$$L(\underset{\sim}{\beta}, \sigma^2 \mid \underset{\sim}{y}, X) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left[-\frac{1}{2\sigma^2}(\underset{\sim}{y} - X\underset{\sim}{\beta})'(\underset{\sim}{y} - X\underset{\sim}{\beta})\right]$$

$$\Theta = \{(\underset{\sim}{\beta}, \sigma^2),\ -\infty < \beta_j < +\infty,\ j = 0, 1, 2 \ldots k,\ \sigma^2 > 0\}$$

$$\Theta_0 = \{(\underset{\sim}{\beta}, \sigma^2) \in \Theta \mid \underset{\sigma^2 > 0}{C\underset{\sim}{\beta} = \underset{\sim}{m}}\}$$

$$\lambda = \frac{\sup\limits_{(\underset{\sim}{\beta}, \sigma^2) \in \Theta_0} L(\underset{\sim}{\beta}, \sigma^2 \mid \underset{\sim}{y}, X)}{\sup\limits_{(\underset{\sim}{\beta}, \sigma^2) \in \Theta} L(\underset{\sim}{\beta}, \sigma^2 \mid \underset{\sim}{y}, X)}$$

$$\sup_{(\underset{\sim}{\beta}, \sigma^2) \in \Theta} L(\underset{\sim}{\beta}, \sigma^2 \mid \underset{\sim}{y}, X) = L(\hat{\underset{\sim}{\beta}}, \hat{\sigma}^2_{mv} \mid \underset{\sim}{y}, X) =$$

$$\hat{\underset{\sim}{\beta}} = (X'X)^{-1} X' \underset{\sim}{Y}$$

$$\hat{\sigma}^2_{mv} = \frac{(\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}})'(\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}})}{n}$$

$$= (2\pi)^{-n/2} (\hat{\sigma}^2_{mv})^{-n/2} e^{-n/2}$$

Cálculo de $\sup\limits_{(\underset{\sim}{\beta}, \sigma^2) \in \Theta_0} L(\underset{\sim}{\beta}, \sigma^2 \mid \underset{\sim}{y}, X)$

Maximizar $L(\beta, \sigma^2 | y, X)$ sujeito à condição

$C\beta = \underset{\sim}{m}$

$(C\beta - m)'\lambda$

$$L = L(\beta, \sigma^2 | y, X) - \underset{\sim}{\lambda}'(C\beta - \underset{\sim}{m})$$ obter $\hat{\beta}_0, \hat{\sigma}_0^2$

$\underset{\sim}{\lambda}$ vetor $p \times 1$ de multiplicadores de Lagrange

$$L(\beta, \sigma^2 | y, X) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left[-\frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta)\right]$$

$y'y - 2y'X\beta + \beta'X'X\beta$

$$\frac{\partial L}{\partial \beta} = -\frac{1}{2\sigma^2}(2X'X\beta - 2X'y)\left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left[-\frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta)\right] - C'\underset{\sim}{\lambda}$$

$$= -\frac{1}{2\sigma^2}(2X'X\beta - 2X'y)\, L(\beta, \sigma^2 | y, X) - C'\underset{\sim}{\lambda}$$

derivada de exp

$$\frac{\partial L}{\partial \sigma^2} = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2}\left(\frac{1}{2\sigma^4}(y - X\beta)'(y - X\beta)\right)\exp\left[-\frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta)\right]$$

$$+ \left(\frac{1}{2\pi}\right)^{n/2}\left[-\frac{n}{2}(\sigma^2)^{-n/2-1}\right]\exp\left[-\frac{1}{2\sigma^2}(y - X\beta)'(y - X\beta)\right]$$

$$= -\frac{n}{2\sigma^2} L(\beta, \sigma^2 | y, X) + \frac{1}{2\sigma^4}(y - X\beta)'(y - X\beta)\, L(\beta, \sigma^2 | y, X)$$

$$\frac{\partial L}{\partial \underset{\sim}{\lambda}} = -(C\beta - \underset{\sim}{m})$$

$$\frac{\partial L}{\partial \sigma^2} = 0$$

$\div \; L(\underset{\sim}{\beta}, \sigma^2 \mid \underset{\sim}{y}, X)$

$$-n\sigma^2 + (\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}}_0)'(\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}}_0) = 0$$

$$\hat{\sigma}_0^2 = \frac{(\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}}_0)'(\underset{\sim}{y} - X\hat{\underset{\sim}{\beta}}_0)}{n}$$

$$\frac{\partial L}{\partial \underset{\sim}{\beta}} = \frac{X'X\hat{\underset{\sim}{\beta}}_0 - X'\underset{\sim}{y}}{\hat{\sigma}_0^2} + \frac{C'\underset{\sim}{\lambda}}{L(\hat{\underset{\sim}{\beta}}_0, \hat{\sigma}_0^2 \mid \underset{\sim}{y}, X)} = \underset{\sim}{0}$$

$$\Rightarrow \quad X'X\hat{\underset{\sim}{\beta}}_0 - X'\underset{\sim}{y} + C'\underset{\sim}{\lambda}^* = 0 \qquad \underset{\sim}{\lambda}^* = +\frac{\hat{\sigma}_0^2\,\underset{\sim}{\lambda}}{L(\hat{\underset{\sim}{\beta}}_0, \hat{\sigma}_0^2 \mid \underset{\sim}{y}, X)}$$

$$X'X\hat{\underset{\sim}{\beta}}_0 + C'\underset{\sim}{\lambda}^* = X'\underset{\sim}{y}$$

$$X'X\hat{\underset{\sim}{\beta}}_0 = X'\underset{\sim}{y} - C'\underset{\sim}{\lambda}^*$$

pular

$$\hat{\underset{\sim}{\beta}}_0 = (X'X)^{-1}X'\underset{\sim}{y} - (X'X)^{-1}C'\underset{\sim}{\lambda}^* = \hat{\underset{\sim}{\beta}} - (X'X)^{-1}C'\underset{\sim}{\lambda}^*$$

$$\frac{\partial L}{\partial \underset{\sim}{\lambda}} = \underset{\sim}{0} \quad \Rightarrow \quad C\hat{\underset{\sim}{\beta}}_0 = \underset{\sim}{m} \Rightarrow C\hat{\underset{\sim}{\beta}} - C(X'X)^{-1}C'\underset{\sim}{\lambda}^* = \underset{\sim}{m}$$

$$\Rightarrow \underset{\sim}{\lambda}^* = \underbrace{[\underset{q\times k+1}{C}\underset{(k+1)\times(k+1)}{(X'X)^{-1}}\underset{(k+1)\times q}{C'}]}_{q\times q}^{-1}(C\hat{\underset{\sim}{\beta}} - \underset{\sim}{m})$$

$r[\;] = r(C) = q$

C é de posto completo
e $X'X$ é simétrica

$$\therefore \hat{\underset{\sim}{\beta}}_0 = \hat{\underset{\sim}{\beta}} - (X'X)^{-1}C'[C(X'X)^{-1}C']^{-1}(C\hat{\underset{\sim}{\beta}} - \underset{\sim}{m})$$

Obs:

$\hat{\sigma}_0^2$ e $\hat{\beta}_0$ são os estimadores de máxima verossimilhança de σ^2 e β sob $H_0: C\beta = m$

Coincidem com os est max veross no modelo reduzido sob H_0.

$$\sup_{\theta \in \Theta_0} L(\beta, \sigma^2 | y, X) = L(\hat{\beta}_0, \hat{\sigma}_0^2 | y, X) = (2\pi)^{-n/2} (\hat{\sigma}_0^2)^{-n/2} e^{-n/2}$$

$$\lambda = \left(\frac{\hat{\sigma}^2_{mv}}{\hat{\sigma}_0^2} \right)^{n/2}$$

$\hat{\sigma}^2_{mv}$

$$RC: \quad \lambda = \left(\frac{\hat{\sigma}^2}{\hat{\sigma}_0^2} \right)^{n/2} \leq \lambda_0$$

Usar a estatística $W = (\lambda^{-2/n} - 1) \dfrac{n - k - 1}{p}$ facilidade de obter a distribuição

- W é uma função monótona de λ

pular

- $\lambda \leq \lambda_0 \iff \lambda^{-1} \geq \lambda_0^{-1} \iff \lambda^{-2/n} \geq \lambda_0^{-2/n} \iff$

$$(\lambda^{-2/n} - 1) \frac{n - k - 1}{p} \geq (\lambda_0^{-2/n} - 1) \frac{n - k - 1}{p} \iff$$

$\iff W \geq w$ forma da região crítica

$$- \quad W = \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_{mv}^2} - 1 \right) \frac{n-k-1}{p} = \frac{\hat{\sigma}_0^2 - \hat{\sigma}_{mv}^2}{\hat{\sigma}_{mv}^2} \; \frac{n-k-1}{p}$$

lembrando que

$$\begin{cases} \hat{\sigma}_0^2 = \dfrac{(\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}}_0)'(\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}}_0)}{n} \\ \hat{\underset{\sim}{\beta}}_0 = \hat{\underset{\sim}{\beta}} - \underbrace{(X'X)^{-1} C' [C(X'X)^{-1}C']^{-1} (C\hat{\underset{\sim}{\beta}} - \underset{\sim}{m})}_{\underset{\sim}{H}_{(k+1)\times 1}} \end{cases}$$

$$= \hat{\underset{\sim}{\beta}} - \underset{\sim}{H}$$

pular

$$\hat{\sigma}_0^2 = \frac{1}{n} (\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}}_0)'(\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}}_0) = \frac{1}{n} \left\{ (\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}} + X\underset{\sim}{H})'(\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}} + X\underset{\sim}{H}) \right.$$

$$= \frac{1}{n} \left\{ (\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}})'(\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}}) + (\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}})' X H + \underset{\sim}{H}' X' (\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}}) + \underset{\sim}{H}' X' X \underset{\sim}{H} \right\}$$

$$\underset{\sim}{H}' X' (\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}}) = \underset{\sim}{H}' (X'\underset{\sim}{Y} - X'X\hat{\underset{\sim}{\beta}}] = \underset{\sim}{H}' [X'\underset{\sim}{Y} - X'\underset{\sim}{Y}] = 0$$

$$(\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}})' X H = [\underset{\sim}{H}' X' (\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}})]' = 0$$

$$= \hat{\sigma}^2 + \frac{\underset{\sim}{H}' X' X \underset{\sim}{H}}{n} = \hat{\sigma}^2 + \frac{1}{n} (C\hat{\underset{\sim}{\beta}} - \underset{\sim}{m})' [C(X'X)^{-1}C']^{-1} C (X'X)^{-1} (X'X)$$

$$(X'X)^{-1} C' [C(X'X)^{-1}C']^{-1} (C\hat{\underset{\sim}{\beta}} - \underset{\sim}{m}) =$$

$$= \hat{\sigma}_{mv}^2 + \frac{1}{n} (C\hat{\underset{\sim}{\beta}} - \underset{\sim}{m})' [C(X'X)^{-1}C']^{-1} (C\hat{\underset{\sim}{\beta}} - \underset{\sim}{m})$$

Substituindo na expressão de W (início pag 93)

$$W = \frac{(C\hat{\underset{\sim}{\beta}} - \underset{\sim}{m})' \left[C(X'X)^{-1}C'\right]^{-1} (C\hat{\underset{\sim}{\beta}} - \underset{\sim}{m})}{n\hat{\sigma}^2_{mv}} \; \frac{n-k-1}{p}$$

$$= \frac{(C\hat{\underset{\sim}{\beta}} - \underset{\sim}{m})' \left[C(X'X)^{-1}C'\right]^{-1} (C\hat{\underset{\sim}{\beta}} - \underset{\sim}{m})}{p \; \dfrac{(\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}})'(\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}})}{n-k-1}} =$$

$$= \frac{(C\hat{\underset{\sim}{\beta}} - \underset{\sim}{m})' \left[C(X'X)^{-1}C'\right]^{-1} (C\hat{\underset{\sim}{\beta}} - \underset{\sim}{m})}{p \quad MSE} \rightarrow \text{estatística de teste}$$

$\downarrow$

Modelo Completo $Y = X\underset{\sim}{\beta} + \underset{\sim}{\varepsilon}$

<u>Distribuição de W</u>

$$\frac{(C\hat{\underset{\sim}{\beta}} - \underset{\sim}{m})' \left[C(X'X)^{-1}C'\right]^{-1} (C\hat{\underset{\sim}{\beta}} - \underset{\sim}{m})}{\sigma^2}$$

$$\hat{\underset{\sim}{\beta}} \sim N_{k+1}(\underset{\sim}{\beta}, (X'X)^{-1}\sigma^2)$$

$$C\hat{\underset{\sim}{\beta}} \sim N_p(C\underset{\sim}{\beta}, C(X'X)^{-1}C'\sigma^2) \rightarrow (C\hat{\underset{\sim}{\beta}} - \underset{\sim}{m}) \sim N_p(C\underset{\sim}{\beta} - \underset{\sim}{m}, C(X'X)^{-1}C'$$

$$A\Sigma = \frac{\left[C(X'X)^{-1}C'\right]^{-1}}{\sigma^2} C(X'X)^{-1}C'\sigma^2 = I \quad (A\Sigma \text{ é idempotente})$$

$$\lambda = \frac{1}{2}(C\underset{\sim}{\beta} - \underset{\sim}{m})' \frac{\left[C(X'X)^{-1}C'\right]^{-1}}{\sigma^2} (C\underset{\sim}{\beta} - \underset{\sim}{m})$$

$$\therefore \frac{(C\hat{\underset{\sim}{\beta}}-\underset{\sim}{m})' \left[C(X'X)^{-1}C'\right]^{-1}(C\hat{\underset{\sim}{\beta}}-\underset{\sim}{m})}{\sigma^2} \sim \chi^2_{p,\lambda}$$

$$\frac{(\underset{\sim}{Y}-X\hat{\underset{\sim}{\beta}})'(\underset{\sim}{Y}-X\hat{\underset{\sim}{\beta}})}{\sigma^2} \sim \chi^2_{n-k-1}$$

$\hat{\underset{\sim}{\beta}}$ e SSE são independentes

$$\therefore \quad W \sim F_{p,n-k-1,\lambda}$$

sob H_0, $C\underset{\sim}{\beta} = \underset{\sim}{m}$, $\lambda=0$, $W \sim F_{p,n-k-1}$ central

O teste da razão de Veross. Generalizada rejeita H_0 p/

$W \geq w$

Tomar $w \mid \quad P(F_{p,n-k-1} \geq w) = \alpha$

Obs:

1) $W = \dfrac{\hat{\sigma}^2_0 - \hat{\sigma}^2_{mv}}{\hat{\sigma}^2_{mv}} \; \dfrac{n-k-1}{p}$

$\hat{\sigma}^2_{mv}$ - est. max. veross. de σ^2 no modelo original

$\hat{\sigma}^2_0$ - est. max. veross. de σ^2 no modelo reduzido sob H_0. (Ambos têm denominador n)

$SST - SSE = SSR$

$SSE_r = SST - SSR_r$

$SSE_c = SST - SSR_c$

Portanto, W pode ser escrita como

$$W = \frac{SSE_r - SSE_c}{SSE_c} \quad \frac{n-k-1}{p}$$

Ho: modelo reduzido

Ha: modelo completo

onde

SSE_c - soma de quadrados do resíduo do modelo completo (modelo original) $= (\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}})'(\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}})$

SSE_r - soma de quadrados do resíduo do modelo reduzido sob Ho $= (\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}}_0)'(\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}}_0) = (\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}})'(\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}}) + (C\hat{\underset{\sim}{\beta}} - \underset{\sim}{m})'[C(X'X)^{-1}C']^{-1}(C\hat{\underset{\sim}{\beta}} - \underset{\sim}{m})$

$SSE_r - SSE_c = (C\hat{\underset{\sim}{\beta}} - \underset{\sim}{m})'[C(X'X)^{-1}C']^{-1}(C\hat{\underset{\sim}{\beta}} - \underset{\sim}{m}) > 0$ pois $C(X'X)^{-1}C'$ é p.d.

(Não escrever $SSE_r - SSE_c = SSR_c - SSR_r$ aqui, pois isto pode não valer. Em alguns casos (raros), a redução sob Ho altera a variável resposta e as somas de quadrados totais não são as mesmas)

Resultados

ii) se A é positiva definida então $r[CAC'] = r(C) \Rightarrow [C(X'X)^{-1}C']$ é inv

iii) se A é p.d $n \times n$ e C é $p \times n$, $r(C) = p$ então CAC' é positiva definida

i) $X_{n \times p}$ $r(X) = p \Rightarrow X'X$ é p.d.

Ex: $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \epsilon$

$\beta_1 = \beta_2 + 4$

Sob Ho $\quad y = \beta_0 + (\beta_2 + 4)x_1 + \beta_2 x_2 + \epsilon$

Saber pag 386

$$y - 4x_1 = \beta_0 + \beta_2(x_1 + x_2) + \epsilon$$

2) $Var(C\hat{\underset{\sim}{\beta}} - \underset{\sim}{m}) = Var(C\hat{\underset{\sim}{\beta}}) = C(X'X)^{-1}C'\sigma^2$

$$\widehat{Var}(C\hat{\underset{\sim}{\beta}}) = C(X'X)^{-1}C' \, MSE$$

3) Para cada λ, o poder do teste é

$$\Pi(\lambda) = \int_{w_\alpha}^{\infty} F(p, n-k-1, \lambda)\, dw ,$$

onde $F(p, n-k-1, \lambda)$ - densidade da estatística $F_{p, n-k-1, \lambda}$

e w_α tal que $P(F_{p, n-k-1} \geq w_\alpha) = \alpha$

4) $\hat{\beta}_0 = \hat{\beta} - (X'X)^{-1} C' [C(X'X)^{-1}C']^{-1} (C\hat{\beta} - m)$, est. max. veross. de β sob $H_0: C\beta = m$ é também o estimador de mínimos quadrados de β sob H_0. [minimiza $(Y - X\hat{\beta}_0)'(Y - X\hat{\beta}_0)$ sujeito à condição $C\hat{\beta}_0 = m$

5) A estatística W para testar $H_0: C\beta = m$ contra $H_a: C\beta \neq m$ fica inalterada se testarmos $H_0: QC\beta = Qm$ contra $H_a: QC\beta \neq Qm$, onde Q é uma matriz $p \times p$ não singular

$C - p \times (k+1)$

$QC - p \times (k+1)$

$C\beta = m \Leftrightarrow QC\beta = Qm$

$$W_1 = \frac{(QC\hat{\beta} - Qm)' [QC(X'X)^{-1}C'Q']^{-1} (QC\hat{\beta} - Qm)}{p\,MSE} =$$

$$= \frac{(C\hat{\beta} - m)' Q' Q'^{-1} [C(X'X)^{-1}C']^{-1} Q^{-1} Q (C\hat{\beta} - m)}{p\,MSE} =$$

$$= \frac{(C\hat{\beta} - m)' [C(X'X)^{-1}C']^{-1} (C\hat{\beta} - m)}{p\,MSE}$$

No exemplo 4, $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \epsilon$

$H_0: \beta_1 = \beta_2 = \beta_3 = \beta_4 \iff C\underset{\sim}{\beta} = \underset{\sim}{m}$

$$C = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{bmatrix}_{3\times 5} \quad \underset{\sim}{m} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} \beta_1 - \beta_2 = 0 \\ \beta_1 - \beta_3 = 0 \\ \beta_1 - \beta_4 = 0 \end{matrix}$$

Ho pode ser escrita alternativamente como

$H_0: C_1 \underset{\sim}{\beta} = \underset{\sim}{m}_1$

$$C_1 = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \quad \underset{\sim}{m}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} \beta_1 - \beta_2 = 0 \\ \beta_2 - \beta_3 = 0 \\ \beta_3 - \beta_4 = 0 \end{matrix}$$

Temos que $C_1 = QC =$ para $Q[\quad]$

$$C_1 = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}_{3\times 3}}_{Q} C \qquad |Q| = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 1$$

$Q_{3\times 3}$ não singular, portanto

W_1 para testar $H_0: \overbrace{C_1 \underset{\sim}{\beta}}^{QC\beta} = \overbrace{\underset{\sim}{m}_1}^{=Q\,m} = 0$ coincide com W que

testa $H_0: C\underset{\sim}{\beta} = \underset{\sim}{m} = 0$ $(\qquad)$

Os graus de liberdade são os mesmos (3 e $n-5$) e assim, as conclusões são as mesmas.

Exemplos

1) $H_0: \underset{\sim}{l}'\underset{\sim}{\beta} = \cancel{l_0}\, b \qquad \underset{\sim}{l}' = [l_0 \; l_1 \; l_2 \ldots l_K]$

$H_a: \underset{\sim}{l}'\underset{\sim}{\beta} \neq \cancel{l_0}\, b \qquad p = 1$

$$W = \frac{(\underset{\sim}{l}'\hat{\underset{\sim}{\beta}} - \cancel{l_0}\, b)' \left[\underset{\sim}{l}'(X'X)^{-1}\underset{\sim}{l}\right]^{-1}_{1\times 1} (\underset{\sim}{l}'\hat{\underset{\sim}{\beta}} - \cancel{l_0}\, b)}{MSE}$$

$$= \frac{(\underset{\sim}{l}'\hat{\underset{\sim}{\beta}} - \cancel{l_0}\, b)^2}{\left[\underset{\sim}{l}'(X'X)^{-1}\underset{\sim}{l}\right]\hat{\sigma}^2} = \frac{(\underset{\sim}{l}'\hat{\underset{\sim}{\beta}} - \cancel{l_0}\, b)^2}{\widehat{Var}(\underset{\sim}{l}'\hat{\underset{\sim}{\beta}})}$$

Rejeitamos H_0 se $W \geq F_{\alpha, 1, n-k-1} = t^2_{\alpha/2, n-k-1}$

$$\iff \frac{\underset{\sim}{l}'\hat{\underset{\sim}{\beta}} - \cancel{l_0}\, b}{\sqrt{\widehat{Var}(\underset{\sim}{l}'\hat{\underset{\sim}{\beta}})}} \geq t_{\alpha/2, n-k-1} \quad \text{ou} \quad \frac{\underset{\sim}{l}'\hat{\underset{\sim}{\beta}} - \cancel{l_0}\, b}{\sqrt{\widehat{Var}(\underset{\sim}{l}'\hat{\underset{\sim}{\beta}})}} \leq -t_{\alpha/2, n-k}$$

RC em termos da estatística t. Permite fazer teste unicaudal.

Equivalentemente, aceitamos H_0 se e só se

$$\cancel{l_0}\, b \in \left[\underset{\sim}{l}'\hat{\underset{\sim}{\beta}} - t_{\alpha/2, n-k-1}\sqrt{\hat{\sigma}^2\, \underset{\sim}{l}'(X'X)^{-1}\underset{\sim}{l}}\;,\; \underset{\sim}{l}'\hat{\underset{\sim}{\beta}} + t_{\alpha/2, n-k-1}\sqrt{\hat{\sigma}^2\, \underset{\sim}{l}'(X'X)^{-1}\underset{\sim}{l}}\right.$$

$\downarrow$

IC com coeficiente $1-\alpha$ para $\underset{\sim}{l}'\underset{\sim}{\beta}$.

Se $\underset{\sim}{l} = [0, 0 \ldots \underset{j}{1} \ldots 0]_{1\times(k+1)}$ e $\cancel{l_0}\, b = 0$, o teste fica

$H_0: \beta_j = 0$

$H_a: \beta_j \neq 0$

$$W = \frac{\hat{\beta}_j^2}{a_{jj}\hat{\sigma}^2}$$

a_{jj} - j-ésimo elemento da diagonal de

$$(X'X)^{-1} = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0k} \\ & a_{jj} & & \\ & & & a_{kk} \end{bmatrix}$$

Rejeitamos H_0 se $W \geq F_{\alpha,1,n-k-1}$

Além disso,

$$W = \frac{SSE_r - SSE_c}{SSE_c}\,\frac{n-k-1}{p} \overset{p=1}{=} \frac{SSE_r - SSE_c}{MSE_c}$$

Modelo Completo: $y = \beta_0 + \beta_1 x_1 + \dots + \beta_j x_j + \beta_{j+1} x_{j+1} + \dots + \beta_k x_k + \epsilon$

Modelo Reduzido: $y = \beta_0 + \beta_1 x_1 + \dots + \beta_{j-1} x_{j-1} + \beta_{j+1} x_{j+1} + \dots + \beta_k x_k + \epsilon$

Como aqui a redução sob H_0 não altera a variável dependente,

$$W = \frac{SSR_c - SSR_r}{MSE_c}$$

↳ Estatística do Teste F-parcial

Intervalo de Confiança para β_j:

$$\hat{\beta}_j \pm t_{\alpha/2,\,n-k-1}\sqrt{a_{jj}MSE}$$

Para $F_{1,n-k-1}$, pode fazer o teste t

2) $\underset{\sim}{\beta} = \begin{bmatrix} \underset{\sim}{\beta}_1 \\ \underset{\sim}{\beta}_2 \end{bmatrix}$ $\quad \underset{\sim}{\beta}_1 \quad (k+1-q)\times 1$

$\underset{\sim}{\beta}_2 \quad q\times 1 \qquad 0 < q \leq k+1$

$H_0: \underset{\sim}{\beta}_2 = \underset{\sim}{b}_2$ → vetor de constantes

$H_a: \underset{\sim}{\beta}_2 \neq \underset{\sim}{b}_2$

$X'X = \begin{bmatrix} X_1' \\ X_2' \end{bmatrix} \begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{bmatrix}$

$k+1-q \times k+1-q \qquad k+1-q \times q$

$X = [\underset{n\times(k+1-q)}{X_1} \quad \underset{n\times q}{X_2}]$

$\underset{\sim}{Y} = X\underset{\sim}{\beta} + \underset{\sim}{\epsilon} = [X_1 \quad X_2]\begin{bmatrix} \underset{\sim}{\beta}_1 \\ \underset{\sim}{\beta}_2 \end{bmatrix} + \underset{\sim}{\epsilon} = X_1\underset{\sim}{\beta}_1 + X_2\underset{\sim}{\beta}_2 + \underset{\sim}{\epsilon}$

$C = [\underset{q\times(k+1-q)}{0} \quad I_q]$ $\qquad \underset{\sim}{m} = \underset{\sim}{b}_2 \rightarrow C\underset{\sim}{\beta} = \underset{\sim}{\beta}_2$

$(k+1-q) \times (k+1-q) \qquad (k+1-q)\times q$

$(X'X)^{-1} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$ $\qquad \hat{\underset{\sim}{\beta}} = \begin{bmatrix} \hat{\underset{\sim}{\beta}}_1 \\ \hat{\underset{\sim}{\beta}}_2 \end{bmatrix}$

$q\times(k+1-q) \qquad q\times q$

$C\hat{\underset{\sim}{\beta}} - \underset{\sim}{m} = \hat{\underset{\sim}{\beta}}_2 - \underset{\sim}{b}_2$

$C(X'X)^{-1}C' = [0 \quad I_q]\underbrace{\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}}_{(X'X)^{-1}}\begin{bmatrix} 0 \\ I_q \end{bmatrix}$

$= T_{22}$

$$W = \frac{(\hat{\underset{\sim}{\beta}}_2 - \underset{\sim}{b}_2)' \, T_{22}^{-1} \, (\hat{\underset{\sim}{\beta}}_2 - \underset{\sim}{b}_2)}{q \, MSE}$$

$$X'X = \begin{bmatrix} X_1' \\ X_2' \end{bmatrix} \begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{bmatrix}$$

$$(X'X)^{-1} = \begin{bmatrix} - & - \\ - & \underbrace{\left[X_2'X_2 - X_2'X_1(X_1'X_1)^{-1}X_1'X_2\right]^{-1}}_{\substack{\shortparallel \\ T_{22}}} \end{bmatrix} \qquad \left[B_{22} - B_{21}B_{11}^{-1}B_{12}\right]^{-1}$$

$$T_{22}^{-1} = X_2'X_2 - X_2'X_1(X_1'X_1)^{-1}X_1'X_2$$

H_0: $\underset{\sim}{\beta}_2 = \underset{\sim}{0}$ (verifica a importância das q últimas variáveis

H_a: $\underset{\sim}{\beta}_2 \neq \underset{\sim}{0}$ independentes)

$$W = \frac{\hat{\beta}_2' \, T_{22}^{-1} \, \hat{\beta}_2}{q \, MSE}$$

Rejeitamos H_0 para $W \geq w$

$P(F_{q, n-k-1} \geq w) = \alpha$

Como

$$W = \frac{SSE_r - SSE_c}{q \, MSE}, \quad SSE_c = \underset{\sim}{Y}'\underset{\sim}{Y} - \hat{\underset{\sim}{\beta}}'X'\underset{\sim}{Y}, \quad SSE_r = \underset{\sim}{Y}'\underset{\sim}{Y} - \hat{\underset{\sim}{\gamma}}'X_1'\underset{\sim}{Y}$$

$\hat{\underset{\sim}{\gamma}} = (X_1'X_1)^{-1}X_1'\underset{\sim}{Y}$, temos que

$$W = \frac{\hat{\underset{\sim}{\beta}}'X'\underset{\sim}{Y} - \hat{\underset{\sim}{\gamma}}'X_1'\underset{\sim}{Y}}{q \, MSE}$$

3) $H_0: \underset{\sim}{\beta}^* = \underset{\sim}{0}$

$$\underset{\sim}{\beta}^* = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix}$$

Caso particular do anterior

$q = k$

$\underset{\sim}{\beta}_1 = \beta_0$

Verificar que, neste caso, $W = \dfrac{MSR}{MSE}$

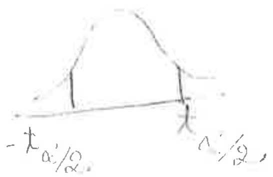
k) Intervalos de Confiança e de Previsão

Intervalo de Confiança para $E(Y_i \mid \underset{\sim}{X}_i) = \underset{\sim}{X}_i \underset{\sim}{\beta}$

$\underset{\sim}{X}_i \underset{\sim}{\beta} = \underset{\sim}{l}' \underset{\sim}{\beta}$ para $\underset{\sim}{l}' = \underset{\sim}{X}_i$

Portanto, o IC com coeficiente $1-\alpha$ é

$$\underset{\sim}{X}_i \hat{\underset{\sim}{\beta}} \pm t_{n-k-1, \alpha/2} \sqrt{\underset{\sim}{X}_i (X'X)^{-1} \underset{\sim}{X}_i' \hat{\sigma}^2}$$



Intervalo de Previsão para $Y \mid \underset{\sim}{X}_h$, onde a observação h não pertence à amostra

Definição

Seja T uma variável aleatória com valor desconhecido.
A var. aleat. $\hat{T}$ é um previsor não viciado para T se

$$E(\hat{T} - T) = 0$$

O erro quadrático médio de $\hat{T}$ é dado por

$$EQM(\hat{T}) = Var(\hat{T} - T) = E(\hat{T}-T)^2 = Var(\hat{T}-T) + E^2(\hat{T}-T)$$

se $\hat{T}$ é não viciado

$\hat{Y}_h = \underset{\sim}{X}_h \hat{\underset{\sim}{\beta}}$ previsor de Y_h

$$E(\hat{Y}_h - Y_h) = \underset{\sim}{X}_h E(\hat{\underset{\sim}{\beta}}) - E(Y_h) = \underset{\sim}{X}_h \underset{\sim}{\beta} - \underset{\sim}{X}_h \underset{\sim}{\beta}$$

$\therefore$ $\hat{Y}_h$ é um previsor não viciado de Y_h

Intervalo de Previsão para $Y | \underset{\sim}{X}_h$

$$\underbrace{\underset{\sim}{X}_h \hat{\underset{\sim}{\beta}}}_{\hat{Y}_h} \pm t_{n-k-1;\alpha/2} \sqrt{\widehat{Var}(\hat{Y}_h - Y_h)}$$

$\widehat{EQM}(\hat{Y}_h)$

$$Var(\hat{Y}_h) = Var(\underset{\sim}{X}_h \hat{\underset{\sim}{\beta}}) = \underset{\sim}{X}_h (X'X)^{-1} \underset{\sim}{X}_h' \sigma^2$$

$$Var(Y_h) = \sigma^2$$

$\hat{Y}_h$ e Y_h são independentes porque o elemento h não pertence à amostra $\Rightarrow$

$$Var(\hat{Y}_h - Y_h) = Var(\hat{Y}_h) + Var(Y_h) = \underset{\sim}{X}_h (X'X)^{-1} \underset{\sim}{X}_h' \sigma^2 + \sigma^2$$

I. Previsão $\underset{\sim}{X}_h \hat{\underset{\sim}{\beta}} \pm t_{n-k-1,\alpha/2} \sqrt{(1 + \underset{\sim}{X}_h (X'X)^{-1} \underset{\sim}{X}_h') \hat{\sigma}^2}$

Intervalo de confiança para σ^2

$$\frac{SSE}{\sigma^2} = \frac{(n-k-1)\,MSE}{\sigma^2} \sim \chi^2_{n-k-1}$$

$$P\left(a \leq \frac{(n-k-1)\,MSE}{\sigma^2} \leq b \right) =$$

$$P\left(\frac{(n-k-1)\,\hat{\sigma}^2}{b} \leq \sigma^2 \leq \frac{(n-k-1)\,\hat{\sigma}^2}{a} \right) = \gamma$$

IC p/ σ^2 : $\left[\frac{(n-k-1)\,MSE}{b} \, , \, \frac{(n-k-1)\,\hat{\sigma}^2}{a} \right]$

a, b tais que $P(a \leq Z \leq b) = \gamma$

$$Z \sim \chi^2_{n-k-1}$$

3.3 - Inferência $\underset{\sim}{\epsilon} \sim N_n(\underset{\sim}{0}, \sigma^2 V)$, V conhecida.

Modelo

$\underset{\sim}{Y} = X \underset{\sim}{\beta} + \underset{\sim}{\epsilon}$

$\underset{\sim}{\epsilon} \sim N_n(\underset{\sim}{0}, \sigma^2 V)$

V matriz positiva definida $(V \neq I)$ conhecida.

V positiva definida $\Rightarrow$ Existe P não singular tal que $V = PP'$

$$\underset{\sim}{Z} = P^{-1}\underset{\sim}{Y} \sim N_n(P^{-1}X\underset{\sim}{\beta}, \sigma^2 I)$$

P'^{-1}

$$Var(\underset{\sim}{Z}) = Var(P^{-1}\underset{\sim}{Y}) = P^{-1}VP^{-1'}\sigma^2 = P^{-1}PP'P^{-1'}\sigma^2$$

$$= \sigma^2 I$$

$\boxed{A^{-1}A = I}$

$(A^{-1}A)' = I$

$(\times A'^{-1})\ A'A^{-1'} = I$

$A^{-1'} = A'^{-1}$

Transformamos o modelo

$$\underset{\sim}{Y} = X\underset{\sim}{\beta} + \underset{\sim}{\epsilon} \qquad \underset{\sim}{\epsilon} \sim N(\underset{\sim}{0}, \sigma^2 V)$$

em

$$P^{-1}\underset{\sim}{Y} = P^{-1}X\underset{\sim}{\beta} + P^{-1}\underset{\sim}{\epsilon}$$

$$\underset{\sim}{Z} = Q\underset{\sim}{\beta} + \underset{\sim}{\eta} \qquad \underset{\sim}{Z} = P^{-1}\underset{\sim}{Y} \quad Q = P^{-1}X \quad \underset{\sim}{\eta} = P^{-1}\underset{\sim}{\epsilon}$$

$$\underset{\sim}{\eta} \sim N_n(\underset{\sim}{0}, \sigma^2 I)$$

O modelo transformado satisfaz às condições do modelo linear geral, caso 1. ($\underset{\sim}{\epsilon} \sim N_n(\underset{\sim}{0}, \sigma^2 I)$)

Os parâmetros de interesse $\underset{\sim}{\beta}$ e σ^2 do modelo original são os parâmetros do modelo transformado.

$\hat{\underset{\sim}{\beta}}$ → est. m.q. e m.v. de $\underset{\sim}{\beta}$ no modelo transformado

$Q = P^{-1}X \qquad Z = P^{-1}\underset{\sim}{Y}$

$$\hat{\underset{\sim}{\beta}} = (Q'Q)^{-1}Q'\underset{\sim}{Z}$$

$$= (X'\underbrace{P^{-1'}P^{-1}}_{V^{-1}}X)^{-1}X'\underbrace{P^{-1'}P^{-1}}_{V^{-1}}\underset{\sim}{Y}$$

$V = PP'$

$V^{-1} = P'^{-1}P^{-1}$

$= P^{-1'}P^{-1}$

$$= (X'V^{-1}X)^{-1}X'V^{-1}\underset{\sim}{Y}$$

$$Var(\hat{\underset{\sim}{\beta}}) = Var\left[(X'V^{-1}X)^{-1}X'V^{-1}\underset{\sim}{Y}\right] =$$

$$= (X'V^{-1}X)^{-1}X'V^{-1}\,V\,V^{-1}X(X'V^{-1}X)^{-1}\sigma^2$$

$$= (X'V^{-1}X)^{-1}\sigma^2$$

$\hat{\underset{\sim}{\beta}}_G = (X'V^{-1}X)^{-1}X'V^{-1}\underset{\sim}{Y}$ → estimador de mínimos quadrados generalizados de $\underset{\sim}{\beta}$ no modelo $\underset{\sim}{Y} = X\underset{\sim}{\beta} + \underset{\sim}{\epsilon}$

Coincide com o estimador de min. quad. usual $[(Q'Q)^{-1}Q'\underset{\sim}{Z}]$ do modelo transformado

$$\hat{\sigma}^2_G = \frac{1}{n-k-1}(\underset{\sim}{Z} - Q\hat{\underset{\sim}{\beta}})'(\underset{\sim}{Z} - Q\hat{\underset{\sim}{\beta}}) = \frac{1}{n-k-1}\left(\underset{\sim}{Z}'\underset{\sim}{Z} - \hat{\underset{\sim}{\beta}}'Q'\underset{\sim}{Z}\right)$$

em função das variáveis transformadas

$$= \frac{1}{n-k-1}\left[y' P^{-1'} P^{-1} y - \underbrace{y' V^{-1} X (X'V^{-1}X)^{-1}}_{\hat{\beta}'} \underbrace{X' P^{-1'}}_{Q'} \underbrace{P^{-1} y}_{z} \right]$$

$$= \frac{1}{n-k-1}\left[y' V^{-1} y - y' V^{-1} X (X'V^{-1}X)^{-1} X' V^{-1} y \right]$$

$$= \frac{1}{n-k-1}\, y' \left[V^{-1} - V^{-1} X (X'V^{-1}X)^{-1} X' V^{-1} \right] y$$

$\hat{\beta}_G$, $\widehat{Var}(\hat{\beta}_G)$ e $\hat{\sigma}^2_G$ são as únicas quantidades necessárias para estimação pontual e por intervalo e testes de hipóteses.

$$W = \frac{(C\hat{\beta}_G - m)' \left[C \overset{(X'V^{-1}X)^{-1}}{(Q'Q)^{-1}} C' \right]^{-1} (C\hat{\beta}_G - m)}{p \ \hat{\sigma}^2_G}$$

~~Também aqui, não trabalhar com $SSR_c - SSR_r$ pois a transformação $P^{-1} y = P^{-1} X \beta + P^{-1} \epsilon$ pode levar a um modelo sem intercepto.~~

Muitas vezes, a transformação $P^{-1} y = P^{-1} X \beta + P^{-1} \epsilon$ pode resultar em um modelo sem intercepto, mesmo que o modelo de partida possua intercepto.

Exemplo 1

$$Y_i = \beta x_i + \epsilon_i$$

$\epsilon_i \sim N(0, \sigma_i^2)$ ϵ_i, ϵ_j v.a. independentes, $i \neq j$

$$\sigma_i^2 = x_i \sigma^2$$

$$Var(\underset{\sim}{\epsilon}) = \begin{bmatrix} x_1 & 0 & 0 \\ & x_2 & 0 \\ & & \ddots \\ & & & x_n \end{bmatrix} \sigma^2 = V \sigma^2$$

$\underset{\sim}{\epsilon}_i \sim N_n(\underset{\sim}{0}, \sigma^2 V)$ V conhecida

$$V^{-1} = \mathrm{diag}\left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots \frac{1}{x_n}\right)$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$X'V^{-1}X = \sum_{i=1}^{n} x_i \qquad X'V^{-1}\underset{\sim}{Y} = \sum_{i=1}^{n} y_i \Rightarrow \hat{\underset{\sim}{\beta}} = \frac{\sum y_i}{\sum x_i} = \frac{\bar{y}}{\bar{x}}$$

Exemplo 2

$$Y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

$$\underset{\sim}{\epsilon} \sim N(\underset{\sim}{0}, \sigma^2 V)$$

$$V = \text{diag}(w_1, w_2 \dots w_n) \implies Var(\epsilon_i) = w_i \sigma^2 \quad w_i > 0$$

conhecidas

ϵ_i, ϵ_j v.a. indep. $i \neq j$

$$V = P P' \implies P = \begin{bmatrix} \sqrt{w_1} & & & \\ & \sqrt{w_2} & & \\ & & \ddots & \\ & & & \sqrt{w_n} \end{bmatrix}$$

$V = P^2$

$\boxed{P = V^{1/2}}$

$$P^{-1} = \text{diag}\left[\frac{1}{\sqrt{w_1}}, \frac{1}{\sqrt{w_2}} \dots \frac{1}{\sqrt{w_n}}\right]$$

$$P^{-1}\underset{\sim}{Y} = \begin{bmatrix} \frac{Y_1}{\sqrt{w_1}} \\ \frac{Y_2}{\sqrt{w_2}} \\ \vdots \\ \frac{Y_n}{\sqrt{w_n}} \end{bmatrix} \qquad P^{-1}X = \begin{bmatrix} \frac{1}{\sqrt{w_1}} & \frac{x_1}{\sqrt{w_1}} \\ \frac{1}{\sqrt{w_2}} & \frac{x_2}{\sqrt{w_2}} \\ & \\ \frac{1}{\sqrt{w_n}} & \frac{x_n}{\sqrt{w_n}} \end{bmatrix}$$

Modelo Transformado : $\dfrac{Y_i}{\sqrt{w_i}} = \dfrac{\beta_0}{\sqrt{w_i}} + \beta_1 \dfrac{x_i}{\sqrt{w_i}} + \overset{\eta_i}{\dfrac{\epsilon_i}{\sqrt{w_i}}}$

sem intercepto

$$SST_{nc} = \sum_{i=1}^{m} \frac{r_i^2}{w_i}$$

Usar a estatística W normalmente.

Se usar ANOVA, trabalhar com SST_{nc} e SSR_{nc} (ver ANOVA para modelo sem intercepto)

Exemplo 3

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

$$* \begin{cases} \epsilon_i = \rho\, \epsilon_{i-1} + v_i \\ \text{com} \quad E(v_i) = 0 \qquad Var(v_i) = \sigma_v^2 \quad \forall i \\ \qquad E(v_i\, v_j) = 0 \quad i \neq j \end{cases}$$

* é um modelo de séries temporais, denominado auto-regressivo de ordem 1 [AR(1)].

Verifica-se neste caso que

$r(\epsilon_i, \epsilon_{i-1}) = \rho$ e, admitindo $|\rho| \neq 1$,

$$E(\epsilon_i) = 0$$

$$Var(\epsilon_i) = \frac{\sigma_v^2}{1-\rho^2} = \sigma_\epsilon^2 \quad \text{e} \quad E(\epsilon_i\, \epsilon_{i-h}) = \rho^h \frac{\sigma_v^2}{1-\rho^2} = \rho^h \sigma_\epsilon^2$$

Portanto

σ^2_ϵ

$$\text{Var}(\underset{\sim}{\epsilon}) = \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{n-1} \\ & 1 & \rho & \cdots & \rho^{n-2} \\ & & 1 & & \rho^{n-3} \\ & & & & \vdots \\ & & & & 1 \end{bmatrix} \frac{\sigma^2_v}{1-\rho^2}$$

$$V = \begin{bmatrix} 1 & \rho & \rho^2 \cdots & \rho^{n-1} \\ & 1 & \rho & \rho^{n-2} \\ & & 1 & \rho^{n-3} \\ & & & \vdots \\ & & & 1 \end{bmatrix}$$

Verifica-se que

$$V^{-1} = \begin{bmatrix} 1 & -\rho & 0 & \cdots & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & & 0 & 0 \\ 0 & -\rho & 1+\rho^2 & & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & & 1+\rho^2 & -\rho \\ 0 & 0 & 0 & & -\rho & 1 \end{bmatrix} \frac{1}{(1-\rho^2)}$$

Se ρ é conhecido, V é conhecida e $\hat{\underset{\sim}{\beta}}_G = (X'V^{-1}X)^{-1}X'V^{-1}\underset{\sim}{y}$

com $X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \\ 1 & x_n \end{bmatrix}$

Observação

O melhor previsor não viciado de Y_h (não presente na amostra) não é $\underset{\sim}{X}_h \hat{\underset{\sim}{\beta}}_G$ como no caso em que $Var(\underset{\sim}{\epsilon}) = \sigma^2 I$.

Se Y_h é correlacionada com $\underset{\sim}{Y}' = (Y_1 \; Y_2 \ldots Y_n)$, verifica-se que o melhor previsor não viciado de Y_h (possui erro quadrático médio mínimo na classe de todos os previsores não viciados de Y_h) é

$$\hat{Y}_h = \underset{\sim}{X}_h \hat{\underset{\sim}{\beta}}_G + W \Sigma^{-1} (\underset{\sim}{Y} - X \hat{\underset{\sim}{\beta}}_G),$$

onde $W = Cov(Y_h, \underset{\sim}{Y}) \quad 1 \times n$

$$\Sigma = Var(\underset{\sim}{\epsilon}) = \sigma^2_\epsilon V$$

Exemplo 3 (continuação)

Prever Y_{n+1} admitindo que o modelo do exemplo 3 é válido para Y_{n+1}.

$h = n+1$

$E(Y_{n+1}) = \beta_0 + \beta_1 x_{n+1}$

$Var(Y_{n+1}) = \sigma^2_\epsilon$

$Cov(\epsilon_{n+1}, \epsilon_i) = Cov(Y_{n+1}, Y_i) = \rho^{n+1-i} \sigma^2_\epsilon \quad i = 1, 2, \ldots, n$

$$\mathrm{Cov}(Y_{n+1}, \underset{\sim}{Y}) = [\rho^n \ \ \rho^{n-1} \ \ \ldots \ \ \rho^2 \ \ \rho]\, \sigma_\epsilon^2 = W$$

$$W\Sigma^{-1} = \sigma_\epsilon^2 \, [\rho^n \ \ \rho^{n-1} \ \ldots \ \rho^2 \ \ \rho] \, \frac{V^{-1}}{\sigma_\epsilon^2} =$$

$$= [\rho^n \ \ \rho^{n-1} \ \ldots \ \rho^2 \ \ \rho] \begin{bmatrix} 1 & -\rho & 0 & \ldots & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & & 0 & 0 \\ 0 & -\rho & 1+\rho^2 & & 0 & 0 \\ & & & & & \\ 0 & 0 & 0 & & 1+\rho^2 & -\rho \\ 0 & 0 & 0 & & -\rho & 1 \end{bmatrix} \frac{1}{1-\rho^2}$$

$$= [0 \ \ 0 \ \ \ldots \ \ 0 \ \ \rho(1-\rho^2)] \frac{1}{1-\rho^2} = [0 \ \ 0 \ \ \ldots \ 0 \ \ \rho]$$

$$\hat{Y}_{n+1} = \underbrace{\hat\beta_0 + \hat\beta_1 x_{n+1}}_{\underset{\sim}{X}_n \hat{\underset{\sim}{\beta}}_G} + \underbrace{[0 \ \ 0 \ \ldots \ 0 \ \ \rho]}_{W\Sigma^{-1}} \underbrace{\begin{bmatrix} Y_1 - \underset{\sim}{X}_1 \hat{\underset{\sim}{\beta}}_G \\ Y_2 - \underset{\sim}{X}_2 \hat{\underset{\sim}{\beta}}_G \\ \vdots \\ Y_n - \underset{\sim}{X}_n \hat{\underset{\sim}{\beta}}_G \end{bmatrix}}_{\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}}_G}$$

$$= \hat\beta_0 + \hat\beta_1 x_{n+1} + \rho\,(Y_n - \hat\beta_0 - \hat\beta_1 x_n)$$

Obs:

- $\hat{\underset{\sim}{\beta}}_G = (X'V^{-1}X)^{-1}X'V^{-1}\underset{\sim}{Y}$ minimiza a soma de quadrados do resíduo ponderada

$$S = (\underset{\sim}{Y} - X\underset{\sim}{\beta})' V^{-1} (\underset{\sim}{Y} - X\underset{\sim}{\beta})$$

- se $V = I$, $\hat{\underset{\sim}{\beta}}_G = (X'X)^{-1}X'\underset{\sim}{Y} = \hat{\underset{\sim}{\beta}}$
- se $V \neq I$, $E(\hat{\underset{\sim}{\beta}}) = (X'X)^{-1}X'E(\underset{\sim}{Y}) = (X'X)^{-1}X'X\underset{\sim}{\beta} = \underset{\sim}{\beta}$

  $\hat{\underset{\sim}{\beta}}$ é não viciado mas não é ótimo pois

  as estatísticas suficientes e completas para o modelo

  $\underset{\sim}{Y} = X\underset{\sim}{\beta} + \underset{\sim}{\epsilon}$ $\quad \underset{\sim}{\epsilon} \sim N(\underset{\sim}{0}, \sigma^2 V)$

  são $\hat{\underset{\sim}{\beta}}_G$ e $\hat{\sigma}^2_G$.