Exercícios do livro texto (Griffiths - Introdução à Eletrodinâmica - 3a. edição). Além dos abaixo, faça o número 10 da primeira lista.

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

$$\int_{-\infty}^{\infty} f(x)\delta(x)\, dx = f(0)\int_{-\infty}^{\infty} \delta(x)\, dx = f(0).$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\, dx = f(a).$$

1. **1.43**

   (a) $\int_2^6 (3x^2 - 2x - 1)\delta(x-3)\,dx$.

   (b) $\int_0^5 \cos x\,\delta(x-\pi)\,dx$.

   (c) $\int_0^3 x^3\delta(x+1)\,dx$.

   (d) $\int_{-\infty}^{\infty} \ln(x+3)\delta(x+2)\,dx$.

$\int_2^6 f(x)\,\delta(x-3) = f(3)$

a) $\int_2^6 (3x^2 - 2x - 1)\,\delta(x-3)\,dx = f(3) = 3\times 3^2 - 2\times 3 - 1 = (3\times 9) - 6 - 1 = 27 - 7 = 20$

$\int_0^5 f(x)\,\delta(x-\pi)$

b) $\int_0^5 \cos x\,\delta(x-\pi)\,d. = f(\pi) = \cos\pi = -1$

$\delta(x-(-1))$ $\quad -1 \notin [0;3]$

(c) $\int_0^3 x^3\delta(x+1)\,dx = 0$

$f(x)\,\delta(x-(-2))$

(d) $\int_{-\infty}^{+\infty} \ln(x+3)\,\delta(x+2)\,dx = \ln(-2+3) = \ln 1 = 0$

2. 1.45

(a) Mostre que

$$x\frac{\mathrm{d}}{\mathrm{d}x}(\delta(x)) = -\delta(x).$$

Sugestão: Use integração por partes.

$$(uv)' = u'v + v'u \Rightarrow u'v = (uv)' - v'u$$

$$\int_{-\infty}^{+\infty} u'v\,dz = \int_{-\infty}^{+\infty} (uv)'\,dz - \int_{-\infty}^{+\infty} uv'\,dz$$

$$\int_{-\infty}^{+\infty} \underbrace{f(x)\,x}_{u(x)} \underbrace{\frac{d}{dx}\delta(x)\,dx}_{v'(x)} = x f(x)\,\delta(x)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \frac{d}{dx}(x f(x))\,\delta(x)\,dx$$

$$= 0 - \int_{-\infty}^{+\infty} \left(f(x) + x\frac{df(x)}{dx}\right)\delta(x)\,dx = \underbrace{\int_{-\infty}^{+\infty} -f(x)\,\delta(x)\,dx}_{-f(0)}$$

$$+ \underbrace{\int_{-\infty}^{+\infty} x\,\delta(x)\,\frac{df(x)}{dx}\,dx}$$

$$= 0 - f(0) = -\int_{-\infty}^{+\infty} f(x)\,\delta(x)\,dx$$

$$\int_{-\infty}^{+\infty} f(x)\,x\frac{d}{dx}(\delta(x))\,dx = -\int_{-\infty}^{+\infty} f(x)\,\delta(x)\,dx \Rightarrow x\frac{d}{dx}(\delta(x)) = -\delta(x)$$

(b) Seja $\theta(x)$ a *função degrau*

$$\theta(x) \equiv \begin{cases} 1 & (x > 0) \\ 0 & (x \leq 0) \end{cases}.$$

Mostre que $d\theta / dx = \delta(x)$.

$$\int_{-\infty}^{+\infty} \underset{u}{f(x)} \underset{v'}{\frac{d\theta}{dx}} dx = \left[\underset{v}{\theta(x)}\, \underset{u}{f(x)}\right] - \int_{0}^{+\infty} \theta(x) \frac{df}{dx} dx$$

$$\left[f(x)\right]_{0}^{\infty} - \int_{0}^{+\infty} \theta(x) \frac{df}{dx} dx = \left[f(x)\right]_{0}^{\infty} - \int_{0}^{+\infty} \theta(x) \frac{df}{dx} = f(\infty) - f(\infty) + f(0)$$

$$\int_{-\infty}^{+\infty} f(x) \frac{d\theta}{dx} dx = f(0) \qquad \text{or} \qquad f(0) = \int_{-\infty}^{+\infty} f(x)\, \delta(x)\, dx$$

$$\int_{-\infty}^{+\infty} f(x) \frac{d\theta}{dx} dx = \int_{-\infty}^{+\infty} f(x)\, \delta(x)\, dx \Rightarrow \frac{d\theta}{dx} = \delta(x)$$

4. **1.47** Efetue as seguintes integrais:

(a) $\int_{\text{espaço}} (r^2 + \vec{r} \cdot \vec{a} + a^2)\delta^3(\vec{r} - \vec{a})\, d\tau$, onde $\vec{a}$ é um vetor fixo e a, seu módulo.

(b) $\int_{\mathcal{V}} |\vec{r} - \vec{b}|^2 \delta^3(5\vec{r})\, d\tau$, onde $\mathcal{V}$ é um cubo de lado 2, centrado na origem, e $\vec{b} = 4\hat{y} + 3\hat{z}$.

4) a) $\int (r^2 + \vec{r} \cdot \vec{a} + a^2)\, \delta^3(\vec{r} - \vec{a})\, d\tau$

$\int_{-\infty}^{+\infty} f(\vec{r})\, \delta^3(\vec{r} - \vec{a}) = f(\vec{a}) = \vec{a}^2 + \vec{a} \cdot \vec{a} + a^2$

b) $\int (\vec{r} - \vec{b})^2 \delta^3(5\vec{r})\, d\tau = \int_{\mathcal{V}} (\vec{r} - \vec{b})^2 \frac{1}{|5|^3} \delta^3(\vec{r})\, d\tau$

$\delta(k(x)) = \frac{1}{|k|} \delta(x)$

b) $\int (\vec{r} - \vec{b})^2 \delta^3(5\vec{r})\, d\tau = \int (\vec{r} - \vec{b})^2 \frac{1}{|5|^3} \delta^3(\vec{r})$

b) $\int \frac{1}{|5|^3} \int (\vec{r} - \vec{b})^2 \delta^3(\vec{r})\, d\tau = f(0) = \frac{1}{125} b^2 = \frac{(4^2 + 3^2)}{125} = \frac{25}{125} = \frac{1}{5}$

c) $\int_{\mathcal{V}} (\vec{r} - \vec{b})^2 \delta^3(5\vec{r})\, d\tau = \frac{16 + 9}{125} = \frac{1}{5}$

$$\delta(kx) = \frac{1}{|k|}\delta(x),$$

$\int_{-\infty}^{+\infty} f(r)\, \delta^3(\vec{r} - \vec{a})\, d\tau = f(a)$

$\delta(Kx) = \frac{1}{|k|} \delta(x)$

(c) $\int_{\mathcal{V}}(r^4 + r^2(\vec{r}\cdot\vec{c})\delta^3(\vec{r}-\vec{c})\,d\tau$, onde $\mathcal{V}$ é uma esfera de raio 6 em torno da origem, $\vec{c} = 5\hat{x} + 3\hat{y} + 2\hat{z}$, e c é sua magnitude.

(d) $\int_{\mathcal{V}}\vec{r}\cdot(\vec{d}-\vec{r})\delta^3(\vec{e}-\vec{r})\,d\tau$, onde $\vec{d} = (1,2,3)$, $\vec{e} = (3,2,1)$, e $\mathcal{V}$ é uma esfera de raio 1.5 centrado em (2,2,2).

$$|c|^2 = 5^2 + 3^2 + 2^2$$
$$= 25 + 9 + 4$$
$$c^2 = 38 \Rightarrow c = \sqrt{38} > 6 \Rightarrow$$
$$\int_V (r^4 + r^2(\vec{r}\cdot\vec{c})\,\delta^3(\vec{r}-\vec{c}) = 0$$

d) $\int_V \vec{r}(\vec{d} - \vec{r})\,\delta^3(\vec{e} - \vec{r})\,d\tau$; $|\vec{e}-\vec{r}|^2 = (3 - 2)^2 + (2-2)^2 + (1-2)^2$

$$= 1^2 + 0^2 + (-1)^2 = 2 < (1,5)^2$$

c) $\int_V (r^4 + r^2(\vec{r}\cdot\vec{c})\,\delta^3(\vec{r} - \vec{c})\,d\tau = 0$

d) $= \int_V \vec{r}\cdot(\vec{d}-\vec{r})\,\delta^3(\vec{e}-\vec{r}) = e(d - e) = (3,2,1)(-2,0,2)$

$$= -6 + 0 + 2 = -4$$

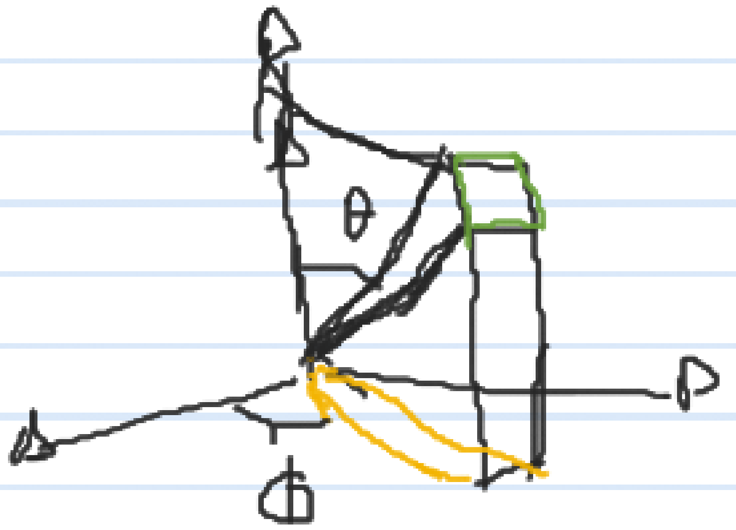
5. **1.53** Verifique o teorema fundamental da divergência para o campo

$$\vec{v} = r^2 \cos\theta\,\hat{r} + r^2 \cos\phi\,\hat{\theta} - r^2 \cos\theta \sin\phi\,\hat{\phi},$$

usando com volume um octante da esfera de raio R (Fig. 1.48). Inclua toda a superfície do octante.

Teorema de Gauss

$$\int_\tau \text{div}\,\vec{v}\,d\tau = \oint \vec{v}\cdot d\vec{S}$$



$$\int_V \text{div}\,\vec{v}\,d\tau = \int_\tau \vec{\nabla}\cdot\vec{v}\,d\tau = \oint_S \vec{v}\cdot d\vec{S}$$

$$d\vec{S} = dS\,\vec{n} = r\,d\theta\,r\,\text{sen}\,\theta\,d\phi = r^2\,\text{sen}\,\theta\,d\theta\,d\phi\,\vec{n}$$

$$\oint_S \vec{v}\cdot d\vec{S} = \oint_S \vec{v}\cdot d\vec{S}\,\hat{r} = \oint_S (r^2\cos\theta)\,r^2\,\text{sen}\,\theta\,d\theta\,d\phi = r^4\int_0^{\pi/2}\cos\theta\,\text{sen}\,\theta\int_0^{\pi/2}d\phi$$

$$= r^4\int_0^{\pi/2}\frac{\text{sen}\,2\theta}{2}\int_0^{\pi/2}d\phi = \frac{r^4}{2}\left[-\frac{\cos 2\theta}{2}\right]\left[\frac{\pi}{2}-0\right]$$

$$= \frac{\pi r^4}{4}\left[\frac{\overset{+1}{-\cos\pi} \overset{+1}{-}(-\cos 0)}{2}\right].$$

$1^2/2$

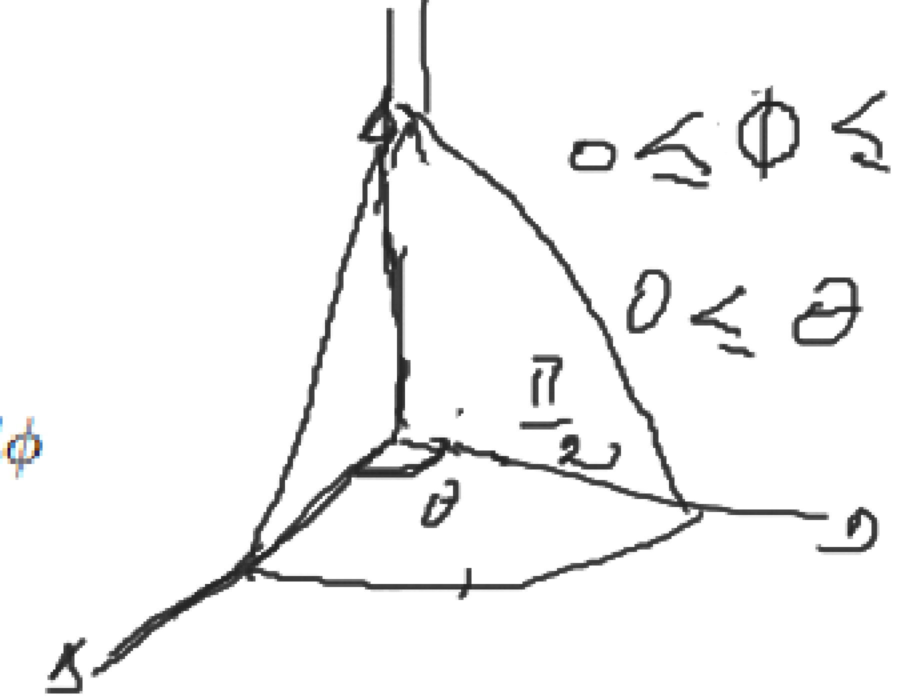
$$\oint_V \vec{v}\cdot d\vec{S} = \frac{\pi r^4}{4};$$

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\overset{v_r}{\left(r^2\, r^2\cos\theta\right)} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\overset{V_\theta}{\left(\sin\theta\, r^2\cos\phi\right)} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\overset{V_\phi}{\left(-r^2\cos\theta\sin\phi\right)}$$

$$= \frac{1}{r^2}4r^3\cos\theta + \frac{1}{r\sin\theta}\cos\theta\, r^2\cos\phi + \frac{1}{r\sin\theta}\left(-r^2\cos\theta\cos\phi\right)$$

$$= \frac{r\cos\theta}{\sin\theta}\left[4\sin\theta + \cos\phi - \cos\phi\right] = 4r\cos\theta.$$

$$\vec{\nabla}\left\{\begin{array}{l} \frac{1}{r^2}\frac{d}{dr}\left(r^2\right) \\ \frac{1}{r\,\mathrm{sen}\,\theta}\frac{d}{d\theta}\left(\mathrm{sen}\,\theta\, r^2 \ldots\right) \end{array}\right.$$



$0 \leq \phi \leq \frac{\pi}{2}$

$0 \leq \theta \leq \frac{\pi}{2}$

$$\int(\nabla\cdot\mathbf{v})\,d\tau = \int(4r\cos\theta)r^2\sin\theta\,dr\,d\theta\,d\phi = 4\int_0^R r^3\,dr\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\int_0^{\pi/2}d\phi$$

$$= \left(R^4\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \boxed{\frac{\pi R^4}{4}.}$$

Surface consists of four parts:

(1) *Curved:* $d\mathbf{a} = R^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}};\ r = R.\quad \mathbf{v}\cdot d\mathbf{a} = \left(R^2\cos\theta\right)\left(R^2\sin\theta\,d\theta\,d\phi\right).$

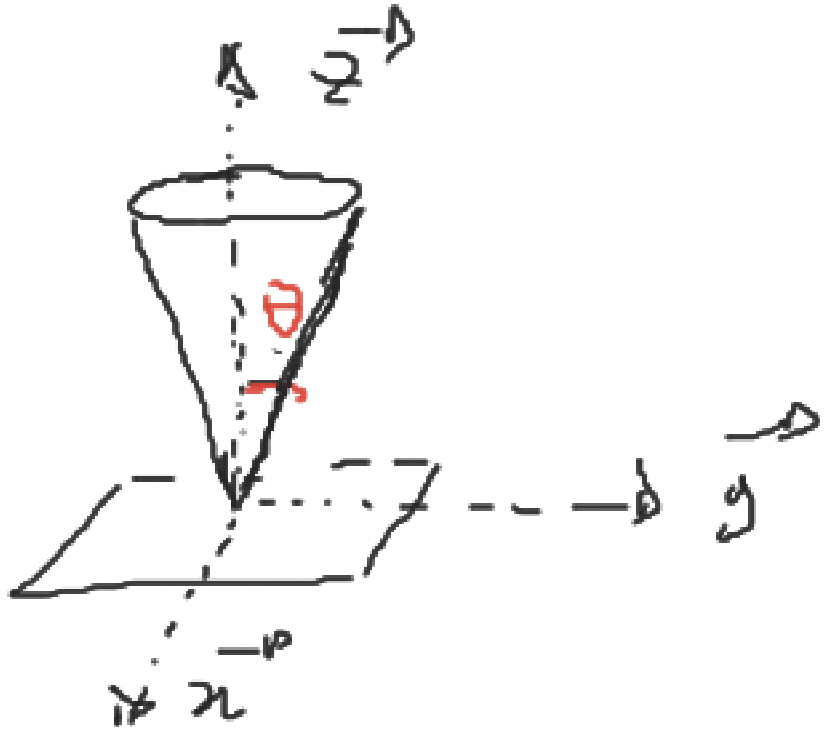
$$\int\mathbf{v}\cdot d\mathbf{a} = R^4\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\int_0^{\pi/2}d\phi = R^4\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \frac{\pi R^4}{4}.$$

7. **1.58** Verifique o teorema fundamental da divergência para o campo

$$\vec{v} = r^2 \sin\theta \,\hat{r} + 4r^2 \cos\theta \,\hat{\theta} + r^2 \tan\theta \,\hat{\phi},$$

usando o volume do sorvete na Fig. 1.52, onde a superfície superior é uma calota esférica de raio R, centrada na origem.

$$\int_V \mathrm{div}\,\vec{V} \cdot d\tau = \oint_S \vec{r}^{\,R} \cdot d\vec{s}$$



$$\frac{1}{r\,\text{sen}\,\theta} \frac{\partial}{\partial \phi}(r^2 \tan\phi) \quad \int_V \vec{\nabla} \cdot \vec{V} \cdot d\tau = \oint_S \vec{V} \cdot d\vec{s}$$

$$\vec{v} = \underbrace{r^2 \,\text{sen}\,\theta}_{v_r} \,\vec{r} + \underbrace{4r^2 \cos\theta}_{v_\theta} \,\hat{\theta} + \underbrace{r^2 \tan\theta}_{= v_\phi} \,\hat{\phi}$$

$$+ \frac{1}{r\,\text{sen}\,\theta\,\partial\theta}\left(\text{sen}\,\theta\, 4r^2 \cos\theta\right) +$$

$0 < r < R$

$0 < \theta < 2\pi$

$0 < \phi < R$

$$\vec{\nabla} \cdot \vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\,\text{sen}\,\theta}\frac{\partial}{\partial\theta}\left(\text{sen}\,\theta\, v_\theta\right) + \frac{1}{r\,\text{sen}\,\theta}\frac{\partial v_\phi}{\partial \phi}$$

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \underbrace{r^2\sin\theta}_{V_r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, \underbrace{4r^2\cos\theta}_{V_\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\underbrace{r^2\tan\theta}_{V_\varphi}\right)$$

$$= \frac{1}{r^2}4r^3\sin\theta + \frac{1}{r\sin\theta}4r^2\left(\cos^2\theta - \sin^2\theta\right) = \frac{4r}{\sin\theta}\left(\sin^2\theta + \cos^2\theta - \sin^2\theta\right)$$

$$= 4r\frac{\cos^2\theta}{\sin\theta}.$$

$\operatorname{div}\vec{V} = \vec{\nabla}\cdot\vec{V}$

$\int \vec{\nabla}\cdot\vec{V}\, d\tau$

1

$$\int(\nabla\cdot\mathbf{v})\,d\tau = \int\left(4r\frac{\cos^2\theta}{\sin\theta}\right)\left(r^2\sin\theta\,dr\,d\theta\,d\phi\right) = \int_0^R 4r^3\,dr\int_0^{\pi/6}\cos^2\theta\,d\theta\int_0^{2\pi}d\phi = \left(R^4\right)(2\pi)\left[\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]\Bigg|_0^{\pi/6}$$

$$= 2\pi R^4\left(\frac{\pi}{12}+\frac{\sin 60^\circ}{4}\right) = \frac{\pi R^4}{6}\left(\pi + 3\frac{\sqrt{3}}{2}\right) = \boxed{\frac{\pi R^4}{12}\left(2\pi + 3\sqrt{3}\right).}$$

Surface coinsists of two parts:

$\oint \vec{V}\cdot d\vec{S}$

(1) *The ice cream:* $r = R$; $\phi : 0 \to 2\pi$; $\theta : 0 \to \pi/6$; $d\mathbf{a} = R^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}$; $\mathbf{v}\cdot d\mathbf{a} = \left(R^2\sin\theta\right)\left(R^2\sin\theta\,d\theta\,d\phi\right) = R^4\sin^2\theta\,d\theta\,d\phi$.

$$\int\mathbf{v}\cdot d\mathbf{a} = R^4\int_0^{\pi/6}\sin^2\theta\,d\theta\int_0^{2\pi}d\phi = \left(R^4\right)(2\pi)\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/6} = 2\pi R^4\left(\frac{\pi}{12}-\frac{1}{4}\sin 60^\circ\right) = \frac{\pi R^4}{6}\left(\pi - 3\frac{\sqrt{3}}{2}\right.$$

$r = R$

2

(2) *The cone:* $\theta = \frac{\pi}{6}$; $\phi : 0 \to 2\pi$; $r : 0 \to R$; $d\mathbf{a} = r\sin\theta\,d\phi\,dr\,\hat{\boldsymbol{\theta}} = \frac{\sqrt{3}}{2}r\,dr\,d\phi\,\hat{\boldsymbol{\theta}}$; $\mathbf{v}\cdot d\mathbf{a} = \sqrt{3}\,r^3\,dr\,d\phi$

$$\int\mathbf{v}\cdot d\mathbf{a} = \sqrt{3}\int_0^R r^3\,dr\int_0^{2\pi}d\phi = \sqrt{3}\cdot\frac{R^4}{4}\cdot 2\pi = \frac{\sqrt{3}}{2}\pi R^4.$$

Therefore $\int\mathbf{v}\cdot d\mathbf{a} = \frac{\pi R^4}{2}\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}+\sqrt{3}\right) = \frac{\pi R^4}{12}\left(2\pi+3\sqrt{3}\right)$. ✓. $\Rightarrow$ $\int_V \vec{\nabla}\cdot\vec{V}\,d\tau = \oint \vec{V}\cdot d\vec{S}$

8. 1.59 Duas elegantes verificações dos teoremas fundamentais:

(a) Combine o corolário 2 do teorema do gradiente com o teorema de Stokes (com $\vec{v} = \vec{\nabla} T$). Mostre que o resultado concorda com o que você já sabe sobre derivadas segundos.

(b) Combine o corolário 2 do teorema do rotacional com o teorema da divergência. Mostre que o resultado concorda com o que você já sabe.

a) $\vec{v} = \vec{\nabla} T$

Stokes : $\oint \vec{\nabla} T \cdot d\vec{l} = \int_S (\vec{\nabla} \wedge \vec{\nabla} T)\, d\vec{s}$

$\oint (\vec{\nabla} T)\, d\vec{l} = 0 \Rightarrow$ corolario 2 de T

$$\oint (\vec{\nabla} T)\, d\vec{l} = \iint (\vec{\nabla} \wedge \vec{\nabla} T)\, d\vec{s} = 0 \Rightarrow \int_S (\vec{\nabla} \wedge \vec{\nabla} T)\, da = 0$$

$$\vec{\nabla} \wedge \vec{\nabla} T = \vec{0} \Rightarrow \text{rot}\, \vec{V} = 0 \qquad \frac{d}{dx}\left(\frac{dT}{dy}\right) = \frac{d}{dy}\left(\frac{dT}{dx}\right)$$

b) $\oint_S (\vec{\nabla} \wedge \vec{v})\, d\vec{s} = 0$; $\oint_S (\vec{\nabla} \wedge \vec{v})\, d\vec{s} = \int \vec{\nabla} \cdot (\vec{\nabla} \wedge \vec{v})\, d\tau = 0$

$$\int_V \vec{\nabla} \cdot (\vec{\nabla} \wedge \vec{v})\, d\tau = 0 \Rightarrow \vec{\nabla} \cdot (\vec{\nabla} \wedge \vec{v}) = \text{div}(\text{rot}\, \vec{v}) = 0$$

$\int_V (\vec{\nabla} \wedge \vec{V})\, d\vec{s} = 0 \quad \Rightarrow$ 2 de Teorema de rotacional

10. **1.62(a)** Encontre a divergência do campo

$$\vec{v} = \frac{\hat{r}}{r}.$$

Para isso,

(a) Calcule diretamente o divergente;

(b) Teste seu resultado por meio do teorema da divergência;

(c) Existe uma função delta na origem, como no caso do campo $\hat{r}/r^2$? Explique.

$$\vec{v} = \frac{\vec{r}}{r}$$

a) $$\operatorname{div}\vec{v} = \vec{\nabla}\cdot\vec{v} = \underbrace{\frac{1}{r^2}\frac{d}{dr}\left(r^2 \times \frac{1}{r}\right)}$$

$$\operatorname{div}\vec{v} = \frac{1}{r^2}\underbrace{\frac{d}{dr}(r)}_{1} = \frac{1}{r^2}$$

b) $$\int \operatorname{div}\vec{v}\, d\tau = \oint_S \vec{v}\cdot d\vec{S}$$

1 $$\int_V \operatorname{div}\vec{v}\, d\tau = \iiint \frac{1}{\cancel{r^2}}\cancel{r^2}\operatorname{sen}\theta\, d\theta\, d\phi\, dr = \int_0^{\pi}\operatorname{sen}\theta\, d\theta \int_0^{2\pi} d\phi \int_0^{R} dr$$

$$= \Big[-\cos\theta\Big]_0^{\pi}\Big[\phi\Big]_0^{2\pi}\Big[r\Big]_0^{R}$$

$$\int \operatorname{div}\vec{v}\cdot d\tau = 2\pi R\Big[\underbrace{-\cos\pi}_{+1} - \underbrace{(-\cos 0)}_{+1}\Big]$$

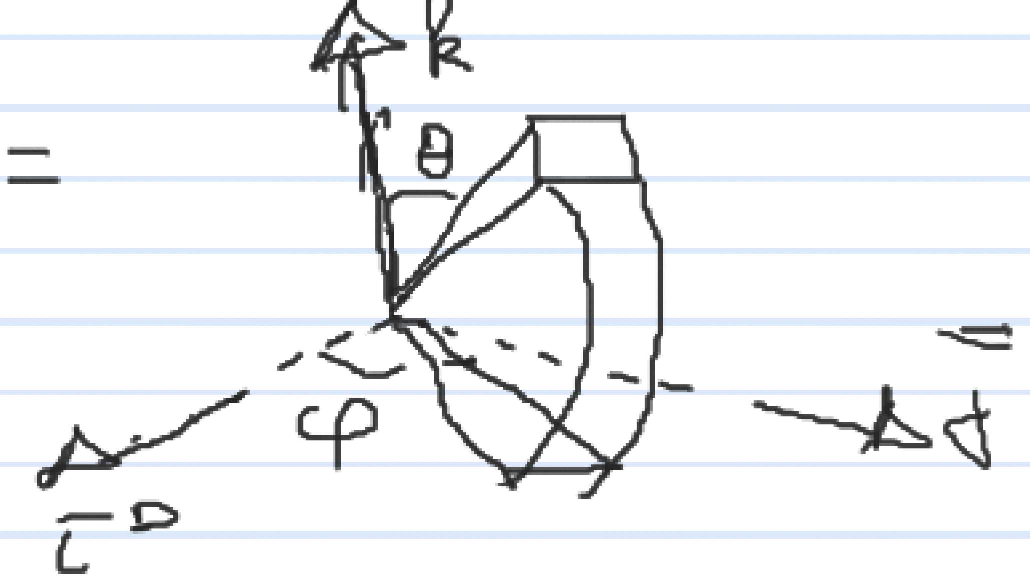
$$\int \operatorname{div}\vec{v}\, d\tau = 4\pi R$$

$$0 < \theta < \pi$$

$$0 < \phi < 2\pi$$

2) $\oint_S \vec{V}\cdot d\vec{S} = \oint \frac{1}{\cancel{r}}\, r^{\cancel{2}}\, \text{sen}\,\theta\, d\theta\, d\phi =$

$\frac{1}{r}\iint R^2\, \text{sen}\,\theta\, d\theta\, d\phi$



$$\oint_S \vec{v}\cdot d\vec{S} = \frac{R^2}{R}\int_0^{\pi} \text{sen}\,\theta\, d\theta \int_0^{2\pi} d\phi = R\,(2)(2\pi) = 4\pi R$$

3) Nao tem δ no origem como o caso do campo

$\frac{\vec{r}}{r}$;;