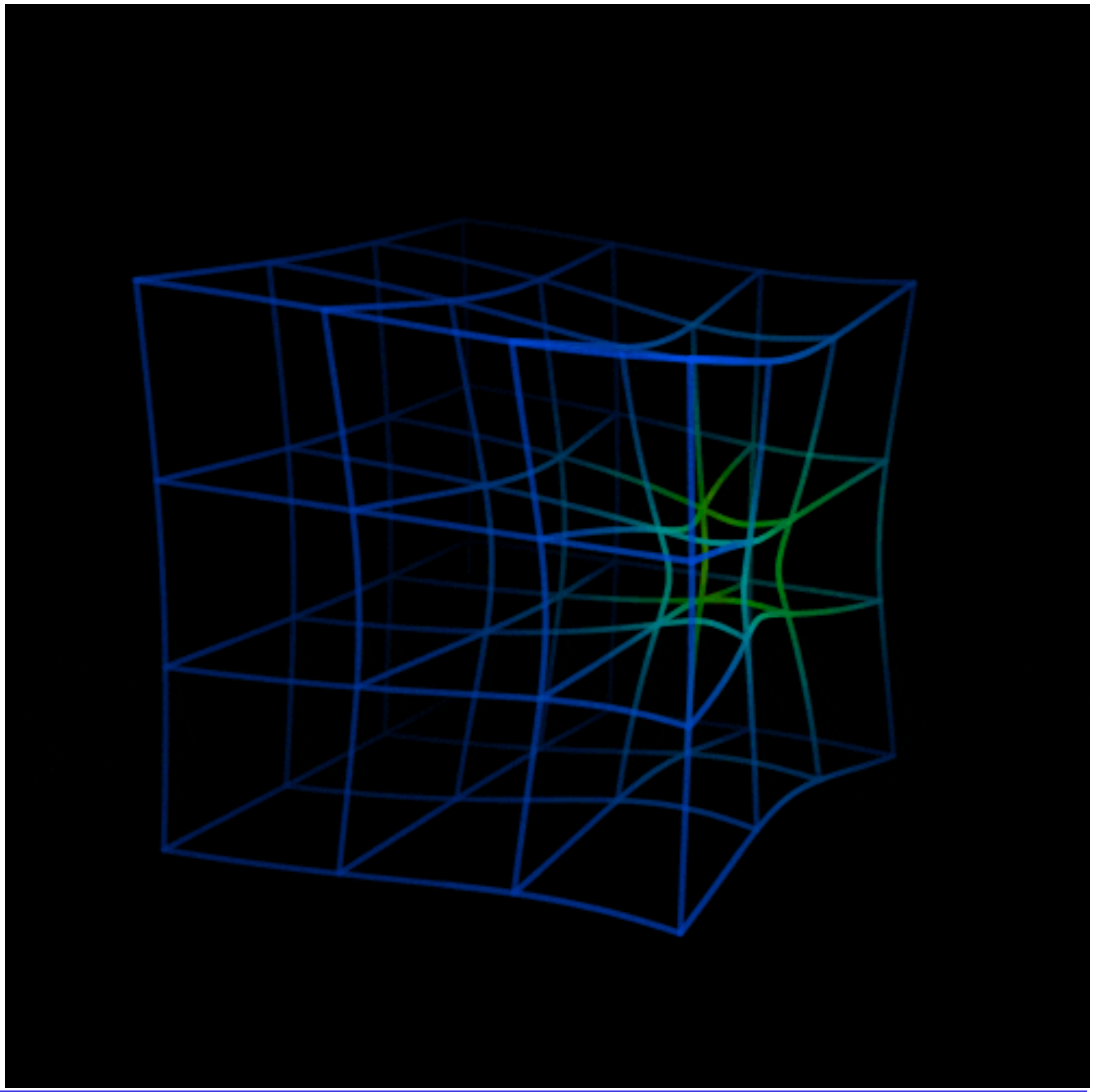
Relativistic Electrodynamics

⚡ **Relativity + Electromagnetic fields**
⚡ **The Faraday tensor**
⚡ **Maxwell Equations: covariant notation**



Maxwell equations

- OK, now let's try to work our way back to Electrodynamics, but using our acquired knowledge about relativity, and how to work with its objects (vectors, tensors etc.)

- In vacuum, the Maxwell equations are (in ***Gauss units*** 😬):

$$\vec{\nabla} \times \vec{B} - \frac{1}{c}\partial_t \vec{E} = \frac{4\pi}{c}\vec{J}_q$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi \rho_q$$

$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\partial_t \vec{B} = 0$$

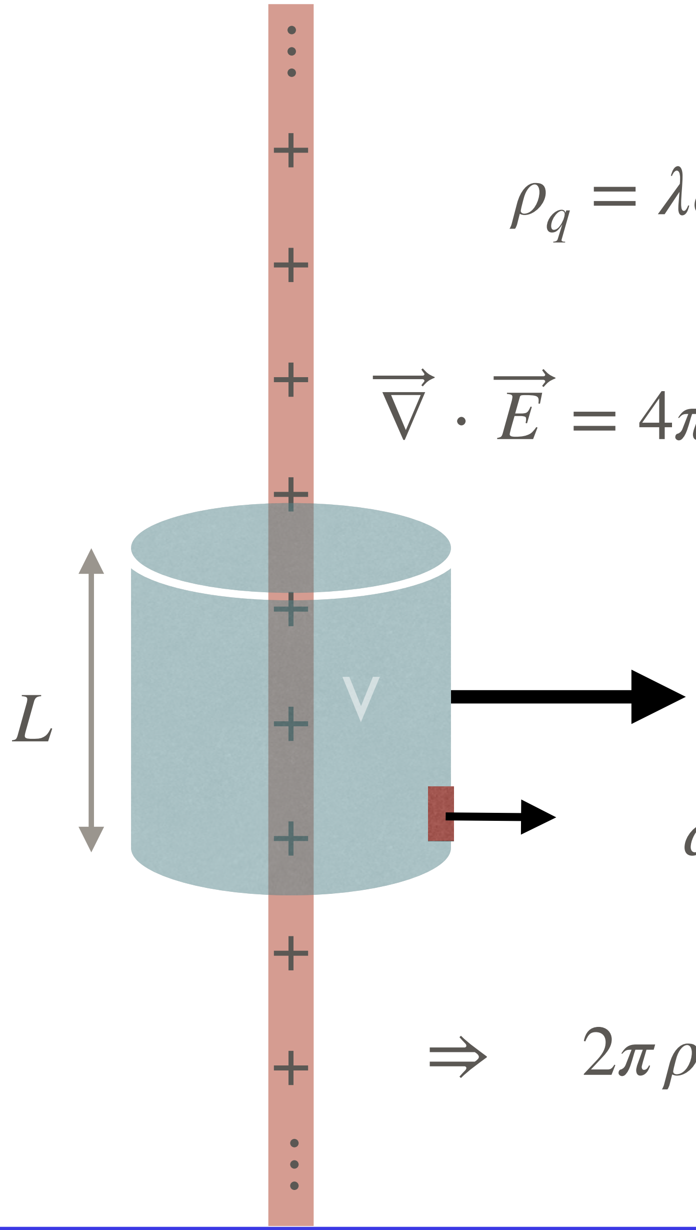
$$\vec{\nabla} \cdot \vec{B} = 0$$

Maxwell equations

- Let's consider the problem of an infinite wire with a linear charge density, and its electric field:

S: observer

at rest



$$\rho_q = \lambda \delta(x)\delta(y) \qquad \lambda = \frac{dq}{dz}$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho_q \quad \Rightarrow \quad \int_{S(V)} d\vec{S} \cdot \vec{E} = 4\pi q_V$$

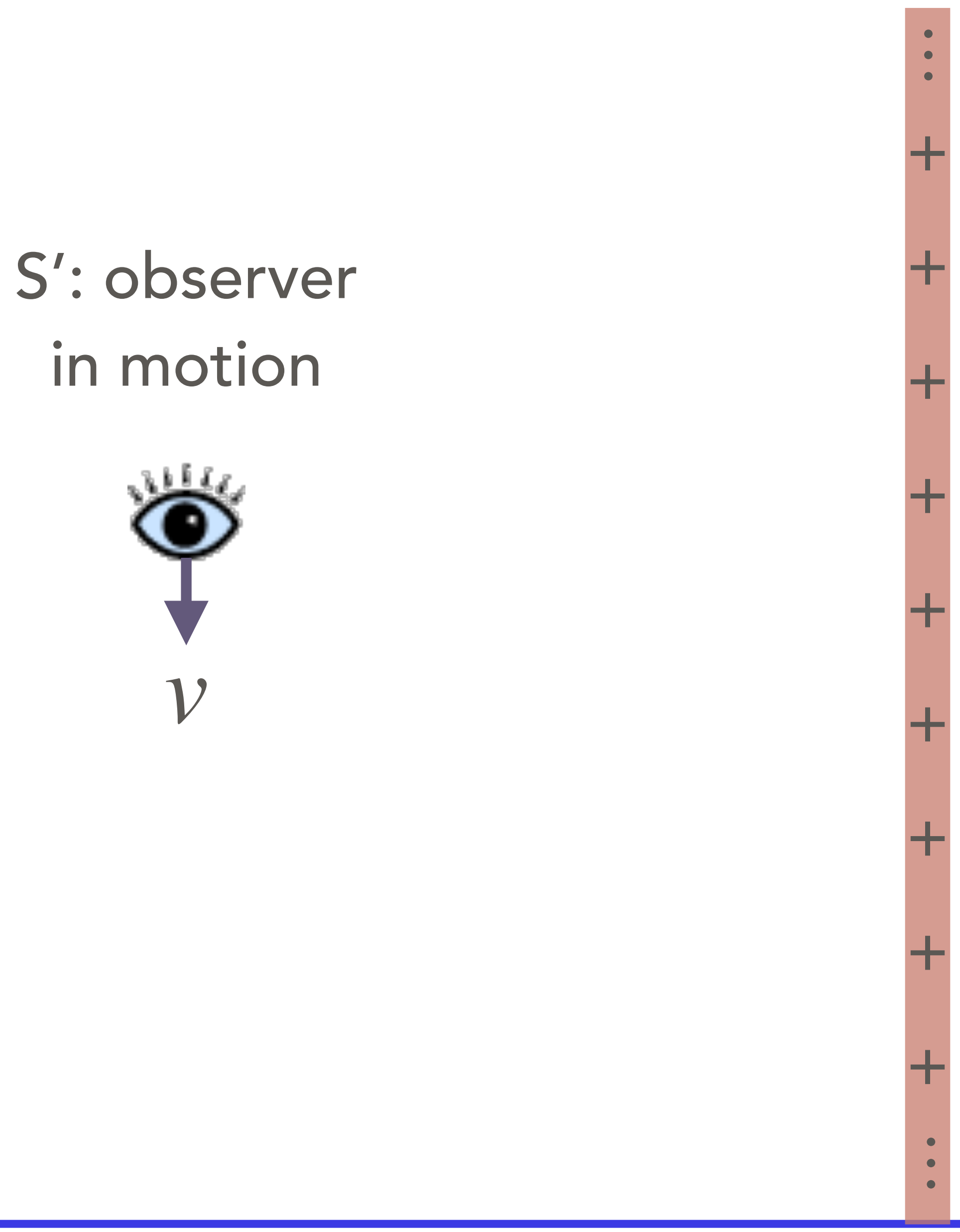
$$\vec{E} = E_\rho \, \hat{\rho}$$

$$d\vec{S} = 2\pi\rho \, dz \, \hat{\rho}$$

$$\Rightarrow \quad 2\pi \rho L E_\rho = 4\pi\lambda L \qquad \boxed{\Rightarrow \vec{E} = \frac{2\lambda}{\rho}\hat{\rho}}$$

Maxwell equations

- This same wire, as seen by an observer in motion parallel to the wire (S′):

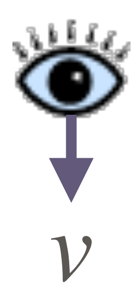


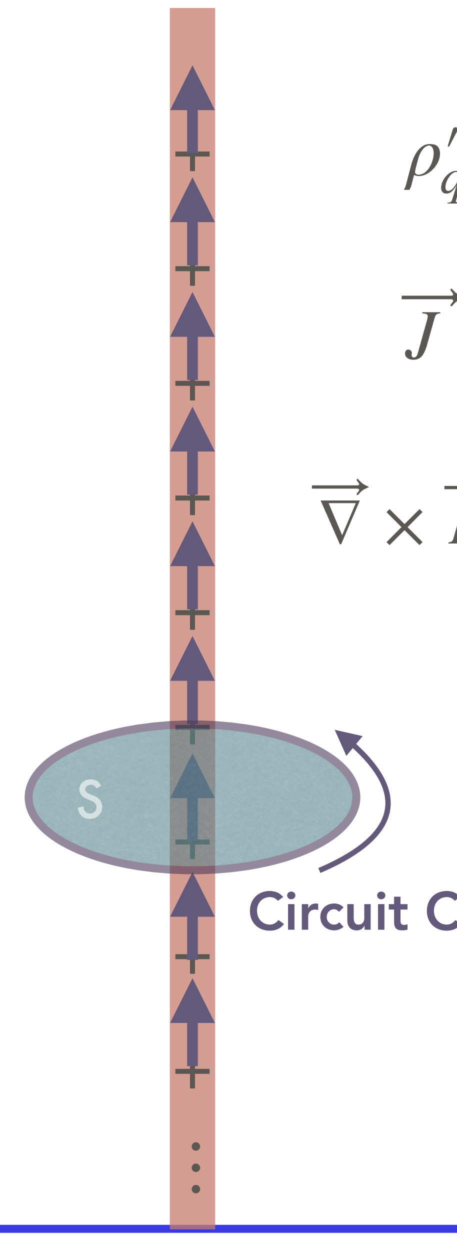
* Note that the directions x, y , or ρ, φ , are the same for both observers

Maxwell equations

- This same wire, as seen by an observer in motion parallel to the wire (S′):

S′: observer
in motion





$$\rho'_q = \lambda' \delta(x)\delta(y) \qquad \lambda' = \frac{dq}{dz'}$$

$$\vec{J}\,' = \lambda' v\, \delta(x)\delta(y)\, \hat{z} \qquad I' = \lambda' v$$

$$\vec{\nabla} \times \vec{B}\,' = \frac{4\pi}{c} \vec{J}\,' \quad \Rightarrow \quad \oint_{C(S)} d\vec{l} \cdot \vec{B}\,' = \frac{4\pi}{c} I'$$

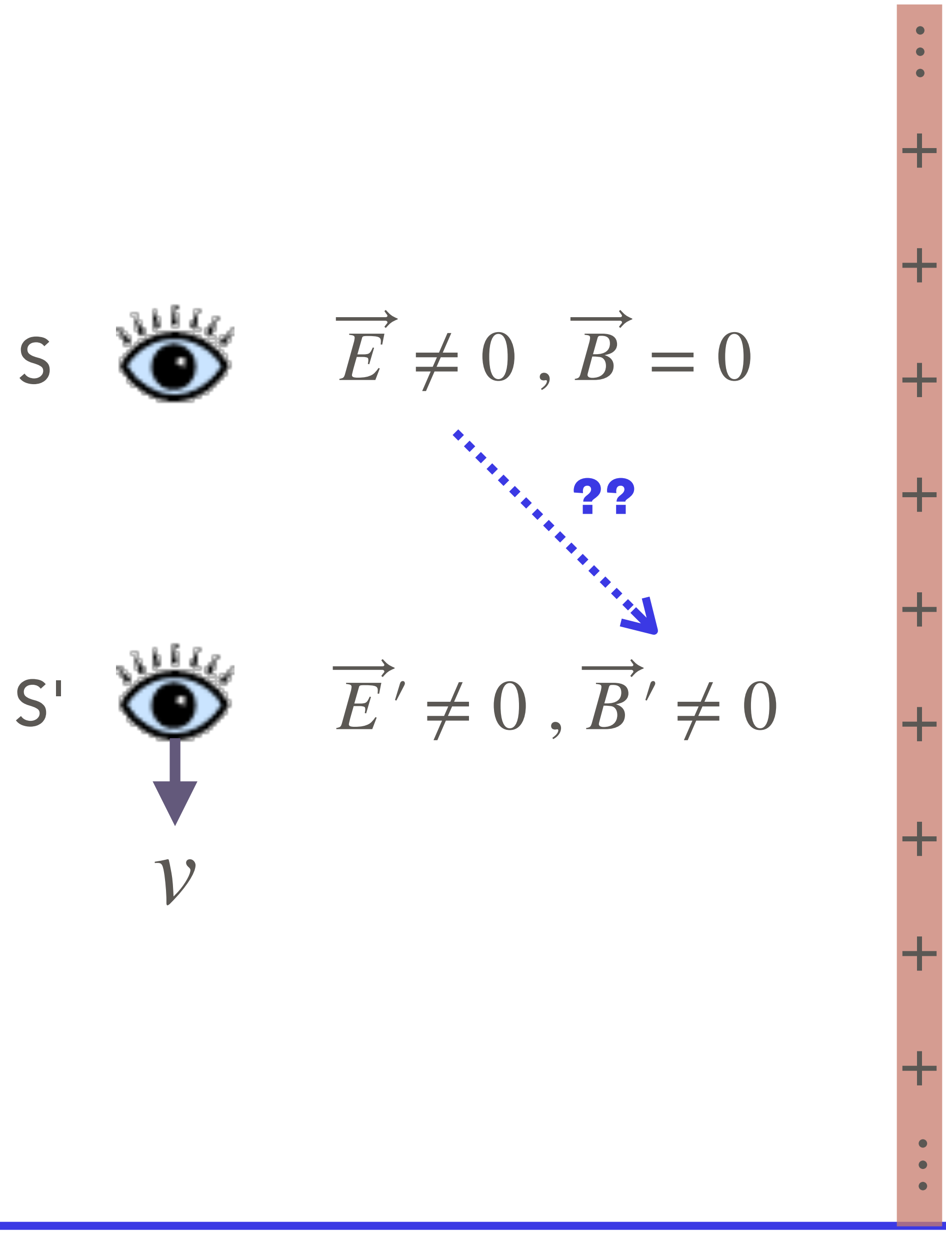
$$\boxed{\Rightarrow \vec{B}\,' = \frac{2I'}{c\,\rho}\,\hat{\varphi}}$$

$$\boxed{\Rightarrow \vec{E}\,' = \frac{2\lambda'}{\rho}\,\hat{\rho}}$$

* Note that the directions x, y , or ρ, φ , are the same for both observers

Maxwell equations

- But the two frames are just related by a Lorentz transformation — so, the fields should also be related by something like a coordinate transformation…



The fields $\vec{E}$ and $\vec{B}$ are 3D vectors, and so are $\vec{E}\,'$ and $\vec{B}\,'$.

But they are ***different fields!***
How can we **connect the two** in the **same transformation**??

Back to Electrodynamics

- In order to do so, let's go back to the Maxwell equations,

$$\boxed{\begin{aligned} &\vec{\nabla} \times \vec{B} - \frac{1}{c}\partial_t \vec{E} = \frac{4\pi}{c}\vec{J}_q \\ &\vec{\nabla} \times \vec{E} + \frac{1}{c}\partial_t \vec{B} = 0 \end{aligned}} \quad \begin{aligned} &, \quad \vec{\nabla} \cdot \vec{E} = 4\pi\rho_q \\ &, \quad \vec{\nabla} \cdot \vec{B} = 0 \end{aligned}$$

- We should first note that some of them contain **combinations** of **spacial** and **time derivatives**. That's an important clue.

- Moreover, notice that taking $\vec{\nabla} \cdot (\ldots)$ in *Ampère's Law*, and combining with $\partial_t(\ldots)$ in *Gauss's Law*, we get:

$$\vec{\nabla} \cdot \left[\cancel{\vec{\nabla} \times \vec{B}} - \cancel{\frac{1}{c}\partial_t \vec{E}} = \frac{4\pi}{c}\vec{J}_q \right] = -\frac{1}{c}\partial_t \left[\cancel{\vec{\nabla} \cdot \vec{E}} = 4\pi\rho_q \right]$$

$$\partial_\mu J^\mu = 0$$

Back to Electrodynamics

- To go one step further, let's recall that both the electric and the magnetic fields can be expressed in terms of potentials (here in Gauss units):

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \quad , \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

- So, the potentials already mix the two fields.

- Therefore, the two 3D fields $\vec{E}$ e $\vec{B}$ actually reduce to one 1D potential (ϕ) and a 3D potential ($\vec{A}$)…

- Well, then, it is like the potentials were part of a single object, like… like… a 4-vector!

$$\{\phi\,, \vec{A}\} \to A^{\mu\,?}_{\mu\,?}$$

Maxwell Equations and potentials

- Let's then re-write the Maxwell Equations in terms of the two potentials, ϕ and $\vec{A}$:

$$\vec{\nabla} \times \vec{B} - \frac{1}{c}\partial_t \vec{E} = \frac{4\pi}{c}\vec{J}_q \Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) - \frac{1}{c}\partial_t \left(-\vec{\nabla}\phi - \frac{1}{c}\partial_t \vec{A} \right) = \frac{4\pi}{c}\vec{J}$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \Rightarrow \vec{\nabla} \cdot \left(-\vec{\nabla}\phi - \frac{1}{c}\partial_t \vec{A} \right) = \rho_q$$

$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\partial_t \vec{B} = 0 \Rightarrow \vec{\nabla} \times \left(\cancel{-\vec{\nabla}\phi} - \frac{1}{c}\cancel{\partial_t \vec{A}} \right) + \frac{1}{c}\partial_t \cancel{(\vec{\nabla} \times \vec{A})} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{\nabla} \cdot \cancel{(\vec{\nabla} \times \vec{A})} = 0$$

Maxwell Equations and potentials

- Simplifying this a bit we get:

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A} + \vec{\nabla}\frac{1}{c}\partial_t\phi + \frac{1}{c^2}\partial_t^2\vec{A} = \frac{4\pi}{c}\vec{J}$$

$$-\vec{\nabla}^2\phi - \frac{1}{c}\partial_t\vec{\nabla}\cdot\vec{A} = 4\pi\rho$$

$$0 = 0$$

$$0 = 0$$

- OK, now let's try to organize all this, remembering that we have $\vec{\nabla} \to \partial_i$, and $(1/c)\,\partial_t \to \partial_0$.

- Keep in mind also that the 4-vector $A \leftrightarrow \{\phi, \vec{A}\}$

Maxwell Equations and potentials

- As we saw earlier, this can be further simplified:

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A} + \vec{\nabla}\frac{1}{c}\partial_t\phi + \frac{1}{c^2}\partial_t^2\vec{A} = \frac{4\pi}{c}\vec{J}$$

$$-\vec{\nabla}^2\phi - \frac{1}{c}\partial_t\vec{\nabla}\cdot\vec{A} = 4\pi\rho$$

- Which gives us:

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) + \vec{\nabla}\partial_0\phi - \partial_\mu\partial^\mu\vec{A} = \frac{4\pi}{c}\vec{J}$$

$$\vec{\nabla}\left(-\vec{\nabla}\phi - \partial_0\vec{A}\right) = 4\pi\rho$$

Maxwell Equations and potentials

- Therefore, if we identify $\phi \to A^0$, we can write:

$$\partial^i(\partial_j A^j) + \partial^i\partial_0\phi - \partial_\mu\partial^\mu A^i = \frac{4\pi}{c}J^i \quad \Rightarrow \quad \partial^i(\partial_\mu A^\mu) - \partial_\mu\partial^\mu A^i = \frac{4\pi}{c}J^i$$

$$\partial_i\left(\partial^0 A^i - \partial^i\phi\right) = 4\pi\rho \quad \Rightarrow \quad \partial_i\left(\partial^0 A^i - \partial^i A^0\right) = 4\pi\rho$$

- Finally, we are able to summarize these expressions to:

$$\left.\begin{aligned} &\partial_\mu\left(\partial^i A^\mu - \partial^\mu A^i\right) = \frac{4\pi}{c}J^i \\ &\partial_i\left(\partial^0 A^i - \partial^i A^0\right) = \frac{4\pi}{c}\rho c \quad J^0 \end{aligned}\right\} \partial_\mu\left(\partial^\nu A^\mu - \partial^\mu A^\nu\right) = \frac{4\pi}{c}J^\nu$$

Maxwell Equations and potentials

- Summarizing: using covariant (relativistic) notation, **Maxwell's equations** are:

$$\partial_\mu \left(\partial^\nu A^\mu - \partial^\mu A^\nu \right) = \frac{4\pi}{c} J^\nu$$

(note the order of the indices!)

- Both the electric and magnetic fields are included in the ***anti-symmetric object***, $\partial^\nu A^\mu - \partial^\mu A^\nu$. We call it the ***Electromagnetic tensor*** (or ***Faraday tensor***):

$$\boxed{F^{\nu\mu} \equiv \partial^\nu A^\mu - \partial^\mu A^\nu} \quad , \quad F^{\mu\nu} = - F^{\nu\mu}$$

(anti-symmetry)

- Note that:

$$\partial^\mu = \eta^{\mu\nu} \partial_\nu = \{ -\partial_0 \, , \vec{\nabla} \}$$

$$\boxed{A^\mu = \{ \phi \, , \vec{A} \}}$$

The Electromagnetic (Faraday) tensor

- In our derivation above we wrote the Faraday tensor as:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \, ,$$

- However, it is more common (and more useful) to write it as:

$$F_{\alpha\beta} = \eta_{\alpha\mu}\eta_{\beta\nu} F^{\mu\nu} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \, ,$$

where the 4-potential is:

$$A_\mu = \eta_{\mu\nu} A^\nu = \{-\phi \, , \vec{A}\} \, .$$

The Electromagnetic (Faraday) tensor

- OK, so now let's derive how the electric and magnetic fields appear inside the Faraday tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad .$$

- Notice that, because of the anti-symmetry, $F_{\mu\nu} = -F_{\nu\mu}$, hence:

$$F_{00} = F_{11} = F_{22} = F_{33} = 0 \quad .$$

- $F_{i0} = \partial_i A_0 - \partial_0 A_i \to \vec{\nabla}(-\phi) - \partial_0 \vec{A} \stackrel{!}{=} \vec{E} \qquad E^i = F_{i0} = -F_{0i}$

- $F_{ij} = \partial_i A_j - \partial_j A_i \leftrightarrow \vec{\nabla} \times \vec{A} = \vec{B}$

The Electromagnetic (Faraday) tensor

- So, finally we get that:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad , \quad \partial_\mu F^{\nu\mu} = \frac{4\pi}{c} J^\nu$$

$$F_{\mu\nu} = \begin{array}{cc} & \nu \longrightarrow \\ \mu \downarrow & \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ +E_x & 0 & +B_z & -B_y \\ +E_y & -B_z & 0 & +B_x \\ +E_z & +B_y & -B_x & 0 \end{pmatrix} \end{array}$$

$$\boxed{F_{i0} = +E^i} \qquad \boxed{F_{12} = \partial_1 A_2 - \partial_2 A_1 = +B_z}$$

$$\boxed{F_{0i} = -E^i} \qquad \boxed{F_{13} = \partial_1 A_3 - \partial_3 A_1 = -B_y}$$

The Electromagnetic (Faraday) tensor

- Por completeness we can also write the Faraday tensor with both indices raised:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}$$

$$F^{\mu\nu} = \begin{array}{cc} & \nu \\ \mu & \begin{pmatrix} 0 & +E_x & +E_y & +E_z \\ -E_x & 0 & +B_z & -B_y \\ -E_y & -B_z & 0 & +B_x \\ -E_z & +B_y & -B_x & 0 \end{pmatrix} \end{array}$$

$$\boxed{F^{i0} = -E^i}$$

$$\boxed{F^{0i} = +E^i}$$

Maxwell Equations

- In short, the ***Maxwell equations with sources*** can be written as:

$$\partial_\mu F^{\nu\mu} = \frac{4\pi}{c} J^\nu$$ (note the order of the indices in $F^{\nu\,\mu}$!)

- The **Maxwell equations without sources,** on the other hand are written as:

HODGE DUAL OF $F_{\mu\nu}$

$$\partial_\alpha F_{\beta\gamma} + \partial_\gamma F_{\alpha\beta} + \partial_\beta F_{\gamma\alpha} = 0 \qquad \text{or} \qquad \epsilon^{\alpha\beta\gamma\delta}\partial_\delta F_{\alpha\beta} = \partial_\delta \left(\epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} \right) = \partial_\delta \, F^{*\gamma\delta} = 0$$

(***Jacobi Identity — Exercise!***)

- Notice also that, due to the ***anti-symmetry*** of $F^{\nu\mu}$, we have:

$$0 = \partial_\nu \partial_\mu F^{\nu\mu} = \partial_\nu \left[\frac{4\pi}{c} J^\nu \right]$$ — but from the continuity equation we know that $\partial_\nu J^\nu = 0$!

Therefore, as before, the continuity equation is a consistency condition (or integrability condition) for Maxwell's equations.

Maxwell Equations

- Let's now make use of the fact that the Faraday tensor is a Minkowski tensor. This means, among other things, that we can take the norm of that tensor:

$$F_{\mu\nu} F^{\mu\nu} \; = \; \eta_{\mu\alpha} \eta_{\nu\beta} F^{\alpha\beta} F^{\mu\nu} \; = \; 2 \left(\vec{B}^2 - \vec{E}^2 \right)$$

- This norm is **invariant** — it has the same value in any reference frame

- Under a Lorentz transformation, that tensor behaves in the following way:

$$F^{\mu\nu} \to F'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} F^{\alpha\beta}$$

$$F_{\mu\nu} \to F'_{\mu\nu} = \Lambda_{\mu}{}^{\alpha} \Lambda_{\nu}{}^{\beta} F_{\alpha\beta}$$

The Electromagnetic field

- Now let's back to our original problem, of the infinite wire

S 

$$\vec{E} = \frac{2\lambda}{\rho}\hat{\rho}$$

$$\vec{B} = 0$$

S' 

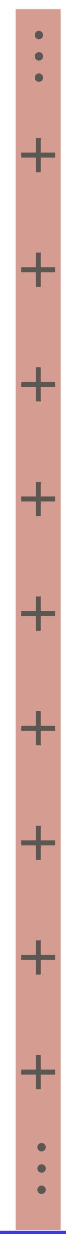
v

$$\vec{v} = -v\hat{z}$$

$$\beta \to -v/c$$

$$\vec{E}' = \frac{2\lambda'}{\rho}\hat{\rho}$$

$$\vec{B}' = \frac{2I'}{c\,\rho}\hat{\varphi}$$



$$F^{\mu\nu} = \begin{pmatrix} 0 & +E_x & +E_y & +E_z \\ -E_x & 0 & +B_z & -B_y \\ -E_y & -B_z & 0 & +B_x \\ -E_z & +B_y & -B_x & 0 \end{pmatrix}$$

$$F^{\mu\nu} \to F'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}F^{\alpha\beta}$$

$$\Lambda^{\mu}{}_{\alpha} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix}$$

The Electromagnetic field

- Now let's back to our original problem, of the infinite wire

S 

$$\vec{E} = \frac{2\lambda}{\rho}\hat{\rho}$$

$$\vec{B} = 0$$

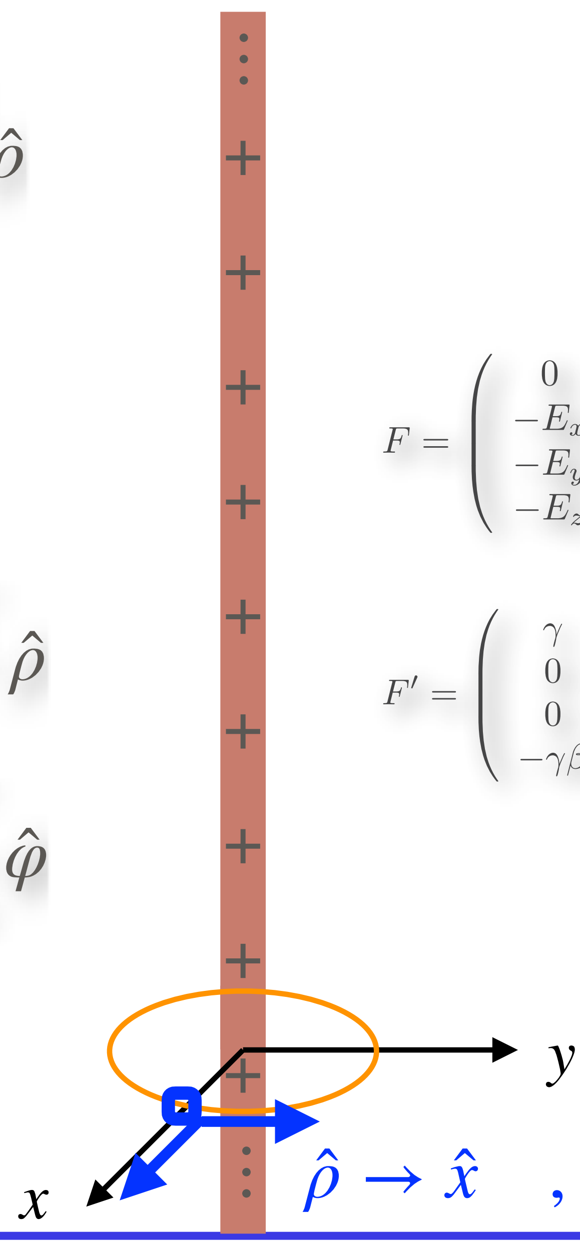
S' 

v

$$\vec{E}' = \frac{2\lambda'}{\rho}\hat{\rho}$$

$$\vec{B}' = \frac{2I'}{c\,\rho}\hat{\varphi}$$

CHECK!!



$\hat{\rho} \to \hat{x}$, $\hat{\varphi} \to \hat{y}$

$$F^{\mu\nu} \to F'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}F^{\alpha\beta}$$

$$F \to F' = \Lambda\, F\, \Lambda^{tr}$$

$$F = \begin{pmatrix} 0 & +E_x & +E_y & +E_z \\ -E_x & 0 & +B_z & -B_y \\ -E_y & -B_z & 0 & +B_x \\ -E_z & +B_y & -B_x & 0 \end{pmatrix}$$

$$F' = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 & +E_x & +E_y & +E_z \\ -E_x & 0 & +B_z & -B_y \\ -E_y & -B_z & 0 & +B_x \\ -E_z & +B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix}$$

$$\beta \to -v/c$$

Relativistic electrodynamics

- In summary, the Maxwell equations are, in **covariant notation**:

$$\boxed{\begin{aligned} \partial_\mu F^{\nu\mu} &= \frac{4\pi}{c} J^\nu \\ \partial_\mu F^{*\nu\mu} &= 0 \end{aligned}}$$

$$\Rightarrow \quad \partial \cdot F = 4\pi J \qquad \text{with} \quad \partial \cdot \partial \cdot F = 0 \leftrightarrow \partial \cdot J = 0$$

$$\Rightarrow \quad \partial \cdot F^* = 0$$

CONTINUITY EQUATION AS AN **INTEGRABILITY CONDITION**

- The Faraday tensor includes both the electric and the magnetic fields — i.e., they are "a single field"!

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad , \qquad F^{\mu\nu} = \begin{array}{cc} & \nu \longrightarrow \\ \mu \downarrow & \begin{pmatrix} 0 & +E_x & +E_y & +E_z \\ -E_x & 0 & +B_z & -B_y \\ -E_y & -B_z & 0 & +B_x \\ -E_z & +B_y & -B_x & 0 \end{pmatrix} \end{array}$$

Relativistic electrodynamics

- In **SI units**, $A^\mu = \{\phi/c, \overrightarrow{A}\}$, and the vacuum Maxwell equations read:

$$\partial_\mu F^{\nu\mu} = \mu_0 J^\nu$$

$$\partial_\mu F^{*\nu\mu} = 0$$

- The Faraday tensor in SI units is given by:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad , \qquad F^{\mu\nu} = \begin{array}{cc} & \longrightarrow \nu \\ \Big\downarrow \mu & \begin{pmatrix} 0 & +E_x/c & +E_y/c & +E_z/c \\ -E_x/c & 0 & +B_z & -B_y \\ -E_y/c & -B_z & 0 & +B_x \\ -E_z/c & +B_y & -B_x & 0 \end{pmatrix} \end{array}$$

Relativistic electrodynamics

- Take a general Lorentz boost, with $\vec{\beta} = \vec{v}/c$:

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta_i \\ -\gamma\beta_j & \delta_{ij} + (\gamma - 1)\frac{\beta_i\beta_j}{\beta^2} \end{pmatrix}$$

- Show that the transformation for $\vec{E}$ and $\vec{B}$ is:

$$\vec{E}' = \gamma \left[\vec{E} + \vec{v} \times \vec{B} - \frac{\gamma - 1}{\gamma} \frac{(\vec{E} \cdot \vec{\beta})\vec{\beta}}{\beta^2} \right] \quad ,$$

$$\vec{B}' = \gamma \left[\vec{B} - \frac{\vec{v}}{c^2} \times \vec{E} - \frac{\gamma - 1}{\gamma} \frac{(\vec{B} \cdot \vec{\beta})\vec{\beta}}{\beta^2} \right]$$

Next class (after exam):

- Field transformations
- Lorentz force
- Jackson, Ch. 11; Zangwill, Ch. 22
- (See also Bo Thidé's book: http://docente.unife.it/guido.zavattini/allegati/251023059-electromagnetic-field-theory-bo-thide.pdf , Ch. 5)