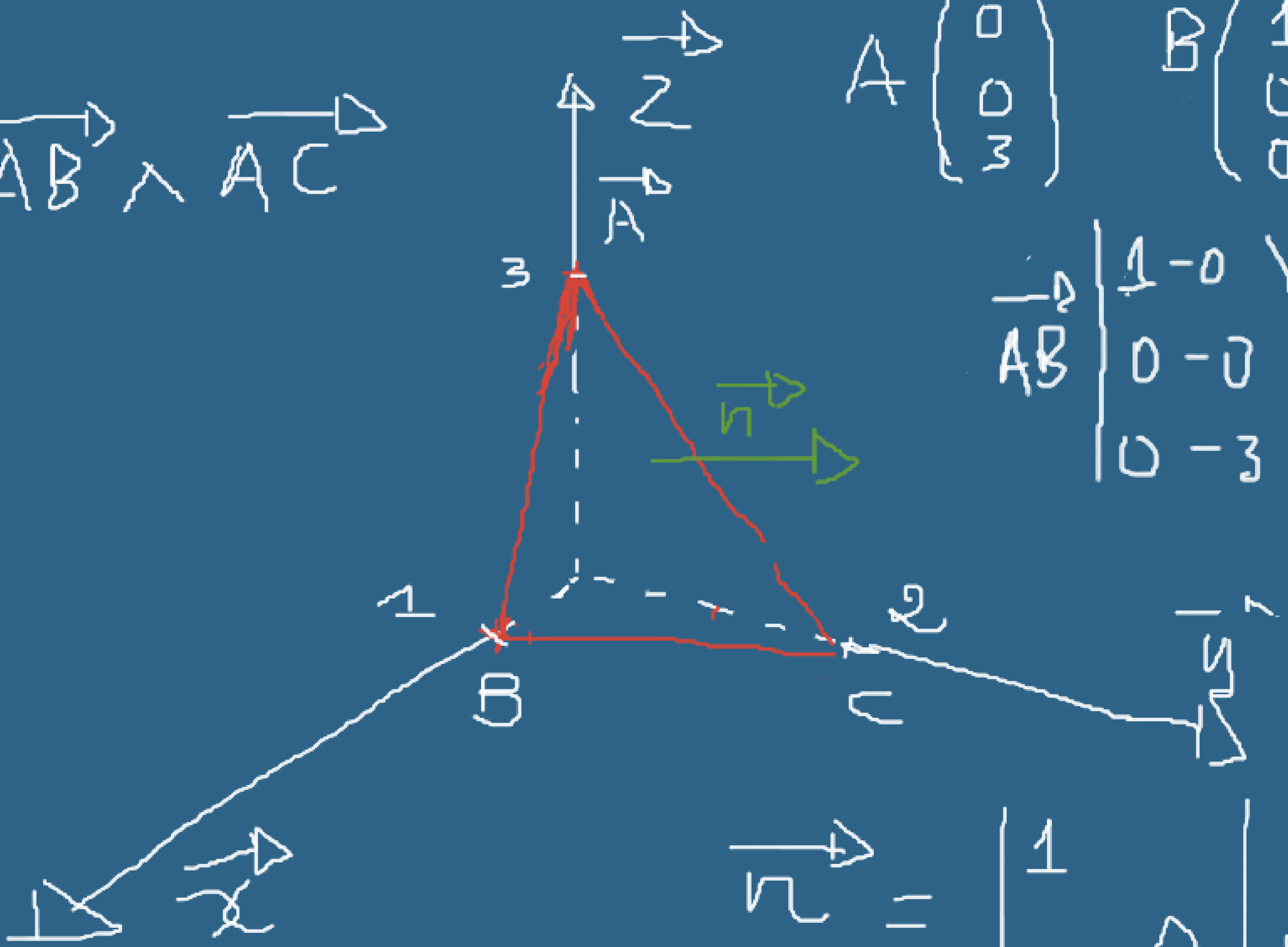
1. 1.4 Use o produto vetorial para encontrar o vetor normal ˆn ao plano indicado na Fig. 1.11.

$$\vec{n} = \vec{AB} \wedge \vec{AC}$$



$$A\begin{pmatrix}0\\0\\3\end{pmatrix} \quad B\begin{pmatrix}1\\0\\0\end{pmatrix} \quad C\begin{pmatrix}0\\2\\0\end{pmatrix}$$

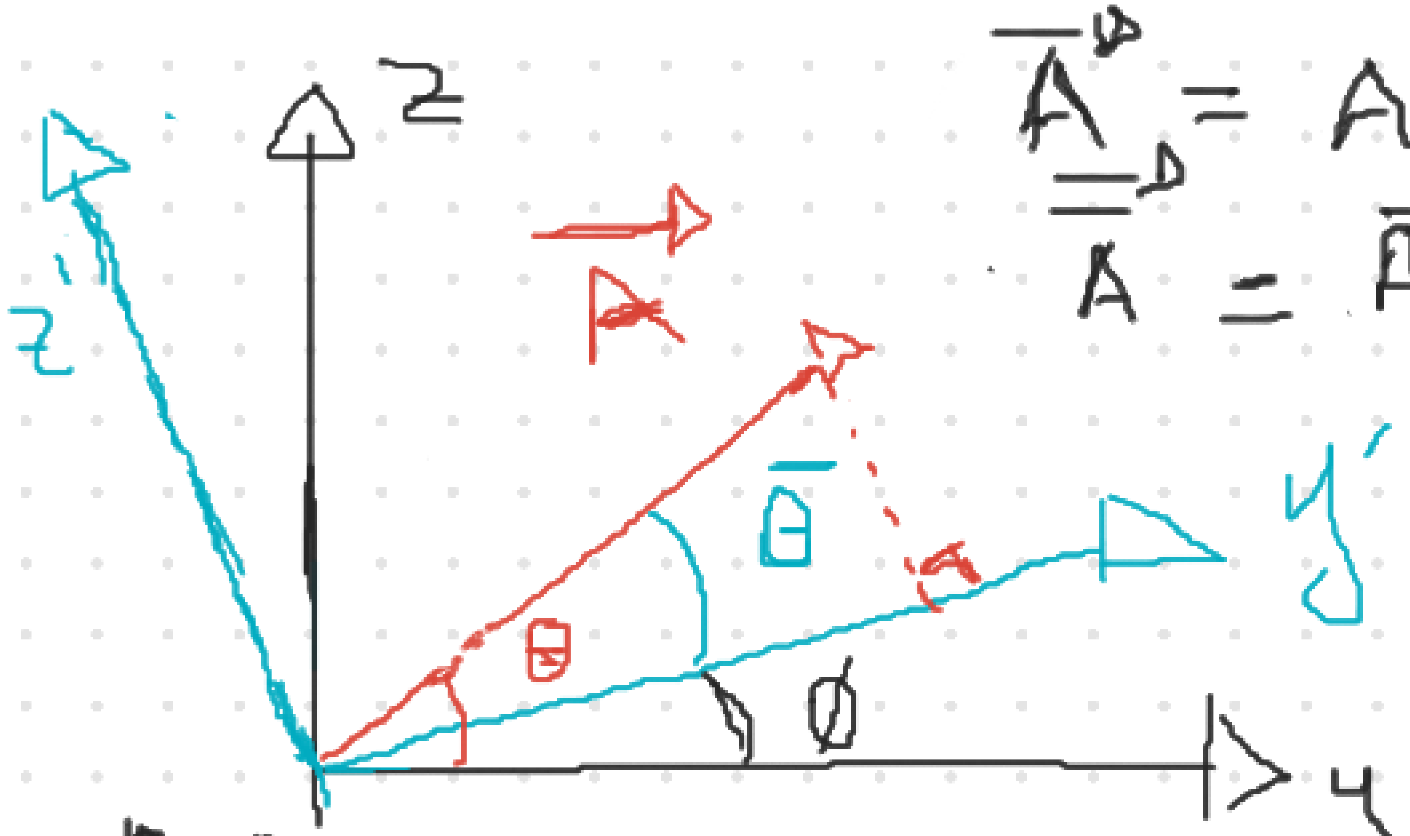
$$\vec{AB}\begin{vmatrix}1-0\\0-0\\0-3\end{vmatrix} \Rightarrow \vec{AB}\begin{pmatrix}1\\0\\-3\end{pmatrix};\ \vec{AC}\begin{pmatrix}0\\2\\-3\end{pmatrix}$$

$$\vec{n} = \begin{vmatrix}1\\0\\-3\end{vmatrix} \wedge \begin{vmatrix}0\\2\\-3\end{vmatrix} = \begin{vmatrix}0+6\\0+3\\2-0\end{vmatrix} = \begin{vmatrix}6\\3\\2\end{vmatrix}$$

$$\vec{n} = 6\vec{i} + 3\vec{j} + 2\vec{k}$$

2.1.8

(a) Mostre que a matriz de rotação (1.29) preserva o produto escalar ,



$\vec{A} = A_x \vec{i} + A_y \vec{j} = A\cos\theta\,\vec{i} + A\,\mathrm{sen}\,\theta\,\vec{j}$

$\vec{\bar A} = \bar A_y \cos\theta\,\vec{i} + \bar A_z\,\mathrm{sen}\,\theta\,\vec{j}$

$$\vec{\bar A}\begin{cases} \bar A_y = A_y\cos\phi + A_z\,\mathrm{sen}\,\phi \\ \bar A_z = -A_y\,\mathrm{sen}\,\phi + A_z\cos\phi \end{cases}$$

$\vec A\cdot\vec A = (A\cos\theta)(A\cos\theta) + (A\,\mathrm{sen}\,\theta)(A\,\mathrm{sen}\,\theta) = A^2(\underbrace{\cos^2\theta + \mathrm{sen}^2\theta}_{=1})$

$\vec A\cdot\vec A = A^2$

$\vec{\bar A}\cdot\vec{\bar A} = \bar A_y^2 + \bar A_z^2 = (A_y\cos\phi + A_z\,\mathrm{sen}\,\phi)^2 + (-A_y\,\mathrm{sen}\,\phi + A_z\cos\phi)^2$

$= A_y^2\cos^2\phi + A_z^2\,\mathrm{sen}^2\phi + \cancel{2A_yA_z\cos\phi\,\mathrm{sen}\,\phi} + A_y^2\,\mathrm{sen}^2\phi + A_z^2\cos^2\phi - \cancel{2A_yA_z\cos\phi\,\mathrm{sen}\,\phi}$

$\vec{\bar A}\vec{\bar A} = A_y^2(\cos^2\phi + \mathrm{sen}^2\phi) + A_z^2(\cos^2\phi + \mathrm{sen}^2\phi) \quad \therefore \quad \vec{\bar A}\cdot\vec{\bar A} = \vec A\,\vec A$

$\vec{\bar A}\vec{\bar A} = A_y^2 + A_z^2 = A^2\cos^2\theta + A^2\,\mathrm{sen}^2\theta = A^2(\cos^2\theta + \mathrm{sen}^2\theta) = A^2$

$$\overset{R}{\begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix}} \overset{R^T}{\begin{pmatrix} R_{xx} & R_{yx} & R_{zx} \\ R_{xy} & R_{yy} & R_{zy} \\ R_{xz} & R_{yz} & R_{zz} \end{pmatrix}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$\vec{A'A'} = \vec{AA}$$

$$\vec{A'A'} = \vec{A}\vec{A} \Rightarrow R R^T = I$$

I : matriz identidade

Seja ~ o vetor separação entre os pontos (x0 , y0 , z0) e (x, y, z), com comprimento $\mathscr{r}$.
Mostre que

(a) $\vec{\nabla}(\mathscr{r}^2) = 2\vec{\mathscr{r}}$;

(b) $\vec{\nabla}(1/\mathscr{r}) = -\hat{\mathscr{r}}/\mathscr{r}^2$;

(c) Qual é a fórmula geral para $\vec{\nabla}(\mathscr{r}^n)$

$\vec{\mathscr{r}} = \vec r - \vec r'$

$\vec r = x\vec i + y\vec j + z\vec k$

$\vec r' = x'\vec i + y'\vec j + z'\vec k$

(3)

$\vec{\mathscr{r}} = \vec r - \vec r' = (x - x')\vec i + (y - y')\vec j + (z - z')\vec k$

$\mathscr{r}^2 = |r - r'|^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$

$\vec\nabla(\mathscr{r}^2) = \vec i\,\frac{d}{dx}(\mathscr{r}^2) + \vec j\,\frac{d}{dy}(\mathscr{r}^2) + \vec k\,\frac{d}{dz}(\mathscr{r}^2)$

$$\frac{d}{dx}\left(r^2\right) = \frac{d}{dx}\left((x-x')^2+(y-y')^2+(z-z')^2\right)$$

$$\frac{d}{dx}\left(r^2\right) = 2(x-x')$$

$$\frac{d}{dy}\left(r^2\right) = 2(y-y')$$

$$\frac{d}{dz}\left(r^2\right) = 2(z-z')$$

$$\vec{\nabla}\left(r^2\right) = 2(x-x')\vec{i}+2(y-y')\vec{j}+2(z-z')\vec{k}$$

$$= 2\left((x-x')\vec{i}+(y-y')\vec{j}+(z-z')\vec{k}\right)$$

$$\vec{\nabla}\left(r^2\right) = 2\vec{r}$$

$$\vec{\nabla}\left(\frac{1}{r}\right) = \vec{\nabla}\left(r^{-1}\right) = \frac{\partial}{\partial x}\left(r^{-1}\right) + \frac{\partial}{\partial y}\left(r^{-1}\right) + \frac{\partial}{\partial z}\left(r^{-1}\right)$$

$$\frac{\partial}{\partial x}\left(r^{-1}\right) = \frac{\partial}{\partial x}\left(\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-\frac{1}{2}}\right)$$

$$= -\frac{2}{2}\left(x-x'\right)\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-\frac{3}{2}}$$

$$= -\left(x-x'\right)\left(r^{2}\right)^{-3/2} = -\left(x-x'\right)r^{-3}$$

$$\frac{\partial}{\partial y}\left(r^{-1}\right) = -\left(y-y'\right)r^{-3}$$

$$\frac{\partial}{\partial z}\left(r^{-1}\right) = -\left(z-z'\right)r^{-3}$$

$$\hat{r} = \frac{\vec{r}}{r}$$

$$\vec{\nabla}\left(\frac{1}{r}\right) = -\left[\left(x-x'\right)\vec{i} + \left(y-y'\right)\vec{j} + \left(z-z'\right)\vec{k}\right]r^{-3} = -\vec{r}\cdot r^{-3} = \frac{\vec{r}}{r^3} = \frac{\hat{r}}{r^2}$$

(c) Qual é a fórmula geral para $\vec{\nabla}(r^n)$.

$$\vec{\nabla}(r^2) = 2\vec{r} = 2r^{(2-2)}\vec{r} = 2r^0\vec{r} = 2\vec{r}$$

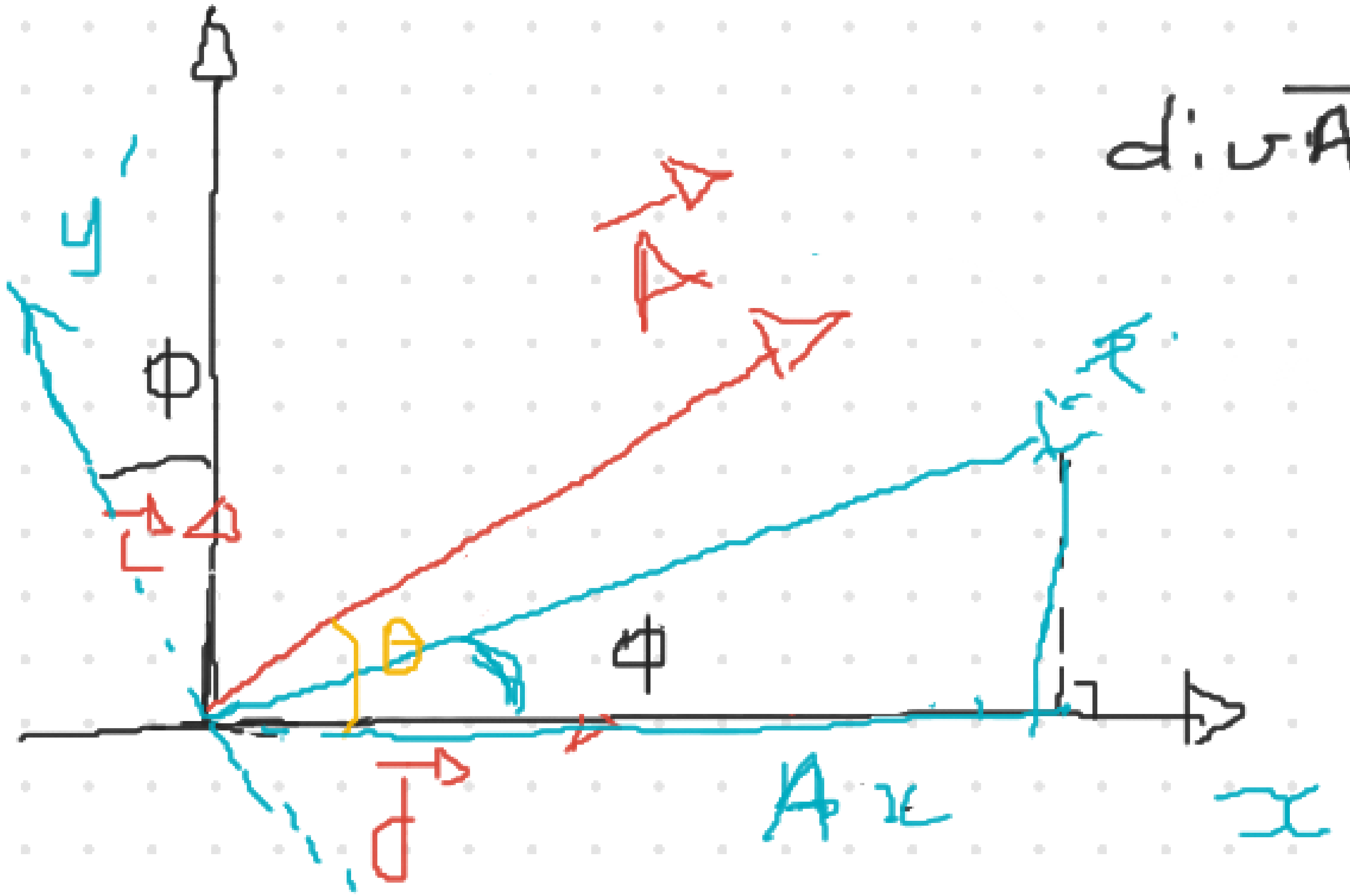
$$\vec{\nabla}(r^{-1}) = -r^{-3}\vec{r} = -1\,r^{-1-2}\vec{r}$$

$$\vec{\nabla}(r^n) = n\,r^{(n-2)}\vec{r}$$

$$\vec{\nabla}(r^n) = n\,r^{(n-2)}\vec{r}$$

4. **1.17** Em duas dimensões, mostre que a divergência se transforma como escalar sob rotações.

derivar Ax e Ay en funcion de x' e y'



$$\vec{A} = A_x \vec{i} + A_y \vec{j}$$

$$\mathrm{div}\,\vec{A} = \vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}$$

$$\frac{\partial A_x}{\partial x} = \frac{\partial A_x}{\partial x'} \times \frac{\partial x'}{\partial x} + \frac{\partial A_x}{\partial y'} \times \frac{\partial y'}{\partial x}$$

$$\frac{\partial A_y}{\partial y} = \frac{\partial A_y}{\partial x'} \times \frac{\partial x'}{\partial y} + \frac{\partial A_y}{\partial y'} \times \frac{\partial y'}{\partial y}$$

$$x' = x \cos\phi + y \,\mathrm{sen}\,\phi$$

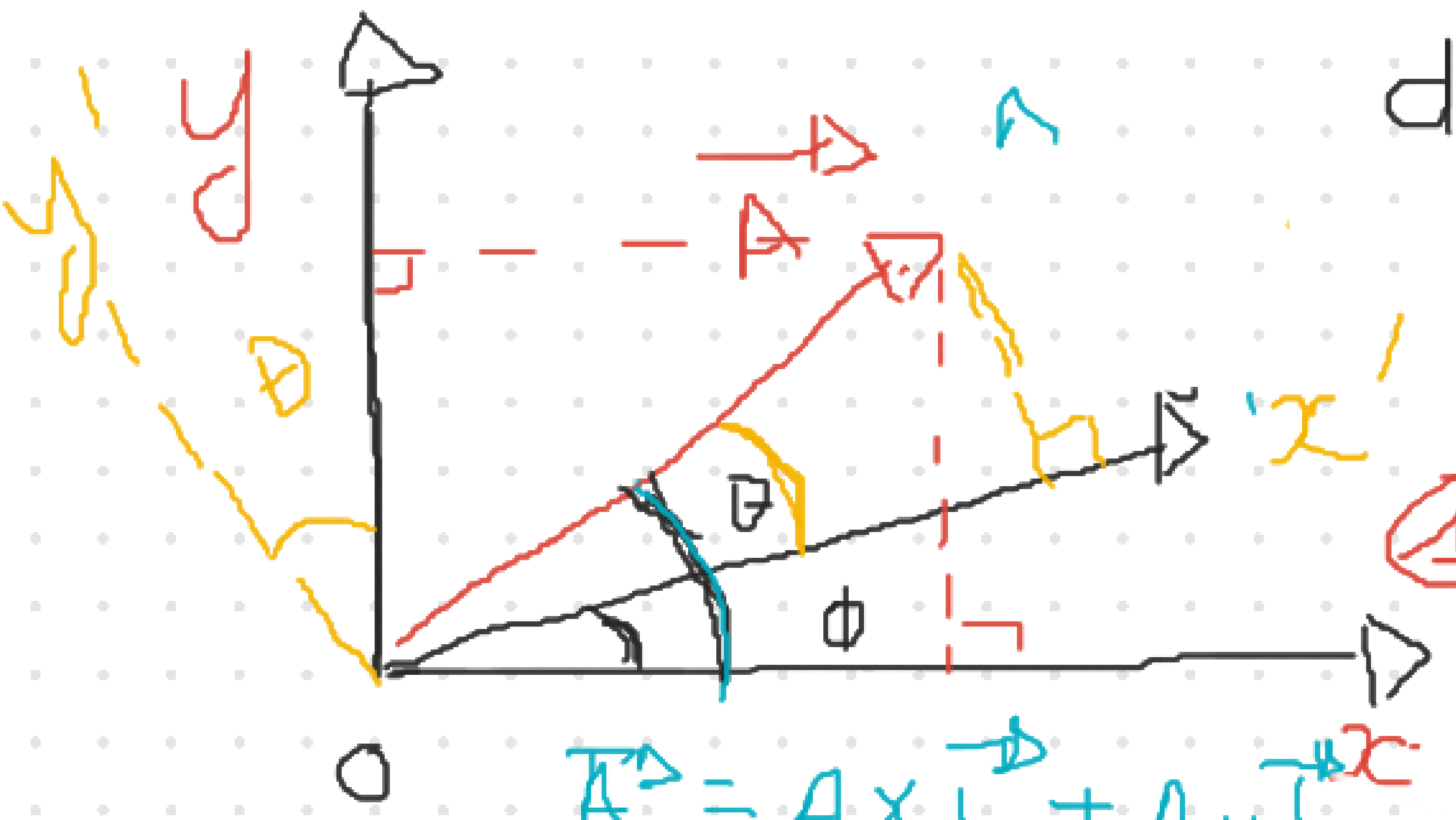
$$y' = -x \,\mathrm{sen}\,\phi + y \cos\phi$$

$$\frac{\partial A_x}{\partial x} = \cos\phi \frac{\partial A_x}{\partial x'} - \mathrm{sen}\,\phi \frac{\partial A_x}{\partial y'}$$

$$\frac{\partial A_y}{\partial y} = \mathrm{sen}\,\phi \frac{\partial A_y}{\partial x'} + \cos\phi \frac{\partial A_y}{\partial y'}$$

$$\mathrm{div}\,\vec{A} = \cos\phi \frac{\partial A_x}{\partial x'} - \mathrm{sen}\,\phi \frac{\partial A_x}{\partial y'} + \mathrm{sen}\,\phi \frac{\partial A_y}{\partial x'} + \cos\phi \frac{\partial A_y}{\partial y'}$$

$$\operatorname{div}\vec{A} = \cos\phi\,\frac{\partial A_x}{\partial x'} - \operatorname{sen}\phi\,\frac{\partial A_x}{\partial y'} + \operatorname{sen}\phi\,\frac{\partial A_y}{\partial x'} + \cos\phi\,\frac{\partial A_y}{\partial y'}$$



$$\vec{A} = A_x\vec{i} + A_y\vec{j}$$

$$\vec{A}\begin{cases} A_x = A\cos\theta \\ A_y = A\operatorname{sen}\theta \end{cases}$$

$$\vec{A'}\begin{cases} A'_x = A\cos(\theta-\phi) \\ A'_y = A\operatorname{sen}(\theta-\phi) \end{cases}$$

$$\operatorname{div}\vec{A} = \cos\phi\cos\theta\,\frac{\partial A}{\partial x'} - \operatorname{sen}\phi\cos\theta\,\frac{\partial A}{\partial y'} + \operatorname{sen}\phi\operatorname{sen}\theta\,\frac{\partial A}{\partial x'} + \cos\phi\operatorname{sen}\theta\,\frac{\partial A}{\partial y'}$$

(1)

$$\operatorname{div}\vec{A} = \cos(\theta-\phi)\,\frac{\partial A}{\partial x'} + \operatorname{sen}(\theta-\phi)\,\frac{\partial A}{\partial y'}$$

$$\operatorname{div}\vec{A'} = \frac{\partial A'_x}{\partial x'} + \frac{\partial A'_y}{\partial y'}$$

(2)

$$\operatorname{div}\vec{A'} = \cos(\theta-\phi)\,\frac{\partial A}{\partial x'} + \operatorname{sen}(\theta-\phi)\,\frac{\partial A}{\partial y'}$$

(1) = (2)

5. **1.19** Encontre uma função vetorial (que não seja uma constante) que tem divergência e rotacional iguais a zero em todo o espaço.

$$\begin{cases} \operatorname{div} \vec{F} = 0 \\ \vec{\operatorname{rot}}\, \vec{F} = \vec{0} \end{cases} \Rightarrow \vec{F} = -3y\,\vec{i} + (3x + z)\,\vec{j} + y\,\vec{k}$$

$$\vec{\operatorname{rot}}\, \vec{F} = \vec{0} \Rightarrow \vec{F} = \vec{\nabla} f = \vec{i}\frac{df}{dx} + \vec{j}\frac{df}{dy} + \vec{k}\frac{df}{dz}$$

$$\frac{df}{dx} = -3y \Rightarrow df = -3y\,dx \Rightarrow f = -3yx + k_1(y,z)$$

$$\frac{df}{dy} = -3x + z \qquad \frac{df}{dy} = -3x + \frac{dk_1}{dy} = -3x + z$$

$$\frac{df}{dz} = y \qquad \frac{dk_1}{dy} = z \Rightarrow k_1 = zy + k_2(z)$$

$$f = -3yx + zy + k_2(z)$$

$$f(x,y,z) = -3yx + zy + k_2(z)$$

$$\frac{df}{dz} = y + \frac{dk_2}{dz}(z) = y \Rightarrow \frac{dk_2}{dz} = 0 \Rightarrow k_2 = cte = K$$

$$f(x,y,z) = -3yx + zy + k$$

7. **1.23** Mostre que

$$\vec{\nabla} \times \left(\frac{\vec{A}}{g}\right) = \frac{g(\vec{\nabla} \times \vec{A}) + \vec{A} \times (\vec{\nabla} g)}{g^2}.$$

$$\vec{\nabla} \wedge \left(\frac{\vec{A}}{g}\right) = \frac{1}{g}\left(\vec{\nabla} \wedge \vec{A}\right) + \vec{A} \wedge \vec{\nabla}\left(\frac{1}{g}\right), \quad \vec{\nabla}\left(\frac{1}{g}\right) = -\frac{\vec{\nabla}(g)}{g^2}$$

$$= \frac{1}{g}\left(\vec{\nabla} \wedge \vec{A}\right) - \vec{A} \wedge \frac{\nabla(g)}{g^2}$$

$$\vec{\nabla} \wedge \left(\frac{\vec{A}}{g}\right) = \frac{g\left(\vec{\nabla} \wedge \vec{A}\right) - \vec{A} \wedge \vec{\nabla}(g)}{g^2}$$

8. **1.26** Mostre que o divergente de um rotacional é sempre zero.

$u\,(u_x, u_y, u_z)$

$$\operatorname{div}(\operatorname{rot} \vec{u}) = \vec{\nabla} \cdot (\vec{\nabla} \wedge \vec{u})$$

$$\vec{\nabla} \times \vec{u} = \begin{vmatrix} \frac{d}{dx} \\ \frac{d}{dy} \\ \frac{d}{dz} \end{vmatrix} \wedge \begin{vmatrix} u_x \\ u_y \\ u_z \end{vmatrix} = \begin{vmatrix} \frac{\partial U_z}{\partial y} - \frac{\partial U_y}{\partial z} \\ \frac{\partial U_z}{\partial x} - \frac{\partial U_z}{\partial x} \\ \frac{\partial U_y}{\partial z} - \frac{\partial U_x}{\partial y} \end{vmatrix}$$

$$\vec{\nabla}\cdot\left(\vec{\nabla}\wedge\vec{u}\right)=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\cdot\begin{pmatrix}\dfrac{\partial u_z}{\partial y}-\dfrac{\partial u_y}{\partial z}\\[2ex]\dfrac{\partial u_x}{\partial z}-\dfrac{\partial u_z}{\partial x}\\[2ex]\dfrac{\partial u_y}{\partial x}-\dfrac{\partial u_x}{\partial y}\end{pmatrix}$$

$$\vec{\nabla}\cdot\left(\vec{\nabla}\wedge\vec{u}\right)=\frac{\partial}{\partial x}\left(\frac{\partial u_z}{\partial y}-\frac{\partial u_y}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}\right)$$

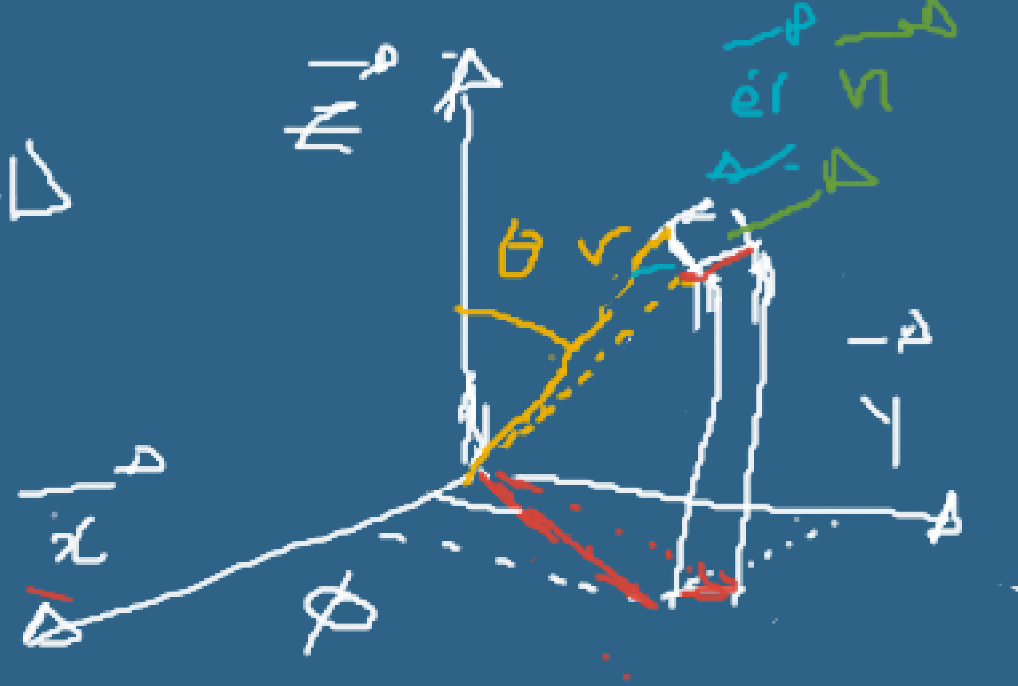
$$\vec{\nabla}\cdot\left(\vec{\nabla}\wedge\vec{u}\right)=\cancel{\frac{\partial^2 u_z}{\partial x\,\partial y}}-\cancel{\frac{\partial^2 u_y}{\partial x\,\partial z}}+\cancel{\frac{\partial^2 u_x}{\partial y\,\partial z}}-\cancel{\frac{\partial^2 u_z}{\partial y\,\partial x}}+\cancel{\frac{\partial^2 u_y}{\partial z\,\partial x}}-\cancel{\frac{\partial^2 u_x}{\partial z\,\partial y}}=0$$

$$\operatorname{div}\left(\overrightarrow{\operatorname{rot}}\,\vec{u}\right)=0$$

10. **1.38** Verifique o teorema de Gauss para a função $\vec{v} = (1/r^2)\hat{r}$, tomando como volume a esfera de raio R, centrada na origem.

Teorema Gauss

$$\int_{\mathcal{V}} \operatorname{div} \vec{V}\, d\tau = \oint_S \vec{V}\, d\vec{S}$$



$$\oint \vec{V} \cdot d\vec{S} = \oint \frac{1}{r^2}\vec{r}\, d\vec{S}\,, \quad d\vec{S} = dS\,\vec{n} = dS\,\vec{e_r}$$

$$d\vec{S} = r\,d\theta \times r\,\text{sen}\,\theta\, d\phi = r^2\,\text{sen}\,\theta\, d\theta\, d\phi\, \vec{e_r}$$

$$\oint \vec{V} \cdot d\vec{S} = \oint\!\!\oint \frac{1}{\cancel{r^2}} \times \cancel{r^2}\,\text{sen}\,\theta\, d\theta\, d\phi\; \vec{r} \times \vec{r}$$

$$\oint \vec{V} \cdot d\vec{S} = \int_0^{\pi} \text{sen}\,\theta\, d\theta \int_0^{2\pi} d\phi = \Big[-\cos\theta\Big]_0^{\pi} \Big[\phi\Big]_0^{2\pi} = 4\pi$$

$$\int_{\mathcal{V}} \vec{\nabla} \cdot \vec{V}\, d\tau = \frac{1}{r}\frac{\partial}{\partial r}\left(r^2 \times V_r\right)$$

$$\int_V \operatorname{div} \vec{v}\, d\tau = \int_V \vec{\nabla} \cdot \vec{v}\, d\tau$$

$$\int_V \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \times \frac{1}{r^2}\right) d\tau = 0$$ verdade excetado no origem

No origem $\delta = 4\pi r$

$$\vec{\nabla} \cdot \vec{v} = \vec{\nabla}\left(\frac{\vec{r}}{r}\right) = 4\pi\, \delta(r) \Rightarrow \int_V \delta(r)\, d\tau = 1$$

$$\int \vec{\nabla} \cdot \vec{v} = \int 4\pi\, \delta(r)\, d\tau = 4\pi \underbrace{\int \delta(r)\, d\tau}_{1} = 4\pi$$

$$\int_V \vec{\nabla} \cdot \vec{v} = 4\pi = \oint_S \vec{v} \cdot d\vec{S}$$