c) Intervalos de Confiança pelo Método de Bonferroni

Desigualdade de Bonferroni

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) \quad \Rightarrow$$

$$P\left(\bigcap_{i=1}^{n} A_i^c\right) = P\left(\left(\bigcup_{i=1}^{n} A_i\right)^c\right) = 1 - P\left(\bigcup_{i=1}^{n} A_i\right) \geq$$

$$\geq 1 - \sum_{i=1}^{n} P(A_i)$$

Se $B_i = A_i^c$, $\underline{P\left(\bigcap_{i=1}^{n} B_i\right) \geq 1 - \sum_{i=1}^{n} P(B_i^c)}$

Desigualdade de Bonferroni

m intervalos de confiança para combinações lineares das componentes de μ

$l_1'\mu \quad l_2'\mu \quad \dots \quad l_m'\mu$

$B_i : \quad a_i \leq l_i'\mu \leq b_i$ com

$P(B_i) = P(a_i \leq l_i'\mu \leq b_i) = 1 - \alpha_i \qquad i = 1, 2, \dots, m$

Nesse caso, o coeficiente de confiança global é

$$P\left(\bigcap_{i=1}^{m} B_i\right) \geq 1 - \sum_{i=1}^{m} P(B_i^c) = 1 - \sum_{i=1}^{m} \alpha_i = 1 - (\alpha_1 + \alpha_2 + \dots + \alpha_m)$$

Portanto, se desejamos um coeficiente de confiança global de $1-\alpha$, cada intervalo individual deverá ser construído com coeficiente $1 - \frac{\alpha}{m} = 1 - \alpha_i \Rightarrow$

$\alpha_i = \frac{\alpha}{m}$. Este fato permite construirmos os intervalos individuais do item a), cada um deles com coeficiente $1 - \alpha/m$, garantindo um coeficiente de confiança global $1-\alpha$.

Assim, para $m=p$, o coeficiente de confiança global dos intervalos para $\mu_1, \mu_2 \dots \mu_p$

$$\left[\bar{X}_1 - t_{n-1}\left(\frac{\alpha}{2m}\right)\sqrt{\frac{s_{11}}{n}}\ ,\ \bar{X}_1 + t_{n-1}\left(\frac{\alpha}{2m}\right)\sqrt{\frac{s_{11}}{n}}\right]$$

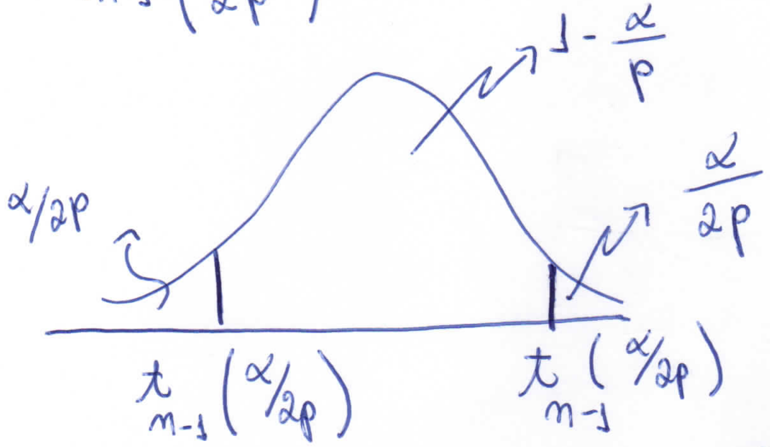
$$\left[\bar{X}_2 - t_{n-1}\left(\frac{\alpha}{2m}\right)\sqrt{\frac{s_{22}}{n}}\ ,\ \bar{X}_2 + t_{n-1}\left(\frac{\alpha}{2m}\right)\sqrt{\frac{s_{22}}{n}}\right]$$

$$\vdots$$

$$\left[\bar{X}_p - t_{n-1}\left(\frac{\alpha}{2m}\right)\sqrt{\frac{s_{pp}}{n}}\ ,\ \bar{X}_p + t_{n-1}\left(\frac{\alpha}{2m}\right)\sqrt{\frac{s_{pp}}{n}}\right]$$

é $1-\alpha$.

$t_{n-1}\left(\frac{\alpha}{2p}\right)$



global

no exemplo $p = m = 3$ $\quad \gamma = 1-\alpha = 0,95 \quad \alpha = 0,05$

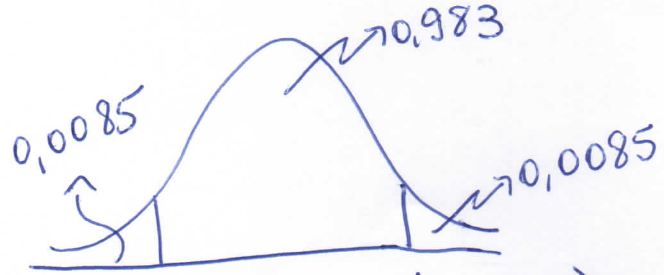
$1-\alpha/m = 1-\frac{0,05}{3} = 1-0,017 = 0,983 \rightarrow$ coef. de conf.

de cada intervalo

$\frac{\alpha}{p} = 0,017$

$\frac{\alpha}{2p} = 0,0085$

soma dos lados $1-0,983 = 0,017$



$t_{86}(0,0085) \simeq 2,5$

IC p/ μ_1 $\quad 527,74 \pm 2,5\sqrt{\frac{5691,34}{87}}$

IC p/ μ_2 $\quad 54,69 \pm 2,5\sqrt{\frac{126,05}{87}}$

IC p/ μ_3 $\quad 25,13 \pm 2,5\sqrt{\frac{23,11}{87}}$

Coef. Global $\gamma = 0,95$

Exercícios

1) $\begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix} \sim N_3(0, \Sigma)$ $\quad \Sigma = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

a) Determine a distribuição Marginal de X_1.

$X_1 \sim N(0, 2)$

b) Determine a dist. conjunta de X_1 e X_2.

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N_2\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \right)$$

c) Determine a dist. condicional de X_0 dado $X_1 = x_1$ e $X_2 = x_2$.

$$\begin{bmatrix} X_0 \\ \hline X_1 \\ X_2 \end{bmatrix} \quad \begin{bmatrix} 0 \\ \hline 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \hline \mu_2 \end{bmatrix} \quad \left[\begin{array}{c|cc} 4 & 1 & 0 \\ \hline 1 & 2 & 1 \\ 0 & 1 & 3 \end{array}\right] = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22} \end{bmatrix}$$

A dist. condicional de X_0 dado $X_1 = x_1$ e $X_2 = x_2$ é

$N_1(\mu_c, \Sigma_c)$

$$\mu_c = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \mu_2 \right) = 0 + [1 \ \ 0] \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} x_1 - 0 \\ x_2 - 0 \end{bmatrix}$$

$$= \frac{1}{5}[1 \ \ 0] \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{5}(3x_1 - x_2).$$

$$\Sigma_c = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} = 4 - [1 \ \ 0]\frac{1}{5}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{17}{5}$$

$$\therefore \ X_0 \mid X_1 = x_1, X_2 = x_2 \sim N\left(\frac{3x_1 - x_2}{5}, \frac{17}{5} \right)$$

d) Determine a dist. de $Z = 4X_0 - 6X_1 + X_2$.

$$Z = \underbrace{\begin{bmatrix} 4 & -6 & 1 \end{bmatrix}}_{A} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix} = AX \quad \therefore \ Z \sim N(\mu, \sigma^2)$$

$$\mu = \begin{bmatrix} 4 & -6 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

$$\sigma^2 = \begin{bmatrix} 4 & -6 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ -6 \\ 1 \end{bmatrix} = 79 \quad \therefore \ Z \sim (0, 79)$$

e) Determine a covariância entre Z_1 e Z_2 para $Z_1 = X_0 - X_1 + 2X_2$ e $Z_2 = 2X_1$.

$$Z_1 = \underbrace{\begin{bmatrix} 1 & -1 & 2 \end{bmatrix}}_{A} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix} \qquad Z_2 = \underbrace{\begin{bmatrix} 0 & 2 & 0 \end{bmatrix}}_{B} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}$$

$$Cov(Z_1, Z_2) = Cov(AX, BX) = A\,Var(X)\,B' =$$

$$\begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = 2$$

2) Se $\begin{bmatrix} X \\ Y \end{bmatrix} \sim N_2(0, \Sigma)$ $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$

a) Obtenha a distribuição de probabilidades de $\begin{bmatrix} Z \\ Y \end{bmatrix}$ onde $Z = X - \dfrac{\sigma_{12}}{\sigma_{22}} Y$.

b) Prove que Z e Y são v. a. independentes.

a) $$\begin{bmatrix} Z \\ Y \end{bmatrix} = \begin{bmatrix} X - \dfrac{\sigma_{12}}{\sigma_{22}} Y \\ Y \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -\dfrac{\sigma_{12}}{\sigma_{22}} \\ 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$\begin{bmatrix} Z \\ Y \end{bmatrix} \sim N_2(0, V)$$

$$E\begin{pmatrix} Z \\ Y \end{pmatrix} = A\,0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$V = \begin{bmatrix} 1 & -\sigma_{12}/\sigma_{22} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\sigma_{12}/\sigma_{22} & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \sigma_{11} - \dfrac{\sigma_{12}^2}{\sigma_{22}} & \sigma_{12} - \dfrac{\sigma_{12}}{\sigma_{22}} \sigma_{22} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{-\sigma_{12}}{\sigma_{22}} & 1 \end{bmatrix} = \begin{bmatrix} \sigma_{11} - \dfrac{\sigma_{12}^2}{\sigma_{22}} & 0 \\ 0 & \sigma_{22} \end{bmatrix}$$

b) $\begin{pmatrix} Z \\ Y \end{pmatrix} \sim N_2$ e $Cov(Z, Y) = 0 \Rightarrow$ Z e Y são independentes

3) Seja $X \sim N_p(\mu, \Sigma)$, V matriz positiva definida tal que $V = PP'$ com P não singular.

a) justifique a existência de P.

b) Prove que $P^{-1}X \sim N_p(P^{-1}\mu, I_p)$.

a) P existe pela decomposição de Cholesky.

b) $P^{-1}X \sim N_p$ porque $AX \sim N_p$.

P'^{-1}

$E(P^{-1}X) = P^{-1}E(X) = P^{-1}\mu$.

$Var(P^{-1}X) = P^{-1}\,Var(X)\,P^{-1'} = P^{-1}PP'P^{-1'}$

$= \underbrace{P^{-1}P}_{I}\,\underbrace{P'P'^{-1}}_{I} = I$.

4) Uma forma quadrática $x'Ax$ é dita positiva definida se a matriz A é positiva definida.
A forma quadrática $3x_1^2 + 3x_2^2 - 2x_1x_2$ é p. d.?

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 & bx_1 + cx_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

$$= (ax_1 + bx_2)x_1 + (bx_1 + cx_2)x_2 = ax_1^2 + 2bx_1x_2 + cx_2^2$$

$\Rightarrow a = 3 \quad c = 3 \quad b = -1$

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

$A_{p \times p}$ simétrica é p.d. $\Leftrightarrow$ suas raízes características são todas positivas (>0)

$A_{p \times p}$ simétrica é p.d. $\Leftrightarrow$ $a_{11} > 0$, $\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}\right) > 0$

$\ldots \det(A) > 0$.

$a_{11} = 3 > 0 \quad \det(A) = 9 - (-1)(-1) = 8 > 0 \quad \therefore A$ é p.d.

R.C. $\det \begin{bmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{bmatrix} = (3-\lambda)^2 - 1 = 0 \Rightarrow 3 - \lambda = \pm 1$

$\Rightarrow \lambda = 2 \quad \lambda = 4$ positivas $\Rightarrow A$ é p. d.

5) Se $X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \sim N_3(\mu, \Sigma)$ $\quad \mu = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ $\quad \Sigma = \begin{bmatrix} 2 & -1/2 & -1 \\ -\frac{1}{2} & 1 & 0 \\ -1 & 0 & 4 \end{bmatrix}$

determine a distribuição condicional de $X_1 | \bar{X} = 6$.

$$\begin{bmatrix} X_1 \\ \bar{X} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} X_1 \\ \bar{X} \end{bmatrix} \sim N_2 \left(\begin{bmatrix} 1 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2 & 1/6 \\ 1/6 & 4/9 \end{bmatrix} \right)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 & -\frac{1}{2} & -1 \\ -\frac{1}{2} & 1 & 0 \\ -1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1/3 \\ 0 & 1/3 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 2 & 1/6 \\ 1/6 & 4/9 \end{bmatrix}$$

$X_1 | \bar{X} = 6 \sim N_1(\mu, \sigma^2) \qquad N_1(3, 31/16)$

$\mu = \mu_1 + \Sigma_{12} \Sigma_{22}^{-1} (\underset{x_2}{6} - \mu_2) = 1 + \frac{1}{6} \cdot \frac{9}{4} \left(6 - \frac{2}{3}\right) = 3$

$g(x_1 | x_2)$ é normal multivariada com vetor de médias μ e matriz de covariância

$\Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{12}' = 2 - \frac{1}{6} \cdot \frac{9}{4} \cdot \frac{1}{6} = \frac{31}{16} = \sigma^2$

$X_1 | \bar{X} = x \sim N_1 \quad \mu = 1 + \frac{1}{6} \cdot \frac{9}{4} \left(x - \frac{2}{3}\right) \; ; \; \sigma^2 = \frac{31}{16}$