A discussion of the above equation with no harvesting where $K=K(t)$ was also periodic (corresponding to a fluctuating environment) was given by B. D. Coleman, Y. Hsieh, and G. P. Knowles in [2]. See also [7] for a more simplified analysis including harvesting.

EXAMPLE 2. The following model for populations of the North American spruce budworm was given by D. Ludwig, D. D. Jones, and C. S. Holling [3]:

$$\dot{x}=rx\left(1-\frac{x}{K}\right)-\frac{\beta x^2}{\alpha^2+x^2}, \quad r, K, \alpha, \beta>0. \tag{3}$$

The second term on the right side of equation (3) models predation by birds and in the absence of predation, the growth is assumed to be logistic. If additional periodic harvesting (say, due to seasonal spraying) were to occur, then the equation would be of the form (1).

In this case

$$g''(x)=-2\left[\frac{r}{K}+\beta\alpha^2\frac{\alpha^2-3x^2}{(\alpha^2+x^2)^3}\right].$$

For $x \geqslant 0, g''(x)$ will be negative if $r/K-\beta/4\alpha^2>0$, and there will be at most two periodic solutions. For appropriate values of the constants there will be a stable equilibrium point x_0 satisfying $K/2<x_0<K$ when there is no periodic harvesting—this can be easily seen by graphing the two expressions comprising the right hand side of the differential equation. Under small amplitude periodic harvesting, the equilibrium point will become a periodic solution.

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The Maximum Brightness of Venus

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Even a casual observer may notice that the planet Venus, at times the dominant object in the evening sky, appears noticeably brighter at some times than at others. While reading a book on popular astronomy [4] one day, I came across the statement (with little explanation) that Venus is at its brightest when the illuminated portion of the apparent disk of the planet is 28% of the whole. Could that fact, I wondered, be determined theoretically? If you reflect for a moment, as I did, on the relative positions of the Earth, Venus and the Sun at various times, you should be able to see that, apart from the observer's local conditions, the brightness of Venus depends primarily on two factors: the apparent size of its disk as seen from Earth, and the fraction of that disk which is illuminated by the Sun's light. Furthermore, those two factors work in opposition: when Venus is closest to Earth, so that its apparent size is greatest, it is then between the Sun and the Earth, so that its illuminated hemisphere is turned away from us; when Venus is fully illuminated from our vantage point, it is on the far side of the Sun and its apparent size is at a minimum (see FIGURE 1). "Aha!" I thought to myself, "a perfect first-year Calculus maximum/minimum problem!" Indeed, I subsequently used the problem as an end-of-the-term group modeling project, as described in [**11**].

Before solving this problem, let me state it in astronomical terminology. Consider the Earthward-facing hemisphere of a planet or moon. As seen from the Earth, that hemisphere appears as a disk. The fraction of the area of that disk which is illuminated by the Sun's light at a given moment is called the **phase**. The **elongation** of a planet viewed from Earth is the size of the angle Sun-Earth-planet. In books and magazines on popular astronomy (e.g., [**9**]), tables giving phases and elongations are common. Thus our problem may be stated as follows:

> *Find the phase and elongation of Venus at the moment it reaches maximum brightness.*

I will begin by assuming that Venus and Earth move in circular orbits with the Sun at the center of each circle. Of course a planet actually moves in what is essentially an ellipse with the Sun at one focus. But the eccentricity of the Earth's orbit is only .017, and that of Venus' is .007. Thus, for example, the difference between the semi-major and semi-minor axes of the Earth's orbit is only about 13,000 miles out of some 93,000,000—an excellent approximation to a circle! The Sun is about one and a half million miles from the center of this near circle, but that is still less than 2% of the semi-major axis.

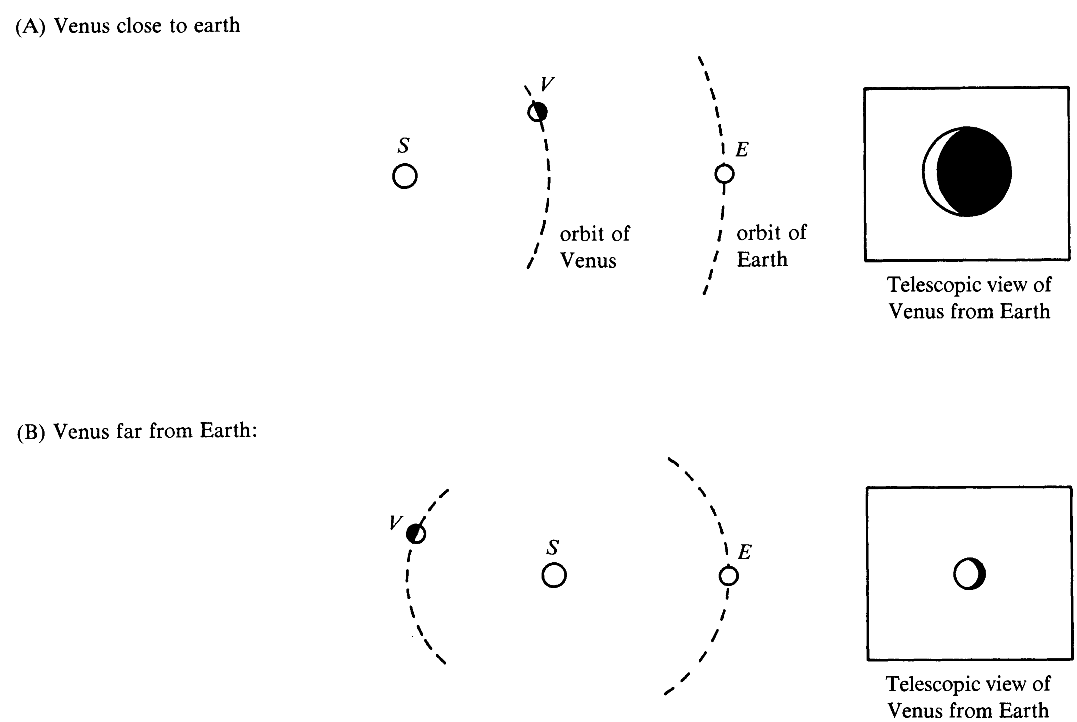

FIGURE 1. **Highly schematic views of the relative positions of Earth (E), Venus (V), and the Sun (S) as seen from above the plane of the Earth's orbit. Representations of the appearance of Venus as viewed from Earth in a telescope appear at the right.**

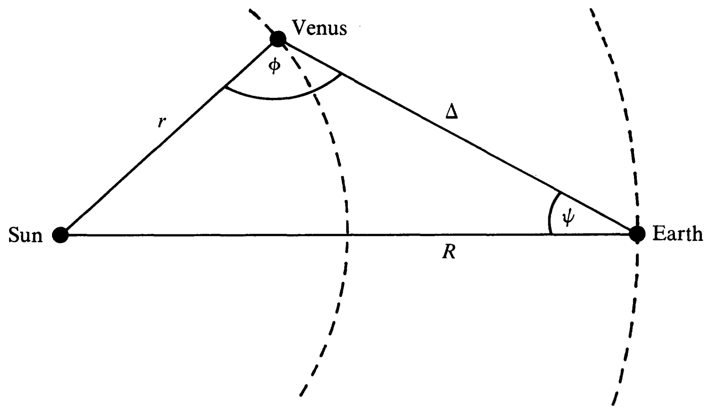

FIGURE 2. **Highly schematic view from above the plane of the solar system.**

I will also assume that the orbits of the planets are coplanar. The orbital plane of Venus is actually inclined by $3°$ to that of the Earth. One consequence of this is that since the angular size of the Sun as seen from the Earth is $\frac{1}{2}°$, Venus rarely passes directly in front of (a **transit**) or behind (an **occultation**) the Sun. For our purposes, however, the very slight inclination makes extremely little difference in the distance between Earth and Venus. For example, should Venus happen to be at its maximum height above the Earth's orbital plane at the same moment that it is at its closest approach, then the difference between its actual distance and its distance calculated as if it were in the same plane is only about 250,000 miles out of some 24 million, i.e., about 1%. Thus we may safely assume that Venus and Earth orbit in the same plane.

Let R denote the (fixed, by assumption) distance from the Earth to the Sun, r the distance from Venus to the Sun, and Δ the distance from the Earth to Venus (see FIGURE 2). Since the apparent diameter of Venus is essentially inversely proportional to Δ, the apparent area of Venus, and hence its brightness (or luminosity), is inversely proportional to Δ^2. Thus if L denotes the luminosity of Venus and p that planet's phase, then the remarks in the first paragraph may be expressed as

$$L = k\frac{p}{\Delta^2},$$

where k is a constant of proportionality.

If we could now obtain Δ as a function of p, we could then differentiate L with respect to p and obtain the desired result, namely, the value of p which maximizes L. However, Δ turns out to be a rather ugly function of p, so instead we will obtain p as a function of Δ and first maximize L with respect to Δ.

The curve which marks the boundary of the lit and unlit portions of Venus (or of any planet or moon) is called the **terminator**. The terminator is always a great circle. (By the way, Venus is virtually a perfect sphere, as are Mercury and the Moon.) As viewed from Earth, though, the terminator is seen projected onto a plane perpendicular to the line from Earth to Venus and thus appears as half of an ellipse. In FIGURE 3, this projection is represented by the arc NAS. The angle Sun-Venus-Earth, which is denoted by ϕ in FIGURE 2, is the angle FCC' in FIGURE 4. Then the angle $A'CA$ in FIGURE 4 is $\pi - \phi$ and so $AC = \rho\cos(\pi - \phi)$, where ρ is the radius of Venus.

Now the area of the half-ellipse $NASCN$ (FIGURE 3) is

$$\frac{1}{2}\pi\rho(AC) = \frac{1}{2}\pi\rho^2\cos(\pi - \phi),$$

so the area of the crescent $NDSAN$ is

$$\frac{1}{2}\pi\rho^2 - \frac{1}{2}\pi\rho^2\cos(\pi - \phi) = \frac{1}{2}\pi\rho^2(1 + \cos\phi).$$

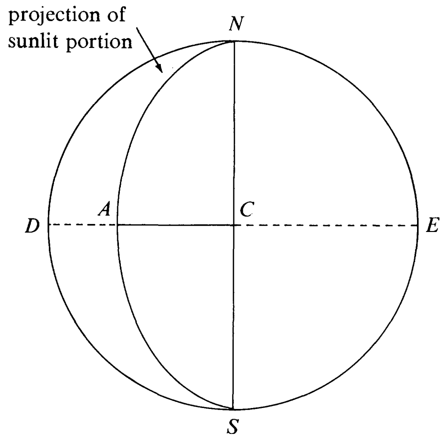

FIGURE 3. **"View" from Earth. Cross-section of Venus outlined by the great circle perpendicular to the line from Earth to Venus. The arc NAS is the projection of the terminator on this cross-section. The points A and C are in the interior of Venus and not actually visible from Earth. The points D, A, C, and E appear again in** FIGURE **4.**

Thus the phase of Venus is given by

$$p = \frac{\frac{1}{2}\pi\rho^2(1+\cos\phi)}{\pi\rho^2} = \frac{1}{2}(1+\cos\phi).$$

I have derived the expression for p for a "crescent" Venus ($\phi > \pi/2$). The reader should check that the expression is still valid for a "gibbous" Venus ($\phi < \pi/2$) even though the derivation has to be modified slightly.

To get p as a function of Δ, apply the law of cosines to the triangle Sun-Venus-Earth (see FIGURE 2) to get

$$R^2 = \Delta^2 + r^2 - 2r\Delta\cos\phi,$$

so

$$\cos\phi = \frac{\Delta^2 + r^2 - R^2}{2r\Delta}$$

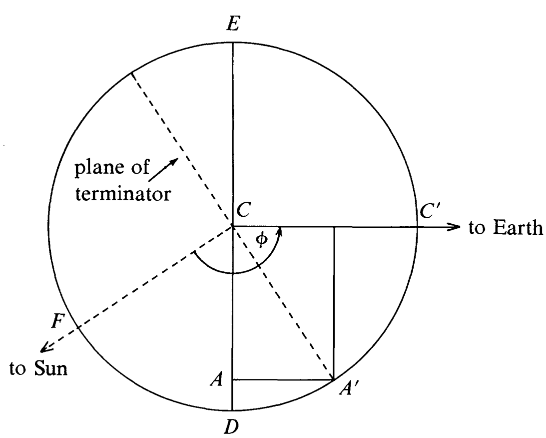

FIGURE 4. **"View" from above. Cross-section of Venus outlined by the great circle parallel to the plane of the Earth's orbit. The line DCE (which is interior to the planet and not actually visible to an observer) is the intersection of the plane of** FIGURE **3 with this cross-section.**

and

$$p=\frac{1}{2}(1+\cos\phi)=\frac{2r\Delta+\Delta^2+r^2-R^2}{4r\Delta}.$$

We can now express the luminosity of Venus as a function of Δ as follows:

$$L=\frac{kp}{\Delta^2}=K\frac{2r\Delta+\Delta^2+r^2-R^2}{\Delta^3},$$

where K is the constant $k/4r$. Then

$$\frac{dL}{d\Delta}=-\frac{K}{\Delta^4}\left(4r\Delta+\Delta^2+3(r^2-R^2)\right)$$

and so $dL/d\Delta=0$ if and only if $\Delta^2+4r\Delta+3(r^2-R^2)=0$. The roots of the quadratic are

$$\Delta=-2r\pm\sqrt{r^2+3R^2}.$$

The negative value for Δ is clearly physically unacceptable. Thus the value $\Delta_M=-2r+\sqrt{r^2+3R^2}$ is the only candidate for a relative extreme for L as a function of Δ. It is not hard to see (by, for example, factoring the quadratic) that the sign of $dL/d\Delta$ is opposite to that of the factor $\Delta-\sqrt{r^2+3R^2}+2r$. Thus L is an increasing function for $\Delta<\Delta_M$ and decreasing for $\Delta>\Delta_M$, and so Δ_M is indeed a relative maximum.

Note that the only physically acceptable values for Δ are those in the interval $[|R-r|, R+r]$. Does Δ_M fall in this interval? It is a straightforward exercise in inequalities to show that if $R>r$, then $\Delta_M>R-r$, but that $\Delta_M<R+r$ if and only if $r/R>1/4$. If $r/R>1$ (i.e., $r>R$), then $\Delta_M<0$. For the case of observing Venus from Earth, r/R does satisfy $1/4<r/R<1$, so Δ_M is in $[R-r, R+r]$.

We should also like the model developed here to be applicable to other planets, e.g., Venus viewed from Jupiter, Mars viewed from Earth, etc. In general, then, R would represent the distance from the Sun to the observer's planet and r the distance from the Sun to the planet under observation. In some cases, e.g., observing Venus from Jupiter, $r/R<1/4$. In such a case $\Delta_M>R+r$, so L increases throughout the interval $[R-r, R+r]$. The extreme values of L then occur, of course, at the endpoints of the interval. On the other hand, observing Mars from Earth gives $r>R$. In this case $\Delta_M<0<r-R$, so L decreases steadily throughout the interval $[r-R, r+R]$.

I will summarize below the outcome of our model, making use of some further astronomical terminology. A planet is called **inferior** (**superior**) if it is closer to (further from) the Sun than the observer's planet. When a superior planet is, from the observer's perspective, 180° away from the Sun it is said to be at **opposition** (see FIGURE 5). When it is again lined up with the Sun and the observer but beyond the Sun, it is in **conjunction**. An inferior planet will also line up with the Sun and the observer in two different configurations. When it is on the near side of the Sun it is in **inferior conjunction**; when it is beyond the Sun it is in **superior conjunction**.

We may then summarize our results as follows:

If $R>r$ *and* $r/R>1/4$, *then the maximum brightness of the observed planet occurs at* $\Delta_M=-2r+\sqrt{r^2+3R^2}$. *Minimum values of brightness occur at the endpoints of the interval* $[R-r, R+r]$, *i.e., at inferior and superior conjunction.*

If $r/R<1/4$, *then the brightness of the observed planet increases steadily from a minimum at* $\Delta=R-r$ *(inferior conjunction) to a maximum at* $\Delta=R+r$ *(superior conjunction).*

If $r>R$, *then the brightness of the observed planet decreases steadily from a maximum at* $\Delta=r-R$ *(opposition) to a minimum at* $\Delta=r+R$ *(conjunction).*

The summary statements above refer to the time interval from inferior to superior conjunction for an inferior planet or from opposition to conjunction for a superior planet. The variation in

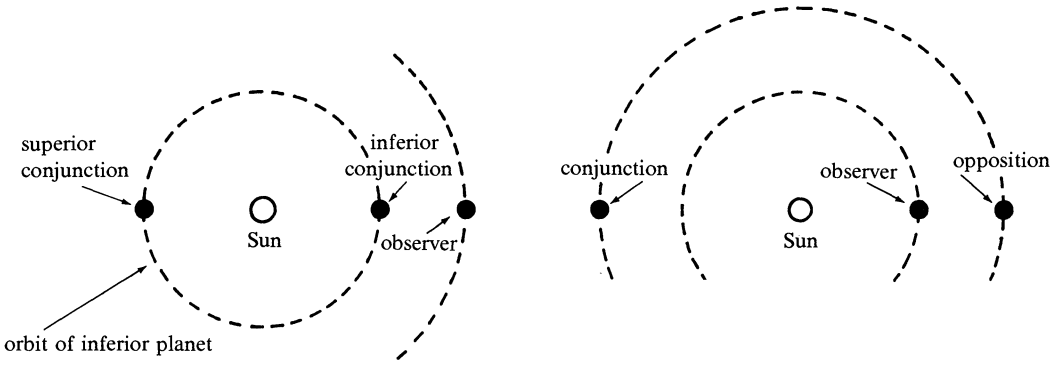

FIGURE 5. **At left, an inferior planet is shown in its two possible alignments (*inferior* and *superior conjunction*) with the line from the observer through the sun. The corresponding alignments (*opposition* and *conjunction*) for a superior planet are shown at right.**

brightness is then of course reversed as the planets shift until they are once again in the original configuration.

How well does this model correspond to observation? For the case of observing Venus ($r = 6.72 \times 10^7$ miles) from Earth ($R = 9.29 \times 10^7$ miles), certainly $1/4 < r/R < 1$, so the calculated value $\Delta_M = 4.00 \times 10^7$ is a relative maximum for L. This value for Δ_M yields $p = 26.6\%$, which is in very good agreement with the figure of 28% quoted at the beginning of this article.

The elongation of Venus (ψ in FIGURE 2) can be calculated from Δ by again using the law of cosines:

$$\cos\psi = \frac{\Delta^2 + R^2 - r^2}{2\Delta R}.$$

For $\Delta = \Delta_M = 4.00 \times 10^7$, the angle ψ is 39.7°. Also, by using the law of sines, ψ can be calculated directly from the phase p as

$$\psi = \arcsin\left(\frac{r}{R}\sin(\arccos(2p-1))\right). \tag{1}$$

To test this model further, I have used data from the *American Ephemeris* [**5**, pp. 4 and 369] for the phase of Venus on some selected dates to compute ψ according to equation (1). TABLE 1 shows the excellent agreement between these computed values and the actual values listed in the *Ephemeris*.

Note that phase and elongation are purely geometric concepts, i.e., they have nothing to do with brightness. The excellent agreement shown in TABLE 1 between the actual and the calculated elongation suggests that even the small difference between our calculated value of 26.6% for the phase of Venus at maximum brightness and the quoted value of 28% is *not* accounted for by our

1977 Date	Phase	ψ actual	ψ calculated from (1)
Jan. 17	.550	47	46.03
Feb. 1	.471	47	46.23
Feb. 16	.376	45	44.49
Feb. 26	.301	42	41.57
Mar. 3	.259	39	39.33
Mar. 8	.214	36	36.39
Mar. 18	.121	28	28.15
Apr. 2	.016	10	10.46

TABLE 1

simplifying geometric assumptions. Have we made some other assumption about brightness that could account for the difference? Yes, in fact, we made an unstated assumption about the *uniformity* of the brightness of the disk of Venus.

When we claim that the luminosity L is proportional to the phase p, we are assuming that if a particular portion of Venus is fully lit during two different phases then it appears equally as bright at each of those times. This is, unfortunately, not the case. Actually, over-all brightness varies non-linearly with the phase (even assuming that distance remains fixed) in a complicated way that depends markedly on the reflective properties of the planet or moon under observation. (By the way, making this non-linear relation explicit for the case of an ideal sphere might be a good project for students in a multivariable calculus class.) For example, when the Earth's moon appears half lit (at First or Last Quarter) it is not half as bright as when it is full but rather only one-eleventh as bright! [**8**, p. 36]. However, the non-linearity of the effect is substantially mitigated on Venus because it is the thick cloud cover and not that planet's surface which reflects sunlight. Thus when the phase is 26.6%, Venus is not quite as bright as our model indicates, but the actual maximum brightness, which must be determined empirically, occurs when the phase is only very slightly greater.

Interestingly, astronomers *define* [**6**, p. 209] the **brilliancy** of Venus as the quantity

$$\frac{2r\Delta + \Delta^2 + r^2 - R^2}{r^3\Delta^3}.$$

This is, of course, essentially what I used for the brightness L. Even though this is a purely theoretical quantity, tables of celestial phenomena actually list the moment of greatest brilliancy, not of greatest brightness (see [**7**, p. 29], [**9**, p. 39]).

I have shown that our model fits the data very well for the case of observing Venus from Earth, two planets with low orbital eccentricity. But what about Mercury, with an eccentricity of .206, or even Mars with an eccentricity of .098? (The figure .098 may seem low, but even before the advent of the telescope Kepler was able to deduce that planets move in ellipses rather than circles by studying the orbit of Mars.) The expressions derived here for phase, elongation and brilliancy will still be accurate if, instead of using values for r and R representing the *mean* distance of those planets from the Sun, values are used which more closely approximate the true values for those planets in the configurations under consideration. Using simple analytic geometry and some knowledge of astronomical coordinate systems, it is not too hard (and it would be a good project for students) to make such approximations for r and R (see for example [**1**]; for a shortcut, see [**3**, p. 297ff]). Even without such refinements, the model still gives a good qualitative account of the variation in brightness of the planet under observation (except that in the case of Mercury, the non-linear variation of brightness with respect to phase has a significant effect).

It did not surprise me that astronomers have long known about the simple model I had worked out on my own to find out whether the phase of Venus at maximum brightness could be calculated theoretically. What did surprise me was that this simple model is actually in very good agreement with observation. I would imagine there are other astronomical phenomena that are amenable to analysis with the level of mathematics used in this note. Indeed, elementary calculus is used at several junctures in texts on positional astronomy [**2**], [**10**].

Once, astronomy and mathematics were practiced by the same people. One sees little evidence of this in today's calculus texts. The instructor who is willing to look at the astronomical literature, or better yet, work out some models for him or herself, most likely will be rewarded with interesting material for use in introductory calculus courses.

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The Circumdisk and its Relation to a Theorem of Kirszbraun and Valentine

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In this note we consider the following problem: *Given a finite set of points in euclidean m-space, characterize the radius R of the smallest disk (closed solid sphere) which contains those points.* We believe that our solution to this problem is new, characterizing R in terms of a well-known quadratic form. Moreover, it provides a new proof of the very appealing theorem of Kirszbraun-Valentine: *If a collection of disks (of varying radii) in* E^m *having nonempty intersection are rearranged so that corresponding distances between centers do not increase, then the rearranged collection also has nonempty intersection.* Whether or not the volume of the intersection can decrease remains a problem which baffles mathematicians.

We first need to review some of the basic ideas that will be required. Let $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n$, $n \geqslant 1$, be a collection of distinct points in euclidean space E^m. Among all those disks (closed solid spheres) which contain these points, there is a unique disk of smallest radius, called the **circumdisk**. In FIGURE 1 we give a simple but very useful example. Here $m = 2$, $n = 2$, and the unbroken circle with diameter $|\mathbf{p}_1 - \mathbf{p}_2|$ bounds the circumdisk.

The existence of a minimal containing disk follows from the Blaschke selection theorem [**12**, p. 37]. However, those readers familiar with sequential compactness in E^m can easily concoct a proof of existence. Such a disk is unique, for if there were two distinct minimal disks, centered at $\mathbf{u}$ and $\mathbf{u}'$, respectively, and having radius R, then a disk of radius $\sqrt{R^2 - \frac{1}{4}|\mathbf{u} - \mathbf{u}'|^2}$ centered at $\frac{1}{2}(\mathbf{u} + \mathbf{u}')$ would also contain all the points $\mathbf{p}_i$.

Some writers use the word circumsphere instead of circumdisk. However, we wish to reserve the former as a term for a generalization of the notion **circumcircle**. Thus, if $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n$ are points in E^m, we define the **circumsphere** to be the sphere of least radius on which all the points lie, provided such a sphere exists. For example, three collinear points have no circumsphere. In FIGURE 1, observe that $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$ lie on a circumsphere (the circle determined by the dashed arc)

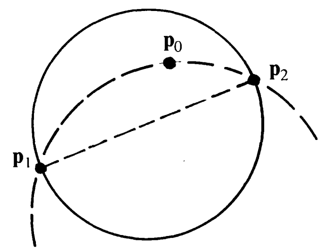

FIGURE 1