v_2}$; $A_1 \in r_1$ e $A_2 \in r_2$

Ângulo: $\cos\theta = \dfrac{|\vec{v_1} \cdot \vec{v_2}|}{|\vec{v_1}|\,|\vec{v_2}|}$ $0 \leq \theta \leq \dfrac{\pi}{2}$

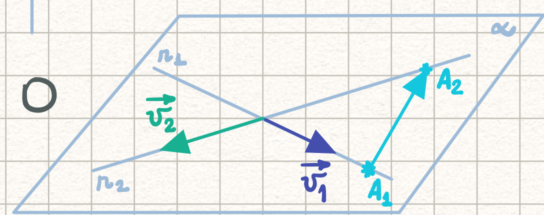
Paralelismo: $\exists K / \vec{v_1} = K\vec{v_2}$ (ou $\vec{v_1} \times \vec{v_2} = \vec{0}$)



Ortogonalidade: $\vec{v_1} \cdot \vec{v_2} = 0$



Coplanaridade: $[\vec{v_1}, \vec{v_2}, \overrightarrow{A_1A_2}] = 0$



Teste da Posição Relativa:

1) Se coplanares, $[\vec{v_1}, \vec{v_2}, \overrightarrow{A_1A_2}] = 0$. Então:

   a) Se paralelas, $\exists K \in \mathbb{R} / \vec{v_1} = K\vec{v_2}$.

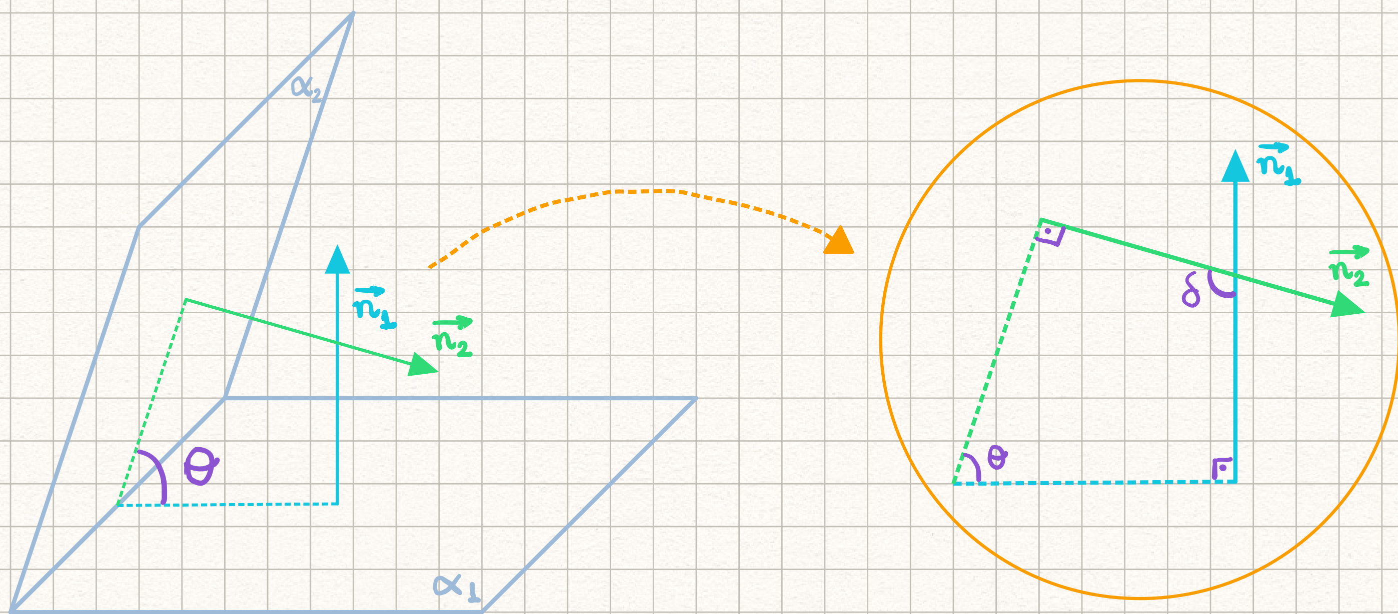
   b) Senão: concorrentes ($\nexists K$; $\exists I = r_1 \cap r_2$)

2) Se reversas: $[\vec{v_1}, \vec{v_2}, \overrightarrow{A_1A_2}] \neq 0$.

Reta Ortogonal a Duas Retas:

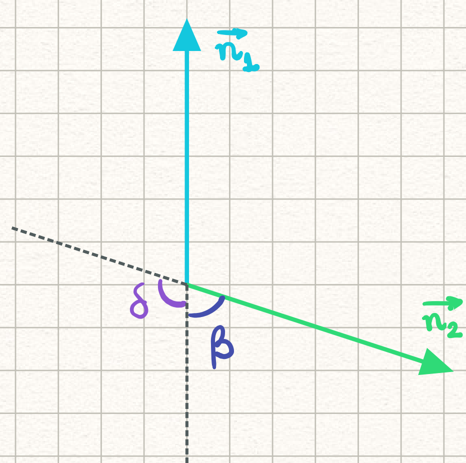
$s \parallel \vec{v}$ é ortogonal a r_1 e a r_2 $\iff$ $\vec{v} \parallel \vec{v_1} \times \vec{v_2}$

Slide 17 - Ângulo entre Dois Planos



$$\Sigma(\text{ângulos internos}) = 2\pi$$

$$\theta + \delta + \frac{\pi}{2} + \frac{\pi}{2} = 2\pi \quad \therefore \quad \delta = \pi - \theta$$



Há 2 $\measuredangle(\vec{n_1}, \vec{n_2})$:

δ ... ângulo obtuso

β ... ângulo agudo

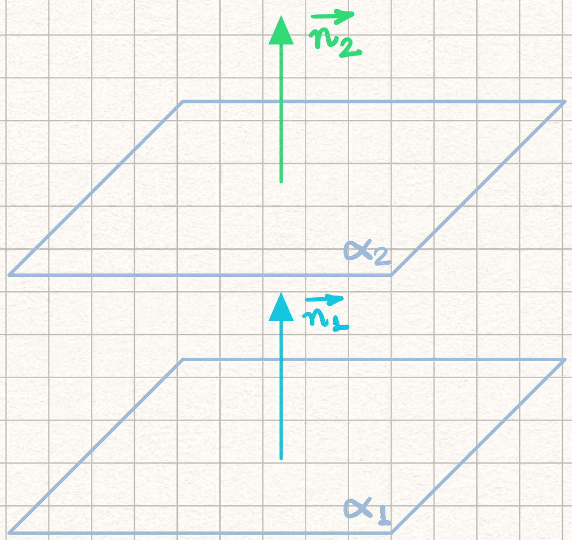
CONVENÇÃO: menor $\longrightarrow$ β

Quanto vale β?

$$\delta + \beta = \pi \longrightarrow \beta = \pi - \delta = \pi - (\pi - \theta) \quad \therefore \quad \boxed{\beta = \theta}$$

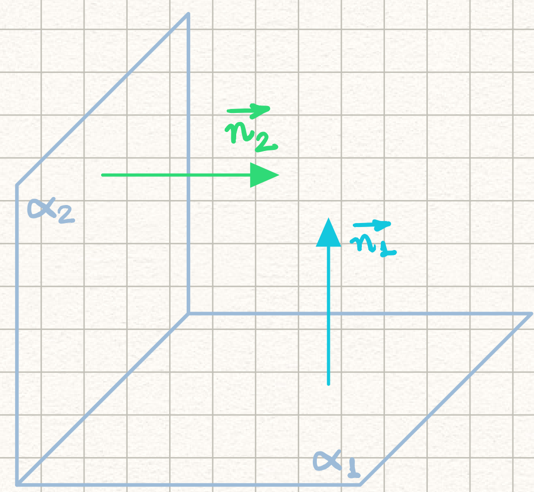
Paralelismo e Ortogonalidade entre Planos

PARALELISMO:



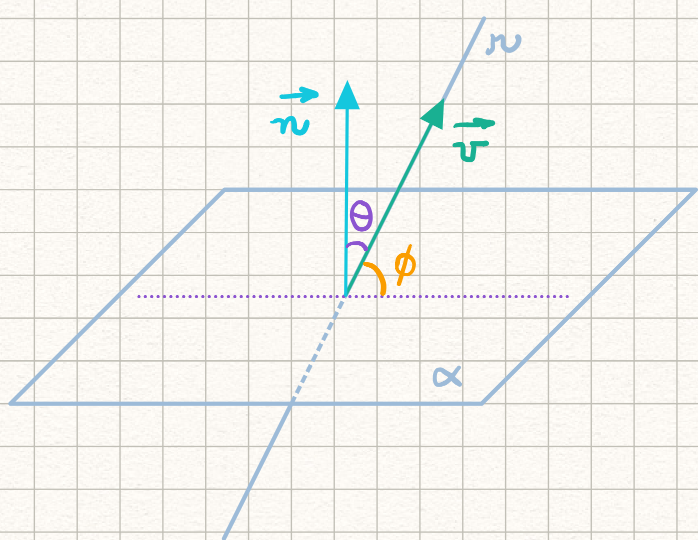
$\exists K \in \mathbb{R} \,/\, \vec{n_1} = K\,\vec{n_2}$ $\quad(\vec{n_1} \parallel \vec{n_2})$

ORTOGONALIDADE:



$\vec{n_1} \cdot \vec{n_2} = 0$ $\quad(\vec{n_1} \perp \vec{n_2})$

Slide 19 – Ângulo entre Reta e Plano



Qual o ângulo entre a reta e o plano?

ϕ

Qual ângulo sei calcular de imediato?

θ

Portanto, deve-se relacionar θ e ϕ:

$$\theta + \phi = \frac{\pi}{2} \quad \therefore \quad \theta = \frac{\pi}{2} - \phi$$

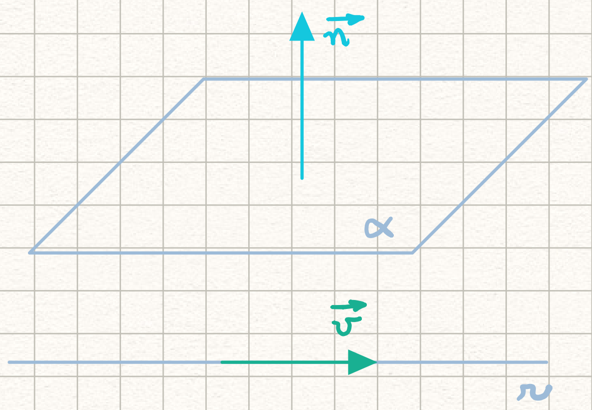
E sabe-se que: $\cos\theta = \cos\left(\frac{\pi}{2} - \phi\right) = \operatorname{sen}\phi$

Logo:

$$\operatorname{sen}\phi = \frac{|\vec{v}\cdot\vec{n}|}{|\vec{v}|\,|\vec{n}|}, \quad 0 \leq \phi \leq \pi/2$$

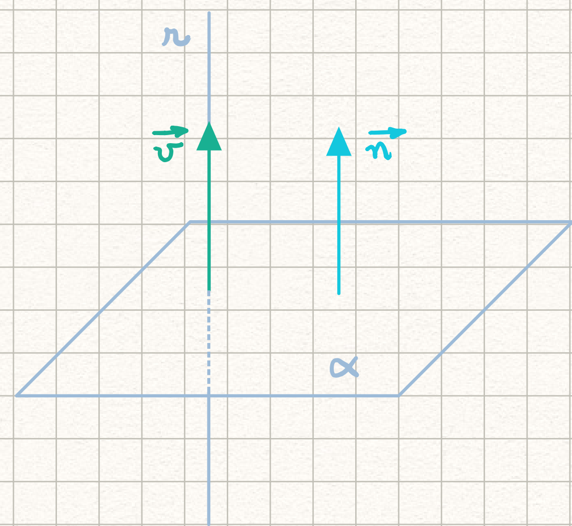
Paralelismo e Ortogonalidade entre Reta e Plano

PARALELISMO:



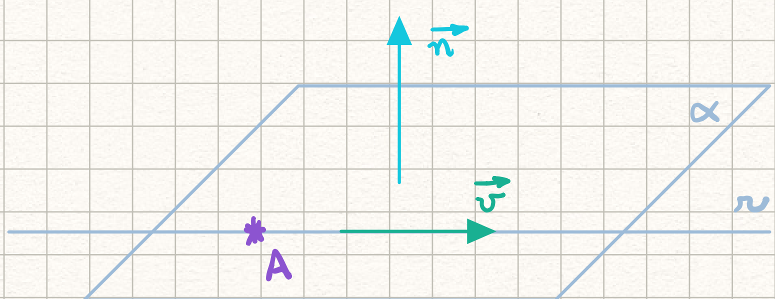
$\vec{v} \cdot \vec{n} = 0 \quad (\vec{v} \perp \vec{n})$

ORTOGONALIDADE:



$\exists\, k \in \mathbb{R} \,/\, \vec{v} = k\vec{n} \quad (\vec{v} \parallel \vec{n})$

Reta contida em um Plano



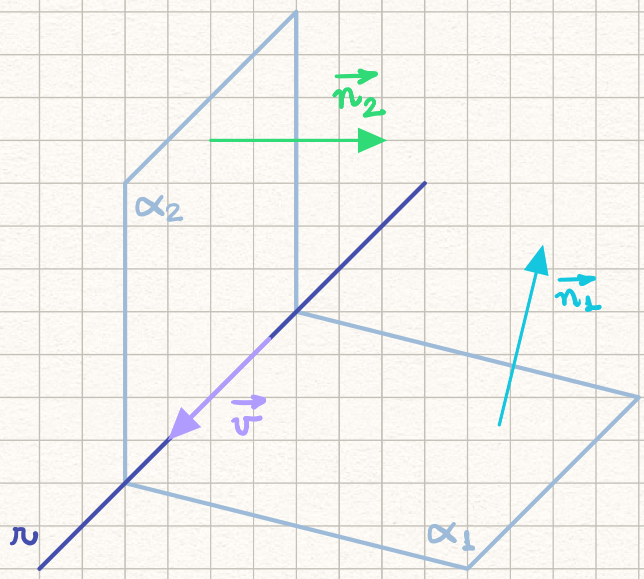
$$r \subset \alpha \iff \left\{ \begin{array}{ll} i. & \vec{v} \cdot \vec{n} = 0 \quad (\vec{v} \perp \vec{n}) \\ ii. & A \in \alpha \text{ e } A \in r \end{array} \right]$$

AO MESMO TEMPO!

Observação:

$r \subset \alpha$ se dois pontos A e B $\in$ à reta e ao plano.

Intersecção entre Dois Planos



$\alpha_1 \cap \alpha_2 = r$

$\downarrow$

satisfaz as eqs. de α_1 e de α_2

OBSERVAÇÃO:

$\vec{v} \parallel r \ , \ \vec{v} \begin{cases} \perp \vec{n_1} \\ \perp \vec{n_2} \end{cases}$

$\therefore \ \vec{v} \parallel \vec{n_1} \times \vec{n_2}$

Slide 23 - Exemplo

$$\left.\begin{aligned} \alpha_1 &: \ 5x - 2y + z + 7 = 0 \\ \alpha_2 &: \ 3x - 3y + z + 4 = 0 \end{aligned}\right\}$$

$r = \alpha_1 \cap \alpha_2$?

r é solução do sistema formado pelas eqs. de α_1 e de α_2:

$$\begin{cases} 5x - 2y + z + 7 = 0 & \text{(i)} \\ 3x - 3y + z + 4 = 0 & \text{(ii)} \end{cases}$$

2 eqs. e 3 incóg. $\therefore$ SPI

(i) - (ii): $2x + y + 3 = 0 \longrightarrow y = -2x - 3$

y em (i): $5x - 2(-2x - 3) + z + 7 = 0$

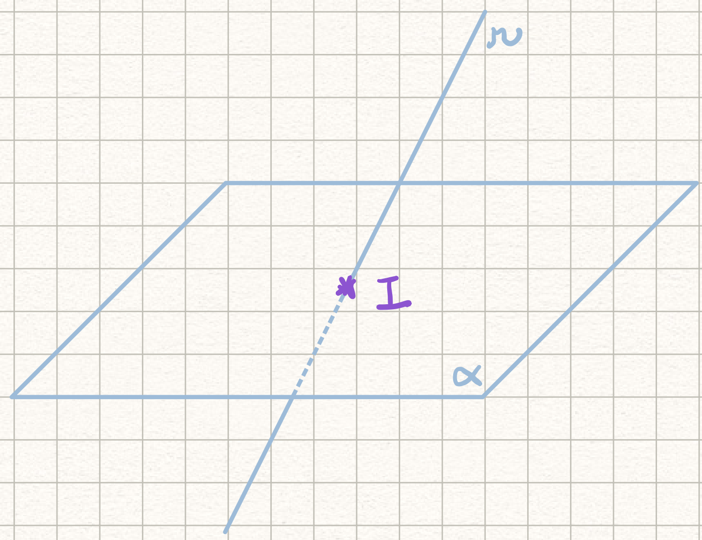
$5x + 4x + 6 + z + 7 = 0 \longrightarrow z = -9x - 13$

Portanto:

$$\begin{cases} y = -2x - 3 \\ z = -9x - 13 \end{cases}$$

representa as eqs. reduzidas em "x" de r, tal que $r = \alpha_1 \cap \alpha_2$.

Interseção entre Reta e Plano



$I = r \cap \alpha$

Solução do sistema:

$$\begin{cases} \text{eqs. reduzidas de } r \\ \text{eq. geral de } \alpha \end{cases}$$

Slide 25 - Exercícios

1

$\alpha_1: 3x + 2y - 6 = 0$

$\alpha_2: Oxz$

$\theta = \measuredangle(\alpha_1, \alpha_2)$?

$$\cos\theta = \frac{|\vec{n_1} \cdot \vec{n_2}|}{|\vec{n_1}|\,|\vec{n_2}|} \quad (1)$$

$\alpha_1 \longrightarrow \vec{n_1} = (3, 2, 0) \quad (\alpha_1 \parallel Oz)$

$\alpha_2: y = 0 \longrightarrow \vec{n_2} = (0, 1, 0)$

Calculando as quantidades para substituir em (1):

$\vec{n_1} \cdot \vec{n_2} = (3,2,0) \cdot (0,1,0) = 2$

$|\vec{n_1}| = \sqrt{\vec{n_1} \cdot \vec{n_1}} = \sqrt{9 + 4} = \sqrt{13}$

$|\vec{n_2}| = 1 \quad (\vec{n_2} = \vec{j}$, versor do eixo $Oy)$

Substituindo em (1):

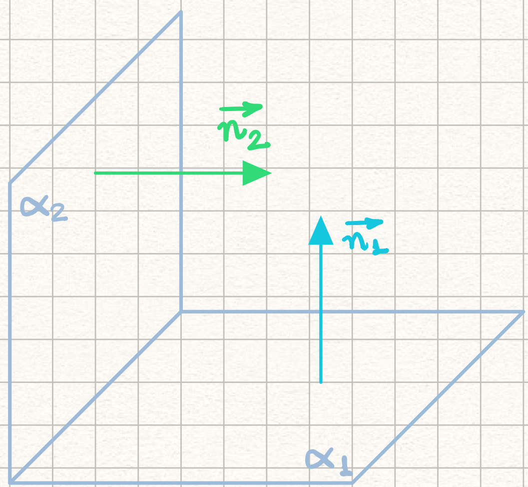
$$\cos\theta = \frac{2}{\sqrt{13}} \quad \therefore \quad \theta = \arccos\left(\frac{2\sqrt{13}}{\sqrt{13}}\right)$$

2

$\alpha_1: \; 2mx + 2y - z = 0$

$\alpha_2: \; 3x - my + 2z - 1 = 0$

$m = ?$ tal que $\alpha_1 \perp \alpha_2$



$\alpha_1 \perp \alpha_2 \iff \vec{n_1} \perp \vec{n_2}$

$\therefore \; \vec{n_1} \cdot \vec{n_2} = 0 \qquad (1)$

$\alpha_1 \longrightarrow \vec{n_1} = (2m, 2, -1)$

$\alpha_2 \longrightarrow \vec{n_2} = (3, -m, 2)$

Substituindo $\vec{n_1}$ e $\vec{n_2}$ em (1):

$$(2m, 2, -1) \cdot (3, -m, 2) = 0$$

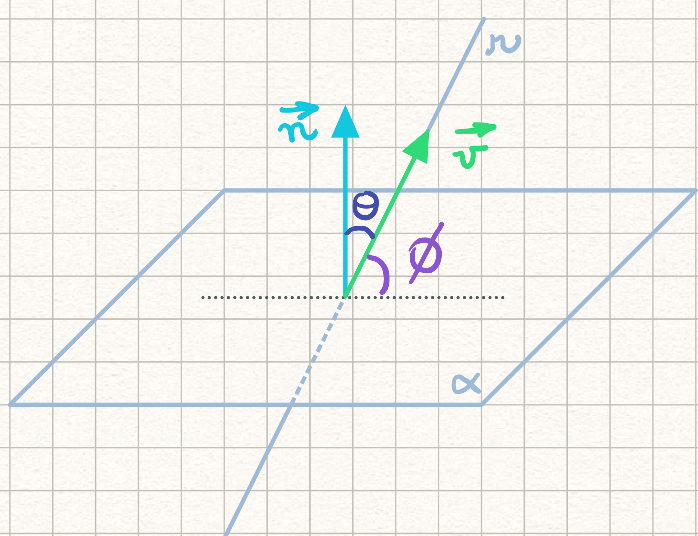
$$6m - 2m - 2 = 0 \quad \therefore \quad m = \frac{1}{2}$$

3

$r: \left\{ \dfrac{x-2}{3} = \dfrac{-y}{4} = \dfrac{z+1}{5} \right.$

$\alpha: \; 2x - y + 7z - 6 = 0$

$\theta = \measuredangle(r, \alpha)$?



$\operatorname{sen}\phi = \cos\theta$

$$\operatorname{sen}\phi = \frac{|\vec{n_1} \cdot \vec{n_2}|}{|\vec{n_1}|\,|\vec{n_2}|} \qquad (1)$$

$r \longrightarrow \vec{v} = (3, -4, 5)$

$\alpha \longrightarrow \vec{n} = (2, -1, 7)$

Calculando as quantidades para substituir em (1):

$$\vec{v} \cdot \vec{n} = (3,-4,5)\cdot(2,-1,7) = 6+4+35 = 45$$

$$|\vec{v}| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{9+16+25} = \sqrt{50}$$

$$|\vec{n}| = \sqrt{\vec{n}\cdot\vec{n}} = \sqrt{4+1+49} = \sqrt{54}$$

Substituindo em (1):

$$\operatorname{sen}\phi = \frac{45}{\sqrt{50}\,\sqrt{54}} = \frac{45}{30\sqrt{3}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

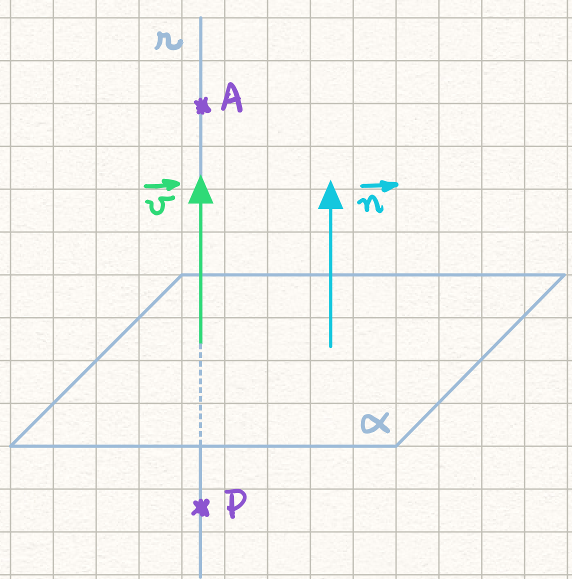
$$\phi = \operatorname{arc\,sen}\left(\frac{\sqrt{3}}{2}\right) \quad \therefore \quad \phi = \frac{\pi}{3}$$

4

$\alpha : x - 3y + 2z - 1 = 0$

$A(2,-1,4)$

$r = ?$ tal que $\begin{cases} r \perp \alpha \\ A \in r \end{cases}$



$\vec{v} \,//\, \vec{n}$

$\alpha \longrightarrow \vec{n} = (1,-3,2)$

$\therefore \quad \vec{v} = k\,\vec{n}\,, \quad k \neq 0$

$k = 1 \longrightarrow \vec{v} = (1,-3,2)$

$P(x,y,z) \in r \iff \overrightarrow{AP} \,//\, \vec{v}$

$\overrightarrow{AP} = t\vec{v}\,, \quad t \in \mathbb{R}$

E. Vetorial: $P = A + t\vec{v}$

$$r : (x,y,z) = (2,-1,4) + t(1,-3,2)\,, \quad t \in \mathbb{R}$$

E Paramétricas : $\quad r: \begin{cases} x = 2 + t \\ y = -1 - 3t \\ z = 4 + 2t \end{cases}, \quad t \in \mathbb{R}$

$(=t)$

E Simétricas : $\quad r: \; x - 2 = \dfrac{-y-1}{3} = \dfrac{z-4}{2}$

E Reduzidas em "x" :

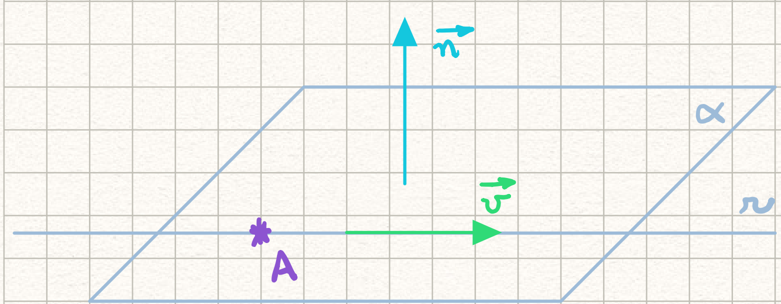
$$x - 2 = \frac{-y-1}{3}$$

$$x - 2 = \frac{z-4}{2} \longrightarrow r: \begin{cases} y = -3x + 5 \\ z = 2x \end{cases}$$

5 $\quad r: \begin{cases} y = 2x - 3 \\ z = -x + 4 \end{cases}$

$\alpha: \; nx + my - z - 2 = 0$

$m, n = ?$ tal que $r \subset \alpha$



$r \subset \alpha$ se e somente se:

$\begin{cases} \text{i.} \quad \vec{v} \cdot \vec{n} = 0 \\ \text{ii.} \quad A \in \alpha \text{ e } A \in r \end{cases}$

$r \longrightarrow \begin{cases} \text{para } x = 0: \; y = -3; \; z = 4. \quad A(0,-3,4) \in r \\ \text{para } x = 1: \; y = -1; \; z = 3. \quad B(1,-1,3) \in r \end{cases}$

$\vec{v} \parallel \vec{AB} \longrightarrow \vec{v} = k\vec{AB} = k(B - A) = k(1, 2, -1)$

$k = 1 \longrightarrow \vec{v} = (1, 2, -1)$

$\alpha \longrightarrow \vec{n} = (n, m, -1)$

Aplicando A, $\vec{v}$ e $\vec{n}$ nas condições para reta contida em plano:

i) $\vec{v} \cdot \vec{n} = 0$

$(1, 2, -1) \cdot (n, m, -1) = 0$

$n + 2m = -1 \quad (a)$

ii) $A \subset r$ e $A \subset \alpha$

$\underset{x\ \ y\ \ z}{A(0,-3,4)} \longrightarrow \alpha : n \cdot 0 + m(-3) - 4 - 2 = 0$

$-3m - 6 = 0$

$\underline{m = -2}$

$m \longrightarrow (a) : n + 2(-2) = -1$

$\underline{n = 3}$

$\therefore m = -2$

$n = 3$

6

$\alpha_1 : z = 3$

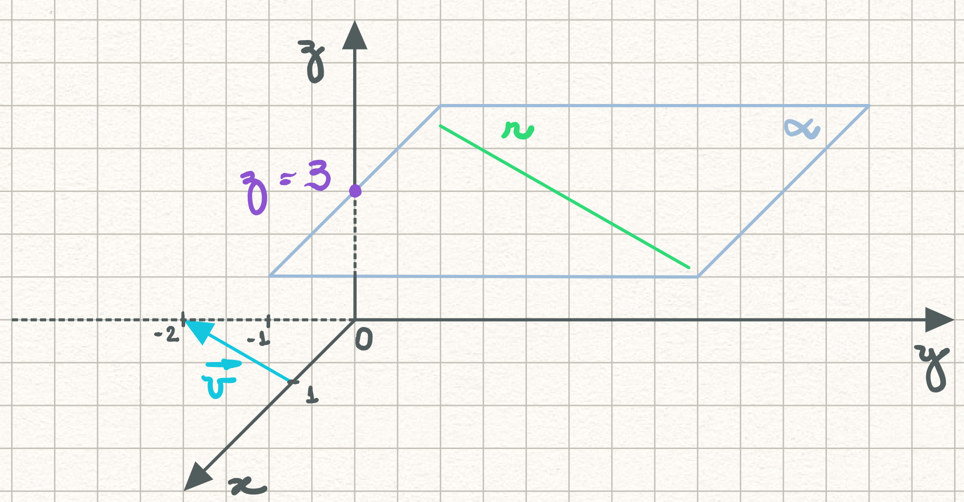
$\alpha_2 : 2x + y - 2 = 0$

$r = ?$ tal que $r = \alpha_1 \cap \alpha_2$

r é a solução do sistema formado pelas eqs. de α_1 e de α_2:

$$r : \begin{cases} z = 3 \\ y = -2x + 2 \end{cases}$$

Eqs. Reduzidas de r na variável $\underline{\underline{x}}$



$r \subset \alpha$

α ... plano // Oxy

que corta

Oz em $z = 3$

Voltando às eq. reduzidas de r :

Para $x = 0$: $y = 2$; $z = 3$ $\therefore$ $A(0,2,3) \in r$

$x = 1$: $y = 0$; $z = 3$ $\therefore$ $B(1,0,3) \in r$

$\vec{v} \parallel \overrightarrow{AB}$ $\therefore$ $\vec{v} = k\,\overrightarrow{AB} = k(B - A) = k(1,-2,0)$

$k = 1 \longrightarrow \vec{v} = (1,-2,0)$

Se r passa por $A(0,2,3)$, qualquer $P(x,y,z) \in r$ se e somente se $\overrightarrow{AP} \parallel \vec{v}$:

$$\overrightarrow{AP} = t\,\vec{v}, \quad t \in \mathbb{R}$$

$P = A + t\,\vec{v}$

E Vetorial : $r: (x,y,z) = (0,2,3) + t(1,-2,0),\ t \in \mathbb{R}$

E Paramétricas : $r: \begin{cases} x = t \\ y = 2 - 2t \\ z = 3 \end{cases},\ t \in \mathbb{R}$

↳ Esta é uma possibilidade, pois a eq. dependerá do ponto escolhido como aquele que se conhece da reta.

7 $r: \begin{cases} x = t \\ y = 1 - 2t \\ z = -t \end{cases},\ t \in \mathbb{R}$ $\qquad I = ?$ tal que $I = r \cap \alpha$

$\alpha: 2x + y - z - 4 = 0$

I é a solução do sistema formado pelas eqs. reduzidas da reta e pela eq. geral do plano.

Escrevendo as eqs. reduzidas de r:

E simétricas: $r: \quad x = \dfrac{1-y}{2} = -z$

E Reduzidas em "y": $r: \begin{cases} x = -\frac{1}{2}y + \frac{1}{2} \\ z = \frac{1}{2}y - \frac{1}{2} \end{cases}$

Sistema:

$$\begin{cases} x = -\frac{1}{2}y + \frac{1}{2} & \text{(i)} \\ z = \frac{1}{2}y - \frac{1}{2} & \text{(ii)} \\ 2x + y - z - 4 = 0 & \text{(iii)} \end{cases}$$

(i), (ii) $\longrightarrow$ (iii): $2\left(-\frac{1}{2}y + \frac{1}{2}\right) + y - \left(\frac{1}{2}y - \frac{1}{2}\right) - 4 = 0$

$$\cancel{-y} + 1 + \cancel{y} - \frac{1}{2}y + \frac{1}{2} - 4 = 0$$

$$\frac{-y + 1 - 6}{2} = 0 \quad \therefore \quad y = -5 /\!/$$

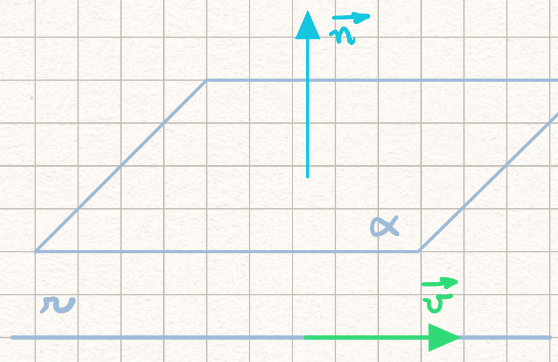
$y \longrightarrow$ (i): $x = -\frac{1}{2}(-5) + \frac{1}{2} \quad \therefore \quad x = 3 /\!/$

$y \longrightarrow$ (ii): $z = \frac{1}{2}(-5) - \frac{1}{2} \quad \therefore \quad z = -3 /\!/$

Logo: $I(3, -5, -3)$

8 $\quad r = ?$ tal que:

* passa por $A(3,6,4)$
* intercepta $Oz \longrightarrow B(0,0,z_0) \in r$
* $/\!/ \alpha: x - 3y + 5z - 6 = 0$



$r \parallel \alpha \Rightarrow \vec{v} \perp \vec{n}$

$\therefore \vec{v} \cdot \vec{n} = 0 \qquad (1)$

$r \begin{cases} \text{passa por } A(3,6,4) \\ \text{intercepta } Oz \longrightarrow B(0,0,z_0) \in r \end{cases} \qquad \therefore \vec{v} \parallel \vec{AB}$

$\vec{v} = k\,\vec{AB} = k(B-A) = k(-3,-6,z_0-4)$

$k=-1 \longrightarrow \vec{v} = (3,6,4-z_0)$

$\alpha \longrightarrow \vec{n} = (1,-3,5)$

Substituindo $\vec{v}$ e $\vec{n}$ em (1):

$(3,6,4-z_0)\cdot(1,-3,5) = 0$

$3-18+5(4-z_0) = 0$

$-15+20-5z_0 = 0 \qquad \therefore z_0 = 1$

Assim:

$\vec{v} = (3,6,3)$ ou $\vec{v} = (1,2,1)$

Um ponto genérico $P(x,y,z) \in r$ se, e somente se:

$\vec{AP} \parallel \vec{v} \quad \therefore \quad \vec{AP} = t\vec{v},\ t \in \mathbb{R}$

$P = A + t\vec{v}$

E vetorial: $r: (x,y,z) = (3,6,4) + t(1,2,1),\ t \in \mathbb{R}$

E Paramétricas : $r: \begin{cases} x = 3 + t \\ y = 6 + 2t \\ z = 4 + t \end{cases}, \quad t \in \mathbb{R}$

E Simétricas : $r: \; x - 3 = \dfrac{y-6}{2} = z - 4$

9 $\quad r: \begin{cases} y = 2x - 3 \\ z = -x + 2 \end{cases}$

$I_1 = ?$ $\qquad$ $I_1 = r \cap Oxy \quad (\alpha_1 : z = 0)$

$I_2 = ?$ $\quad$ tais que $\quad$ $I_2 = r \cap Oxz \quad (\alpha_2 : y = 0)$

$I_3 = ?$ $\qquad$ $I_3 = r \cap Oyz \quad (\alpha_3 : x = 0)$

$I_1 = r \cap Oxy \quad \begin{cases} y = 2x - 3 \\ z = -x + 2 \\ z = 0 \end{cases} \quad \therefore \quad I_1 (2, 1, 0)$

$I_2 = r \cap Oxz \quad \begin{cases} y = 2x - 3 \\ z = -x + 2 \\ y = 0 \end{cases} \quad \therefore \quad I_2 \left(\frac{3}{2}, 0, \frac{1}{2} \right)$

$I_3 = r \cap Oyz \quad \begin{cases} y = 2x - 3 \\ z = -x + 2 \\ x = 0 \end{cases} \quad \therefore \quad I_3 (0, -3, 2)$

Logo, as intersecções são : $I_1 (2, 1, 0)$

$I_2 \left(\frac{3}{2}, 0, \frac{1}{2} \right)$

$I_3 (0, -3, 2)$