14 - Derivada da Função Inversa

Seja $y = f(x)$ uma função definida em um intervalo aberto (a,b). Suponhamos que $f(x)$ admita uma função inversa $x = g(y)$ contínua. Se $f'(x)$ existir e ser diferente de zero para todo $x \in (a,b)$, então $g = f^{-1}$ é derivável e valerá

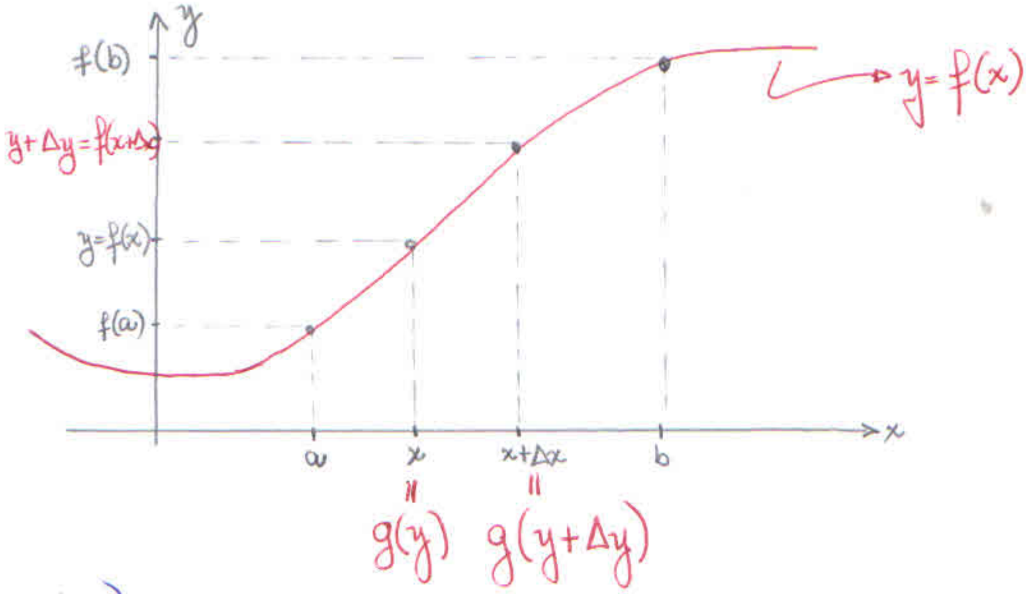
$$g'(y) = \frac{1}{f'(x)} = \frac{1}{f'[g(y)]}$$

Demonstração

Hipótese: $y = f(x)$ admite a função inversa, $f'(x)$ existe $x \in (a,b)$

Tese: $g'(y) = \dfrac{1}{f[g(y)]}$

Considere a figura seguinte,



$$\Delta x = g(y+\Delta y) - x$$

$$x + \Delta x = g(y + \Delta y) \Rightarrow \text{se } \Delta y \to 0 \text{ então } \Delta x \to 0$$

$$\Delta y \to 0 \Leftrightarrow \Delta x \to 0$$

$$\Delta x = g(y+\Delta y) - x$$

$$\boxed{\Delta x = g(y + \Delta y) - g(y)}$$

$$\Delta y = f(x+\Delta x) - y$$

$$\boxed{\Delta y = f(x+\Delta x) - f(x)}$$

Então,

$$g'(y) = \lim_{\Delta y \to 0} \frac{\overbrace{g(y+\Delta y) - g(y)}^{\Delta x}}{\Delta y}$$

$$g'(y) = \lim_{\substack{\Delta y \to 0 \\ \Delta x \to 0}} \frac{\Delta x}{\Delta y} = \lim_{\substack{\Delta y \to 0 \\ \Delta x \to 0}} \frac{1}{\frac{\Delta x}{\Delta y}} = \frac{1}{\underbrace{\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta y} = f(x+\Delta x) - f(x)}_{f'(x)}}$$

Derivada das Funções Trigonométricas

Função arco tangente

Seja $f: \mathbb{R} \longrightarrow \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ definida por $f(x) = \operatorname{arctg} x$. Então $y = f(x)$ é derivável e $y' = \frac{1}{1+x^2}$.

$\operatorname{sen}\dfrac{\pi}{6} = \dfrac{1}{2}$ $\quad$ $\operatorname{arc\,sen}\dfrac{1}{2} = \dfrac{\pi}{6}$

$$\operatorname{arc\,sen} x = y$$
$$x = \operatorname{sen} y$$

$y = \operatorname{arc\,tg} x \iff x = \operatorname{tg} y \,,\; y \in \left(-\pi/2, \pi/2\right)$

Como $\operatorname{tg} y$ existe e é diferente de zero para qualquer $y \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, aplicando o teorema da derivada da função inversa tem-se:

$$y' = \frac{1}{(\operatorname{tg} y)'} = \frac{1}{\sec^2 y}$$

Como $\sec^2 y = 1 + \operatorname{tg}^2 y$, obtém-se:

$$y' = \frac{1}{1 + \operatorname{tg}^2 y}$$

Substituindo $\operatorname{tg} y$ por x, tem-se:

$$y' = \frac{1}{1 + x^2} \Rightarrow \boxed{y' = \frac{1}{1 + u^2}\,du}$$

Derivada das Funções Trigonométricas Inversas

$$d(\operatorname{arc\,cos} u) = \frac{-1}{\sqrt{1 - u^2}}\,du \,,\; |u| > 1$$

$$d(\operatorname{arc\,sec} u) = \frac{1}{|u| \cdot \sqrt{u^2 - 1}}\,du$$

$$d(\operatorname{arc\,cossec} u) = \frac{1}{|u| \cdot \sqrt{u^2 - 1}}\,du \,,\; |u| > 1$$

$$d(\operatorname{arc\,sen} u) = \frac{1}{\sqrt{1 - u^2}}\,du$$

$$d(\text{arc cotg } u) = \frac{-1}{1+u^2} \cdot du$$

Exemplos

a) $y = tg^{-1} \underbrace{\sqrt{x}}_{u} \iff y = \text{arc tg} \sqrt{x}$

$$y' = \frac{1}{1+u^2} du \qquad u = x^{1/2}$$

$$u' = \frac{1}{2} x^{-1/2}$$

$$y' = \frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2} x^{-1/2}$$

$$y' = \frac{1}{2\sqrt{x}(1+x)}$$

b) $G(x) = \underbrace{\sqrt{1-x^2}}_{u} \cdot \underbrace{\arccos x}_{v}$ $\qquad d(u.v) = u\,dv + v\,du$

$u = (1-x^2)^{1/2}$

$u' = \frac{1}{\cancel{2}}(1-x^2)^{-1/2} \cdot (-\cancel{2}x)$

$u' = -x \cdot (1-x^2)^{-1/2}$

$v = \arccos x$

$v' = \frac{-1}{\sqrt{1-x^2}}$

$$G'(x) = (1-x^2)^{1/2} \cdot \left(\frac{-1}{\sqrt{1-x^2}}\right) + \arccos x \cdot \left[-x \cdot (1-x^2)^{-1/2}\right]$$

$$G'(x) = \cancel{\sqrt{1-x^2}} \cdot \left(\frac{-1}{\cancel{\sqrt{1-x^2}}}\right) - \frac{x \arccos x}{\sqrt{1-x^2}}$$

$$G'(x) = -1 - \frac{x \arccos x}{\sqrt{1-x^2}}$$

c) $y = \text{arc cossec} \underbrace{(2x+3)}_{u}$ $\qquad u = 2x+3$, $du = 2$

$$y' = \frac{2}{|2x+3| \cdot \sqrt{(2x+3)^2 - 1}}$$

$$y' = \frac{2}{|2x+3| \cdot \sqrt{4x^2 + 12x + 9 - 1}}$$

$2x+3 < -1 \qquad\qquad 2x+3 > 1 \quad , \quad |u| > 1$

$2x < -4 \qquad\qquad 2x > -2$

$x < -2 \qquad\qquad x > -1$

$$Df = \{x \in \mathbb{R} \;/\; x < -2 \text{ ou } x > -1\}$$

d)

$$\frac{d}{dx}\left[\underbrace{x}_{u}\underbrace{\operatorname{arctg} x}_{v} - \frac{1}{2}\ln\left(1+x^2\right)\right] = \operatorname{arctg} x \qquad \begin{array}{ll} u = x & v = \operatorname{arctg} x \\ u' = 1 & v' = \dfrac{1}{1+x^2} \end{array}$$

$$= \frac{x}{1+x^2} + \operatorname{arctg} x - \frac{1}{2} \cdot \frac{1}{1+x^2} \cdot 2x$$

$$= \cancel{\frac{x}{1+x^2}} + \operatorname{arctg} x \cancel{- \frac{x}{1+x^2}}$$

$$= \operatorname{arctg} x \quad \text{cqd.}$$