Derivada da Função Exponencial

Se $f(x) = a^x$, $(a > 0$ e $a \neq 1)$ então $f'(x) = a^x \ln a$.

Hipótese: $f(x) = a^x$, $a > 0$ e $a \neq 1$

Tese: $f'(x) = a^x \ln a$

$$f'(x) = \lim_{\Delta x \to a} \frac{a^{x+\Delta x} - a^x}{\Delta x}$$

$$= \lim_{\Delta x \to a} \frac{a^x . a^{\Delta x} - a^x}{\Delta x}$$

$$= \lim_{\Delta x \to a} \frac{a^x \left(a^{\Delta x} - 1\right)}{\Delta x} \quad \ln a$$

$$\boxed{\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a}$$

$$= a^x \ln a$$

Se $a = e$ $f(x) = e^x$,

$$f'(x) = e^x$$

$\ln e = \log_e e$

$y \Leftarrow\Rightarrow e^y = e^1$

$y = 1$

$$\boxed{\begin{array}{c} v = e^u \\ \dfrac{dv}{du} = e^u . du \end{array}}$$

Exemplos:

a) $\underbrace{f(x)}_{v} = 2e^{3x^2 + 6x}$ $\Rightarrow$ $u = 3x^2 + 6x$

$v = 2 . e^u$ $\qquad \dfrac{du}{dx} = 6x + 6$

$f'(x) = 2e^{3x^2+6x} . (6x + 6) \rightarrow 6(x+1)$

$f'(x) = (12x + 12) . e^{3x^2 + 6x}$

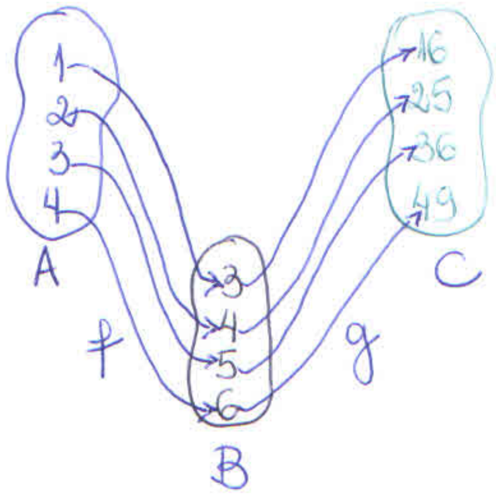
b) $y = 3e^x + \dfrac{4}{\sqrt[3]{x}}$ $\Longrightarrow$ $y = 3e^x + 4x^{-1/3}$

$y' = 3e^x . (1) - \frac{1}{3} 4 x^{-4/3}$

$y' = 3e^x - \dfrac{4}{3\sqrt[3]{x^4}}$

Regra da Cadeia

Função Composta



$f: A \to B$

$x \longmapsto y = \underbrace{x+2}$

$g: B \to C$

$x \longmapsto y = (x+1)^2$

$g(f(x)) = (x+2+1)^2$

$g(f(x)) = (x+3)^2$

$$\boxed{h = g \circ f}$$

Então $h(x) = g(f(x))$

$h: A \to C$

$x \longmapsto y = (x+3)^2$

Então para a função a derivada será:

$h(x) = g(f(x))$

$$\boxed{h'(x) = g'(f(x)) \cdot f'(x)}$$

Exemplos

a) $y = \operatorname{sen} 2x$

$f(x) = 2x$

$g(x) = \operatorname{sen} x$

$d(\operatorname{sen} u) = \cos u \cdot du$

$\Rightarrow \underbrace{g(f(x))}_{g'(f(x)) \cdot f'(x)} = \operatorname{sen} 2x$

$\underbrace{g'(f(x)) \cdot f'(x)}_{y' = \cos 2x \cdot 2}$

b) $y = (x^2 + 7x + 1)^7$

$f(x) = x^2 + 7x + 1$

$f'(x) = 2x + 7$

$g(x) = x^7$

$g'(x) = 7x^6$

$y' = 7(x^2 + 7x + 1)^6 . (2x + 7)$

$y' = (14x + 49) . (x^2 + 7x + 1)^6$

c) $y = \text{sen}^6 (3x^2 + 1)$

$d(\text{sen}\, u) = \cos u \, du$

$f(x) = 3x^2 + 1 \longrightarrow f'(x) = 6x$

$g(x) = \text{sen}\, x \longrightarrow g'(x) = \cos x$

$h(x) = x^6 \longrightarrow h'(x) = 6x^5$

$h(g(f(x)))$

$\Longrightarrow h'(g(f(x))) . g'(f(x)) . f'(x)$

$y' = 6 \,\text{sen}^5 (3x^2 + 1) . \cos(3x^2 + 1) . 6x$

$y' = 36x \,\text{sen}^5 (3x^2 + 1) . \cos(3x^2 + 1)$

Se $y = g(u)$, $u = f(x)$ e as derivadas $\frac{dy}{du}$ e $\frac{du}{dx}$ existem, então a função composta $y = g[f(x)]$ terá derivada que é dada por:

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$ ou seja,

$$y'(x) = g'(u) . f'(x)$$

Demonstração

Hipótese $y = g(u)$, $u = f(x)$ $\quad \frac{dy}{du}, \frac{du}{dx}$ existem

Tese:

$$y = g(f(x))$$

$$y' = \frac{dy}{dw} \cdot \frac{dw}{dx}$$

$$y' = g'(x) \cdot f'(x)$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{g[f(x+\Delta x)] - g[f(x)]}{\Delta x}$$

Pela hipótese $w = f(x)$ $\quad y = g(w)$

$\Delta w = f(x+\Delta x) - f(x)$ $\qquad$ Se $\Delta x \to 0$ então $\Delta w \to 0$

$\Delta w = f(x+\Delta x) - w$

$f(x+\Delta x) = \Delta w + w$

$$\frac{dy}{dx} = \lim_{\Delta w \to 0} \frac{g[\Delta w + w] - g[w]}{\Delta x} \cdot \frac{\Delta w}{\Delta w}$$

$$= \lim_{\Delta w \to 0} \frac{g[\Delta w + w] - g[w]}{\Delta w} \cdot \frac{\Delta w}{\Delta x}$$

$$= \lim_{\Delta w \to 0} \frac{g[\Delta w + w] - g[w]}{\Delta w} \cdot \frac{f(x+\Delta x) - \overbrace{f(x)}^{w}}{\Delta x}$$

Logo,

$$\frac{dy}{dx} = \underbrace{\lim_{\Delta w \to 0} \frac{g[x+\Delta x] - g[x]}{\Delta w}}_{g'(x)} \cdot \underbrace{\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}}_{f'(x)}$$

$$\frac{dy}{dx} = g'(x) \cdot f'(x)$$

d) $g(x) = \dfrac{x^3}{\sqrt[3]{3x^2-1}}$ u v $\quad u = x^3 \quad v = (3x^2-1)^{-1/3}$

$\dfrac{du}{dx} = 3x^2 \quad \dfrac{dv}{dx} = -\dfrac{1}{3}(3x^2-1) \cdot 6x$

$g(x) = x^3 \cdot (3x^2-1)^{-1/3}$

$g'(x) = u \cdot dv + v \cdot du$

$g'(x) = x^3 \cdot \left[-\dfrac{1}{\cancel{3}} (3x^2-1)^{-4/3} \cdot \cancel{6}^{2} x \right] + (3x^2-1)^{-1/3} \cdot 3x^2$

$g'(x) = -2x^4 (3x^2-1)^{-4/3} + 3x^2 (3x^2-1)^{-1/3}$

$g'(x) = x^2 (3x^2-1)^{-4/3} \cdot \left[-2x^2 + 3(3x^2-1) \right]$

$g'(x) = x^2 (3x^2-1)^{-4/3} \cdot (7x^2-3)$

$g'(x) = \dfrac{7x^4 - 3x^2}{(3x^2-1)^{4/3}}$