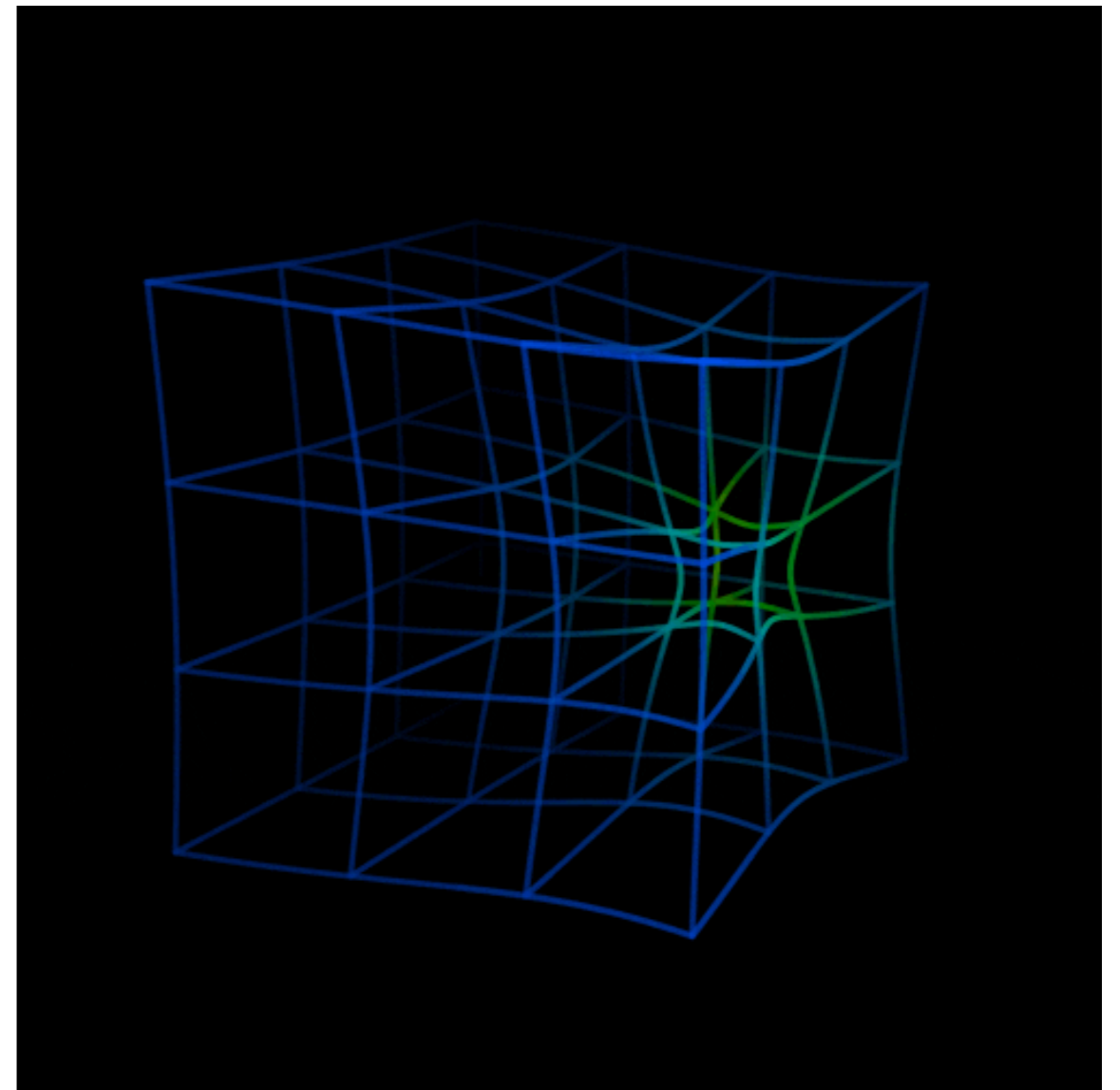


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# Relativity

- ⚡ Lorentz invariance
- ⚡ Vectors and tensors
- ⚡ Lorentz transformations
- ⚡ Relativity, Electricity and Magnetism



# Relativity and the wave equation

- As we saw earlier, Maxwell's Equations can be reduced to a very simple set of equations for the potentials (in the Lorentz gauge):

$$-\square \phi = \frac{\rho}{\epsilon} \quad , \quad \text{and} \quad -\square \vec{A} = \mu \vec{J}$$

where the D'Alembertian is  $\square = -\frac{1}{c_s^2} \frac{\partial^2}{\partial t^2} + \nabla^2$  .

- The D'Alembertian can in fact be written in terms of the **Minkowski metric**,  $\eta_{\mu\nu} = \text{diag}\{-1, 1, 1, 1\}$  , as:

$$\square = -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2 = \eta_{\mu\nu} \partial_\mu \partial_\nu$$

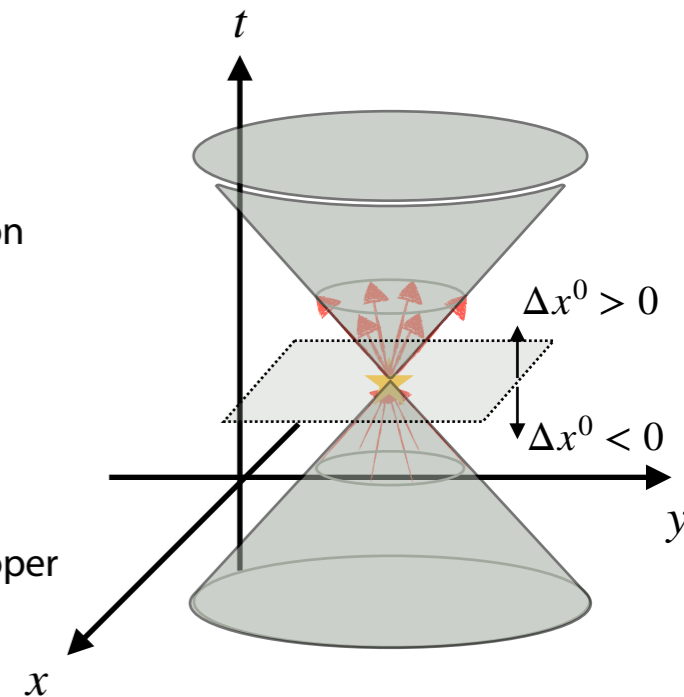
where  $x^\mu = \{ct, \vec{x}\}$  ,  $\partial_\mu = \{\partial/\partial(ct), \vec{\nabla}\}$  .

- As a result of using this "covariant notation", many of our expressions are also further simplified, e.g. Green's function becomes ( $\Delta x^\mu = \{ct - ct', \vec{x} - \vec{x}'\}$ ):

$$G(x^\mu; x'^\mu) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik_\mu \Delta x^\mu}}{||k||^2 + m^2} = \frac{\theta(\Delta x^0)}{2\pi} \delta(\Delta \tau^2) \quad ,$$

where  $k_\mu \Delta x^\mu = \eta_{\mu\nu} k^\mu \Delta x^\nu = -\omega c \Delta t + \vec{k} \cdot \Delta \vec{x}$  , and  $\Delta \tau^2 = -\eta_{\mu\nu} \Delta x^\mu \Delta x^\nu = c^2 \Delta t^2 - \Delta \vec{x}^2$  is the proper time (invariant) interval between two events, which also determines the **light cone**.

Einstein sum convention:  
repeated indices are  
assumed to be summed  
 $\sum_\mu A_{\mu\nu} B^{\mu\alpha} \rightarrow A_{\mu\nu} B^{\mu\alpha}$



---

# Relativity

- At this point it is clear that we must start taking relativity into account into everything that we do regarding Electrodynamics. Here is a brief review.
- First, let's remember the notion of a **4-vector**:

"Regular" notation  $V^\mu = \{V^0, V^1, V^2, V^3\}$

"Dual" notation :  $V_\mu = \eta_{\mu\nu} V^\nu = \{-V^0, V^1, V^2, V^3\}$

"2xDual" notation :  $V^\mu = \eta^{\mu\nu} V_\nu$

- The 4-norm of a vector is then written as:

$$||V||^2 = \eta_{\mu\nu} V^\mu V^\nu = \eta^{\mu\nu} V_\mu V_\nu = V^\mu V_\mu = V_\mu V^\mu$$

Notice that, depending on the vector, we could have  $||V||^2 > 0$  (space-like vector) ,  $||V||^2 < 0$  (time-like vector) or  $||V||^2 = 0$  (light-like, or null, vector).

- The inverse of the Minkowski metric,  $\eta^{\mu\nu}$  , is identical to the Minkowski metric:

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \text{diag}\{-1, 1, 1, 1\} \quad , \quad \eta^{\mu\alpha} \eta_{\alpha\nu} = \delta_\nu^\mu$$

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

# Relativity

- Special Relativity is built around the notion of invariance: the invariance of the 4-norms of 4-vectors.
- E.g., consider the light cone, which is defined by the condition that  $\Delta\tau^2 = -\eta_{\mu\nu}\Delta x^\mu\Delta x^\nu = 0$  (a null vector). The statement, then, is that in any reference frame this 4-norm is zero — in other words: if two events are on the light cone of each other in a coordinate frame, they are on the light cone in all frames.
- A general, linear coordinate transformation will change the components of a 4-vector as:

$$V^\mu \longrightarrow V'^\mu = \Lambda^\mu_\nu V^\nu$$

where the transformation matrix  $\Lambda^\mu_\nu$  is a function only of things like the rotation between the two frames, or a relative velocity between the two frames.

- In particular, for a pure rotation,

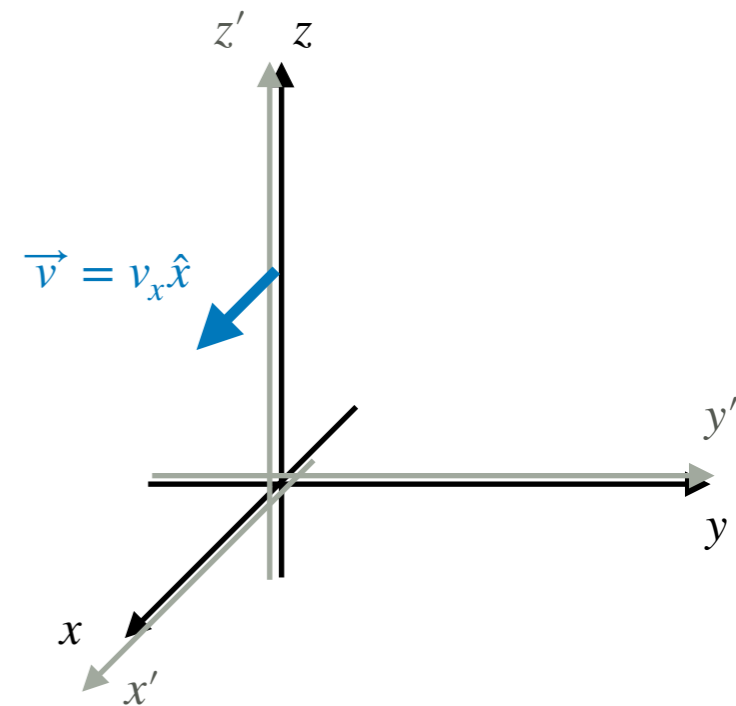
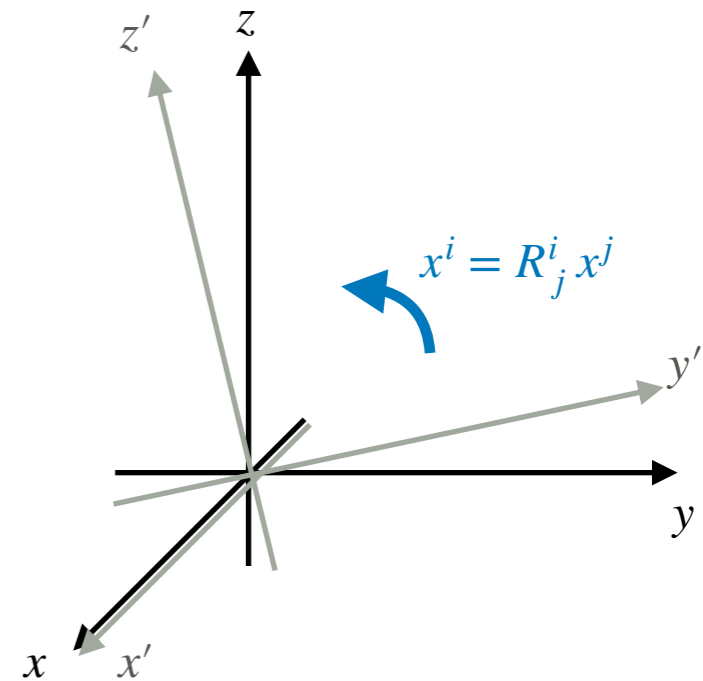
$$\Lambda^\mu_\nu \rightarrow \Lambda^0_0 = 1, \quad \Lambda^i_0 = \Lambda^0_1 = 0, \quad \Lambda^i_j = R^i_j \quad \text{where } R^i_j \text{ is the Euler matrix.}$$

- For a pure "boost" in the x-direction we have the usual Lorentz transformations:

$$\Lambda^\mu_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where  $\beta = v/c$ ,  $\gamma = (1 - \beta^2)^{-1/2}$ . You can check that, with this matrix, you will get that:

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu \quad \Rightarrow \quad ct' = \gamma(ct - \beta x), \quad x' = \gamma(x - \beta ct)$$



# Lorentz invariance

- The invariance of the "norms" in this 1+3-dimensional "spacetime" is at the core of Modern Physics, and has been verified experimentally with astonishing precision (see, e.g., [https://en.wikipedia.org/wiki/Modern\\_searches\\_for\\_Lorentz\\_violation](https://en.wikipedia.org/wiki/Modern_searches_for_Lorentz_violation))

- Let's see what this means for a general coordinate transformation. We have that a change of coordinate frame leads to:

$$V^\mu \longrightarrow V'^\mu = \Lambda^\mu_\nu V^\nu$$

- But the norm of that object is invariant, i.e.,

$$||V||^2 = ||V'||^2 \Rightarrow \eta_{\mu\nu} V^\mu V^\nu = \eta_{\mu\nu} V'^\mu V'^\nu$$

$$\eta_{\mu\nu} V'^\mu V'^\nu = \eta_{\mu\nu} (\Lambda^\mu_\alpha V^\alpha) (\Lambda^\nu_\beta V^\beta) = \eta_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta V^\alpha V^\beta = ||V||^2$$

- Now, remember that the indices inside these sums are arbitrary, we can always write:

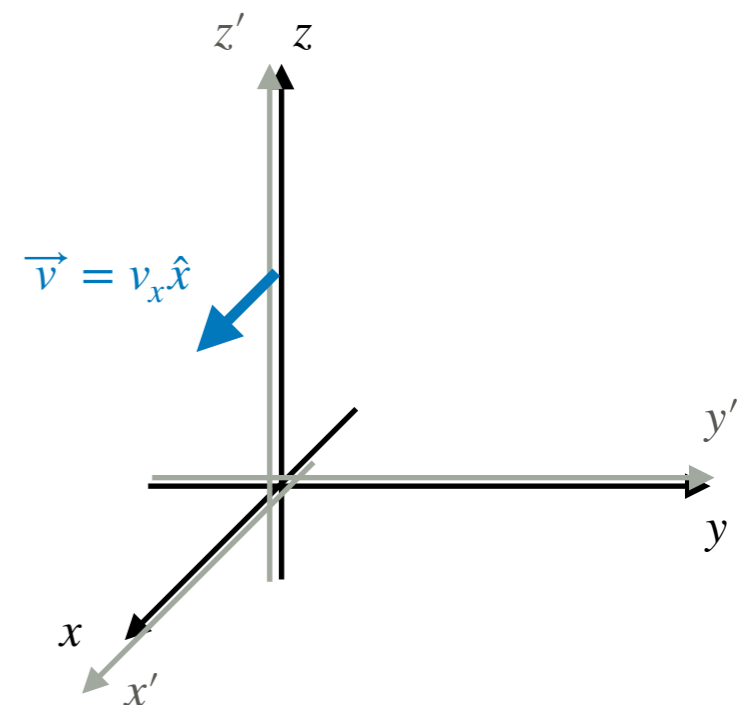
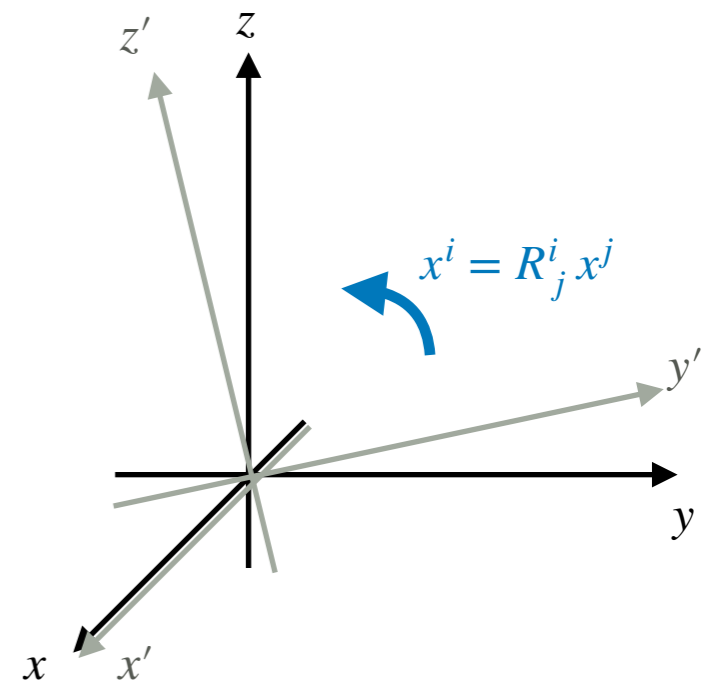
$$||V||^2 = \eta_{\mu\nu} V^\mu V^\nu = \eta_{\alpha\beta} V'^\alpha V'^\beta, \text{ etc.}$$

- Therefore, we can rephrase the equation above as:

$$\eta_{\mu\nu} V'^\mu V'^\nu = \eta_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta V^\alpha V^\beta = \eta_{\alpha\beta} V^\alpha V^\beta$$

In other words,

$$\eta_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta = \eta_{\alpha\beta} \quad \text{which we could write in matrix notation as } \Lambda^t \cdot \eta \cdot \Lambda = \eta$$



---

# Minkowski tensors

- Vectors in Minkowski space (4-vectors) transform in the usual way:

$$V'^{\mu} = \Lambda^{\mu}_{\nu} V^{\nu}$$

- We can conceive of objects ("tensors") such as  $S^{\mu\nu} = U^{\mu}P^{\nu}$ , which transform as:

$$S^{\mu\nu} \rightarrow S'^{\mu\nu} = \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} S^{\alpha\beta}$$

- We can also think about objects such as  $T_{\mu\nu} = \partial_{\mu}A_{\nu}$ , which transform as:

$$T_{\mu\nu} \rightarrow T'_{\mu\nu} = \Lambda^{\alpha}_{\mu} \Lambda^{\beta}_{\nu} T_{\alpha\beta}$$

- And let's not forget about scalars, such as  $q = \partial_{\mu}V^{\mu}$ , which do not transform at all:

$$q \rightarrow q' \stackrel{!}{=} q$$

- Notice, however, that the transformation matrix  $\Lambda^{\mu}_{\nu}$  is not a tensor: while the tensor components change under a coordinate transformation, the matrix  $\Lambda^{\mu}_{\nu}$  is the transformation itself, so it is doesn't act on itself!

---

# Manipulating Minkowski tensors

- Some 4-vectors are intrinsically "normal" (e.g.,  $x^\mu$ ,  $\Delta x^\mu$ ), and some others are intrinsically "dual", e.g.:

$$\frac{\partial}{\partial x^\mu} \equiv \partial_\mu$$

- OK, sure, this is a rather peculiar "vector": but in order to make it more "real", just make it act on some function, like:

$$A_\mu = \partial_\mu \phi \quad \text{or} \quad \partial_\mu x^\mu = \partial_0 x^0 + \partial_1 x^1 + \partial_2 x^2 + \partial_3 x^3 = 4$$

In this latter case we get something which is obviously invariant.

- Let's check how this object changes under a Lorentz transformation:

$$\frac{\partial}{\partial x^\mu} \rightarrow \frac{\partial}{\partial x'^\mu} = \frac{\partial}{\Lambda^\mu_\nu \partial x^\nu} = (\Lambda^\mu_\nu)^{-1} \frac{\partial}{\partial x^\nu}$$



- But this is just the transformation for the dual vectors:

$$(\Lambda^\mu_\nu)^{-1} = \Lambda^\nu_\mu \quad \Rightarrow \quad \partial'_\mu = \Lambda^\nu_\mu \partial_\nu \quad \Lambda(\text{dual}) = \Lambda^{-1} = \Lambda(v \rightarrow -v, R \rightarrow R^{\text{tr}})$$

# Manipulating Minkowski tensors

- OK, But how about those indices up or down? And how do we compare that with the matrix notation?
- The following "tips" are simply a tool to help you visualize the operations, as you grow more familiar with tensor notation.
- Let's start by recalling how we operate with regular matrices, in Euclidean space, when  $V = V(\text{dual})$ :

$$A \cdot B = C \quad \Leftrightarrow \quad A_{ij} \cdot B_{jk} = C_{ik} \quad \Leftrightarrow \quad \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

- $A_{1\blacksquare} B_{\blacksquare 1} + A_{1\blacksquare} B_{\blacksquare 1} = C_{11}$   

- $A_{1\blacksquare} B_{\blacksquare 2} + A_{1\blacksquare} B_{\blacksquare 2} = C_{12}$   


IN MATRIX NOTATION, THE FIRST INDEX IS FOR THE LINES, AND THE SECOND IS FOR THE COLUMNS



---

# Manipulating Minkowski tensors

- OK, but what about the correspondence with the tensor notation?
- First, bear in mind that you must always know if your vector is a "normal" or a "dual" vector, since they behave differently.
- In tensor notation, we denote this distinction by showing the indices on the top for "normal" vectors ( $dr \rightarrow dr^\mu$ ), and on the bottom for dual vectors ( $\partial \leftrightarrow \partial_\mu$ )
- It is now possible to make an *analogy* in terms of the lines and columns of matrix notation.

(1) Associate normal vectors to the **column vectors** of matrix notation (i.e., the index denotes the *lines* of the column vector);

(2) Associate dual vectors to **line vectors** in matrix notation (i.e., the index denotes the columns inside the line vector)

$$dr^\mu = \begin{pmatrix} dr^0 \\ dr^1 \\ dr^2 \\ dr^3 \end{pmatrix}$$

$$\partial_\mu = (\partial_0, \partial_1, \partial_2, \partial_3)$$

---

# Manipulating Minkowski tensors

- It is useful to think of the Minkowski metric as an **operator**, which not only generates the norm of a vector, but that takes a regular (column) vector into a dual (line) vector, and vice-versa
- Notice that in matrix algebra, this exchange of lines/columns is the **transpose**:
- $dr_\mu = \eta_{\mu\nu} dr^\nu \quad \leftrightarrow \quad dr(\text{dual}) = (\eta \cdot dr)^{\text{tr}}$
- Or, equivalently:
- $dr^\mu = \eta^{\mu\nu} dr_\nu \quad \leftrightarrow \quad dr = [dr(\text{dual}) \cdot \eta]^{\text{tr}}$

# Manipulating Minkowski tensors

- Some operations are easy that way, e.g.:

$$dr'^{\mu} = \Lambda^{\mu}_{\nu} dr^{\nu} \Leftrightarrow \begin{pmatrix} dr'^0 \\ dr'^1 \\ dr'^2 \\ dr'^3 \end{pmatrix} = \begin{pmatrix} \Lambda^0_0 & \Lambda^0_1 & \Lambda^0_2 & \Lambda^0_3 \\ \Lambda^1_0 & \Lambda^1_1 & \Lambda^1_2 & \Lambda^1_3 \\ \Lambda^2_0 & \Lambda^2_1 & \Lambda^2_2 & \Lambda^2_3 \\ \Lambda^3_0 & \Lambda^3_1 & \Lambda^3_2 & \Lambda^3_3 \end{pmatrix} \cdot \begin{pmatrix} dr^0 \\ dr^1 \\ dr^2 \\ dr^3 \end{pmatrix}$$

- Now consider the dual vector, which transforms under the dual transformation:

$$\partial'_{\mu} = \Lambda^{\nu}_{\mu} \partial_{\nu}$$

- Note that, in matrix notation, the **first** index is usually for the lines, and the **second** is usually for the columns. Well, we have a bit of an ambiguity there, right? If in tensor notation the upper index is for the lines, and the lower one for the columns, how do we operate on the matrix  $\Lambda^{\nu}_{\mu}$ ?
- First, bear in mind that in true **tensorial notation** (where we only deal with indices), **there is no ambiguity**. When in doubt, go back to the tensor notation!
- In order to proceed with our analogy, let's remember that, with matrices, the **transpose** exchanges lines  $\leftrightarrow$  columns .

# Manipulating Minkowski tensors

- Using the transpose we can write:

- $\partial'_\mu = \partial'_\nu \Lambda_\mu^\nu$ , which in matrix notation is  $\partial'_\mu = \partial_\nu (\Lambda_\mu^\nu)^{\text{tr}} \leftrightarrow \partial' = \partial \cdot \Lambda^{\text{tr(dual)}}$

$$\partial'_\mu = \Lambda_\mu^\nu \partial_\nu \leftrightarrow (\partial'_0, \partial'_1, \partial'_2, \partial'_3) = (\partial_0, \partial_1, \partial_2, \partial_3) \cdot \begin{pmatrix} \Lambda_0^0 & \Lambda_0^1 & \Lambda_0^2 & \Lambda_0^3 \\ \Lambda_1^0 & \Lambda_1^1 & \Lambda_1^2 & \Lambda_1^3 \\ \Lambda_2^0 & \Lambda_2^1 & \Lambda_2^2 & \Lambda_2^3 \\ \Lambda_3^0 & \Lambda_3^1 & \Lambda_3^2 & \Lambda_3^3 \end{pmatrix}^{\text{tr}}$$

- This can also be written as:

$$\partial'_\mu = (\partial'_0, \partial'_1, \partial'_2, \partial'_3) = (\partial_0, \partial_1, \partial_2, \partial_3) \cdot \begin{pmatrix} \Lambda_0^0 & \Lambda_1^0 & \Lambda_2^0 & \Lambda_3^0 \\ \Lambda_0^1 & \Lambda_1^1 & \Lambda_2^1 & \Lambda_3^1 \\ \Lambda_0^2 & \Lambda_1^2 & \Lambda_2^2 & \Lambda_3^2 \\ \Lambda_0^3 & \Lambda_1^3 & \Lambda_2^3 & \Lambda_3^3 \end{pmatrix}$$

---

# Manipulating Minkowski tensors

- Some other simple examples using this analogy:

(a) Two Lorentz transformations, one on top of the other:

$$dr''^\mu = \Lambda^\mu_\nu dr'^\nu = \Lambda^\mu_\nu \tilde{\Lambda}^\nu_\alpha dr^\alpha \quad \leftrightarrow \quad dr'' = \Lambda \cdot dr' = \Lambda \cdot \tilde{\Lambda} \cdot dr$$

(b) Lorentz transformation for the dual:

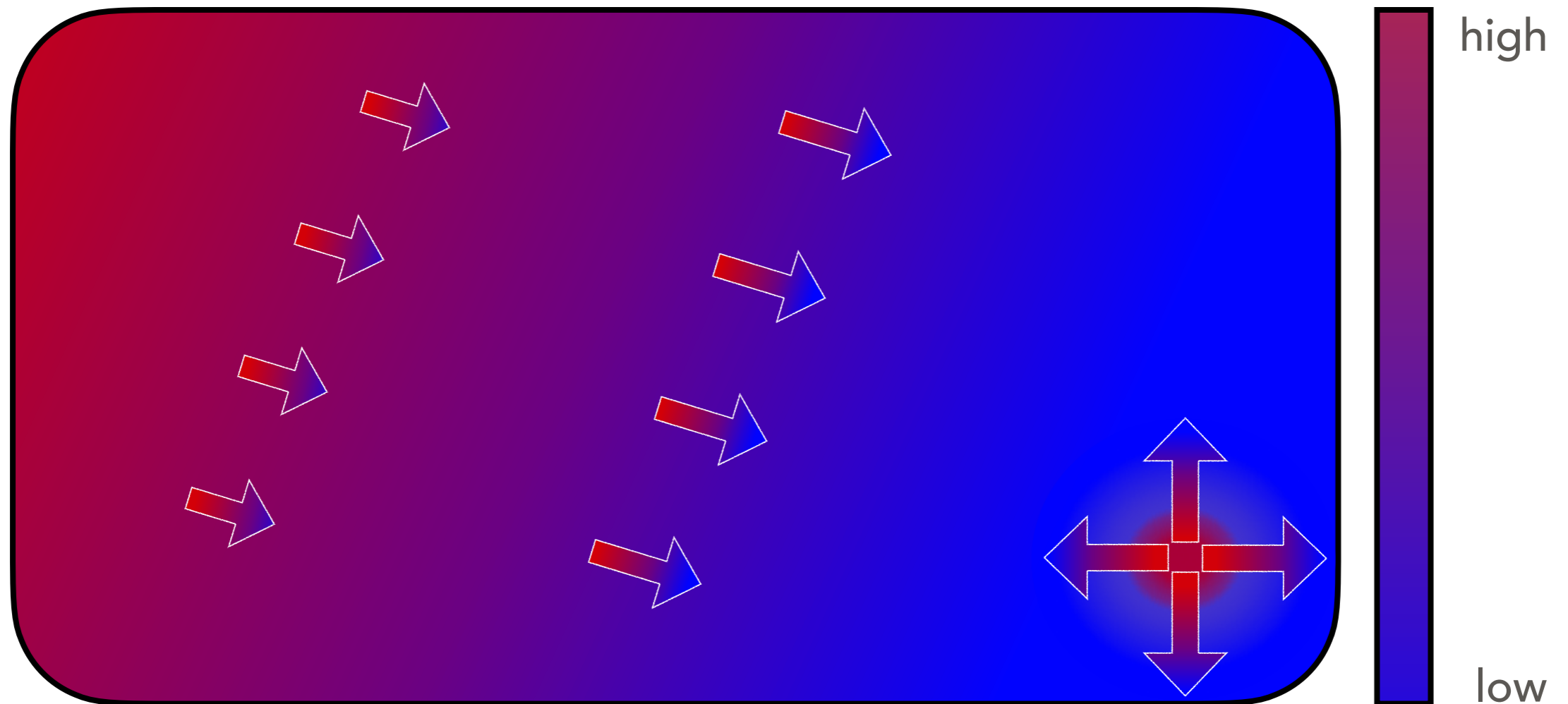
$$A_\mu = \Lambda^\nu_\mu A_\nu \quad \leftrightarrow \quad A' = A \cdot \Lambda^{\text{tr}}(\text{dual})$$

(c) Lorentz invariance:

$$\Lambda^\mu_\sigma \eta_{\mu\nu} \Lambda^\nu_\alpha = \eta_{\sigma\alpha} \quad \leftrightarrow \quad \Lambda^{\text{tr}} \cdot \eta \cdot \Lambda = \eta$$

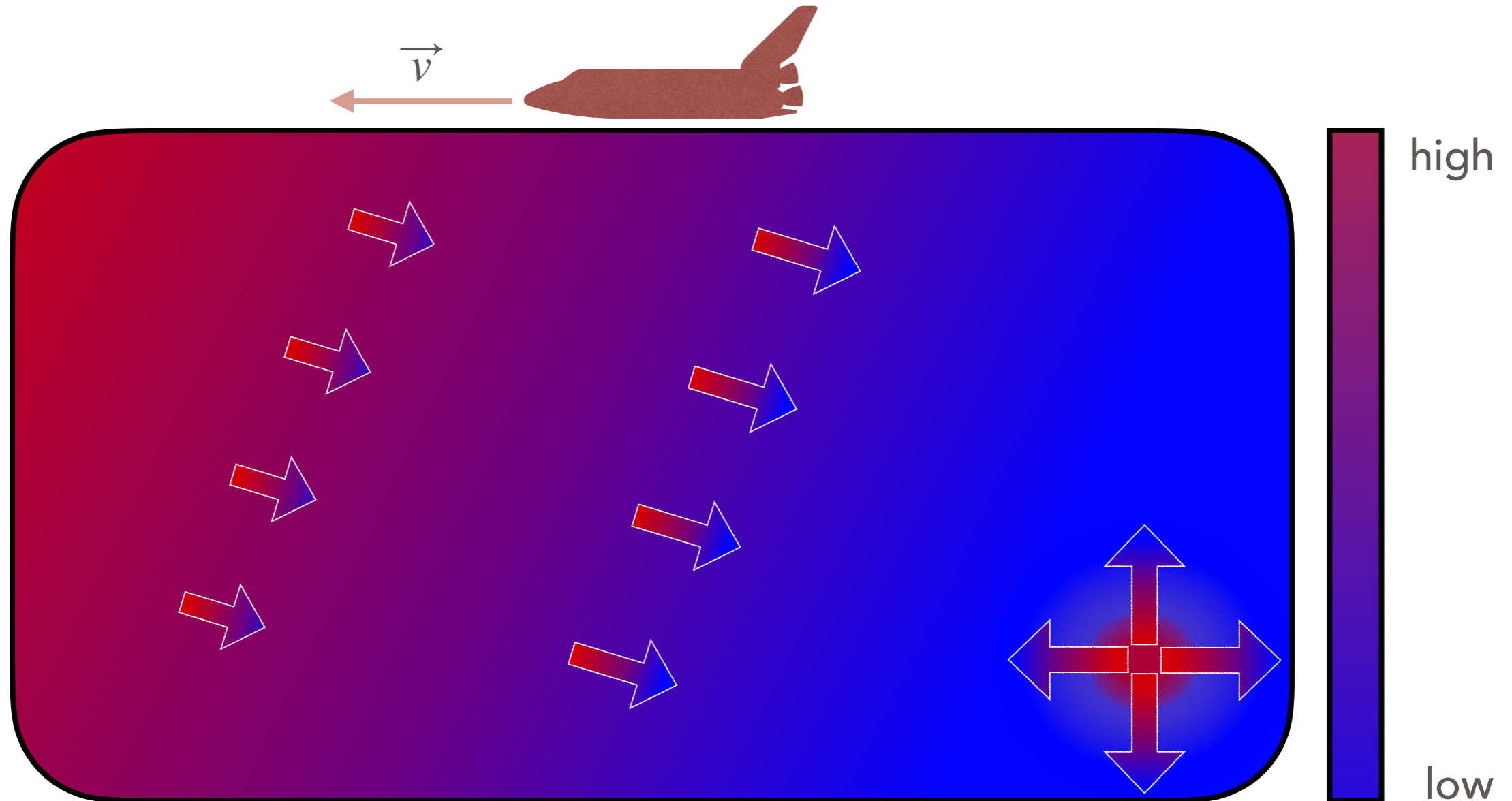
# Currents and densities

- We can think of mass or charge distributions in terms of *densities* and *currents*:



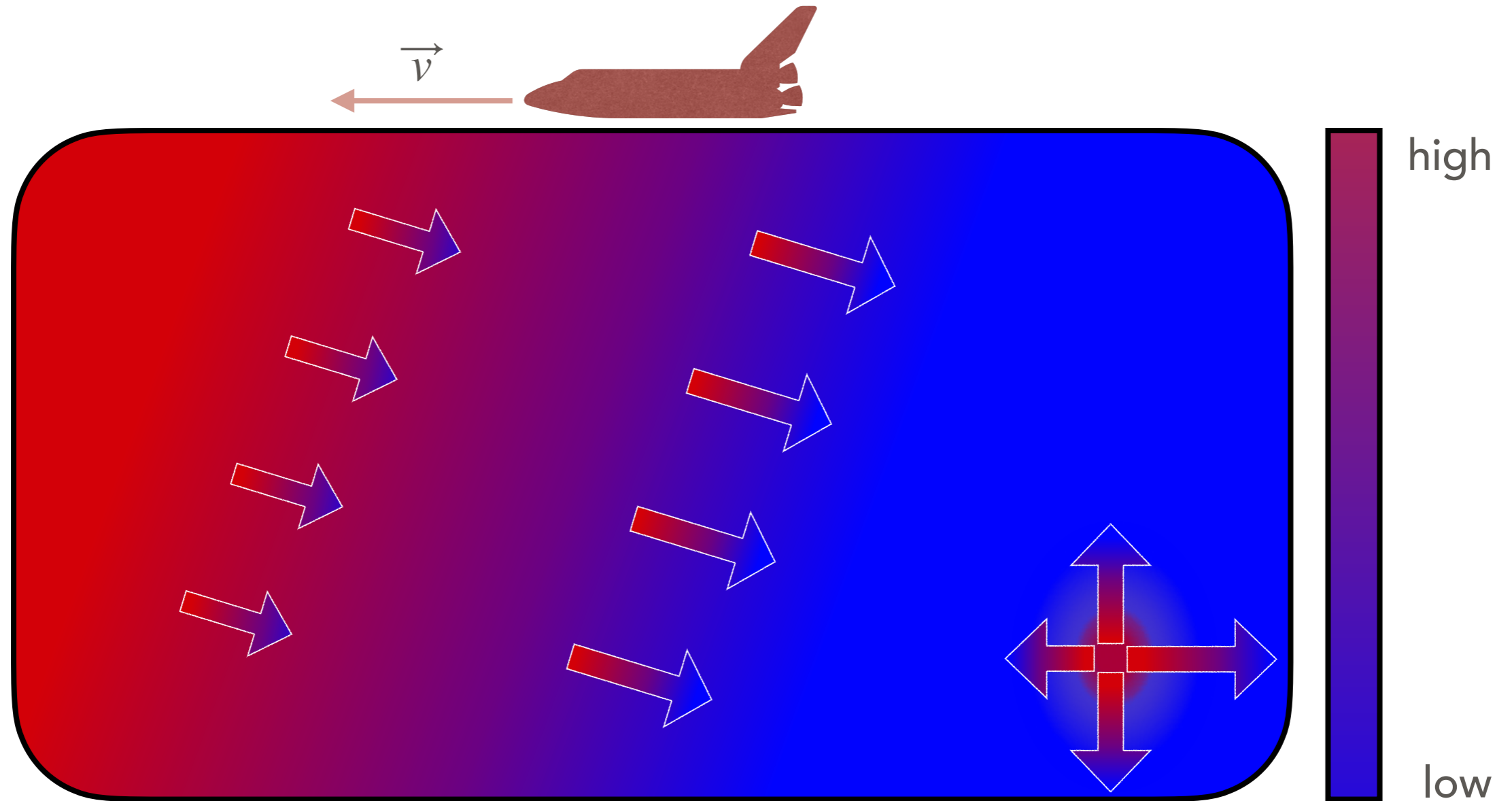
# Currents and densities

- But how do they transform between different frames?



# Currents and densities

- But how do they transform between different frames?





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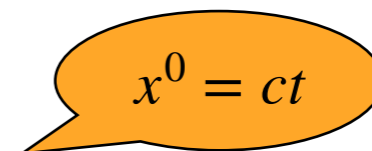
# Currents and densities

- Let's start by establishing the relationship between the two (densities and currents) — the continuity equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \quad , \text{ or better still: } \partial_t \rho + \partial_i J^i = 0$$

- We should immediately realize that the equation above can be recast in the following way:

$$\partial_\mu J^\mu = 0 \quad ,$$


$$x^0 = ct$$

where the 4-current is given by:

$$J^\mu = \{ \rho c, \vec{J} \}$$

# Currents and densities

- The 4-current is a true 4-vector, just like:

$$\text{4-velocity} \quad U^\mu = \frac{dr^\mu}{d\tau} = \gamma(V)\{c, \vec{V}\} \quad \left[ d\tau = \frac{1}{\gamma} dt \right]$$

$$\text{4-momentum} \quad P^\mu = mU^\mu = \{E/c, \vec{P}\}$$

$$\text{4-current} \quad J^\mu = \{\rho c, \vec{J}\}$$

- With that clear, the continuity equation becomes even more simple and elegant:

$$\partial_\mu J^\mu = 0$$

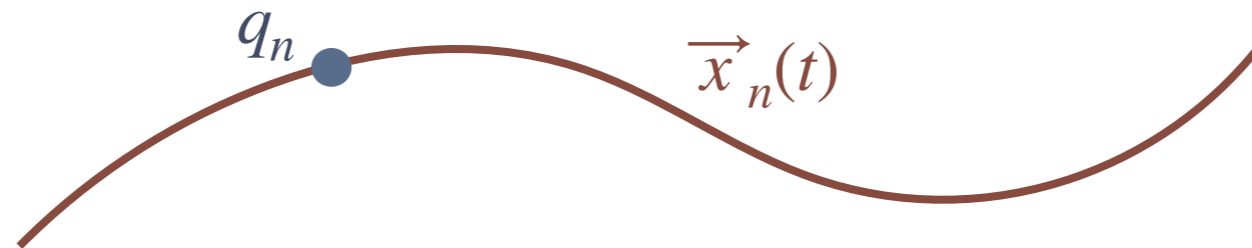
- Notice that we could even write this equation in the rather contrived way:

$$\partial^\mu J_\mu = 0, \text{ where } \partial^\mu = \eta^{\mu\nu} \partial_\nu = \{-\partial_0, \vec{\nabla}\} \text{ e } J_\mu = \{-\rho c, \vec{J}\}$$

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# Currents and densities

- In a more concrete way, we can think of charge densities and currents as *distributions*:

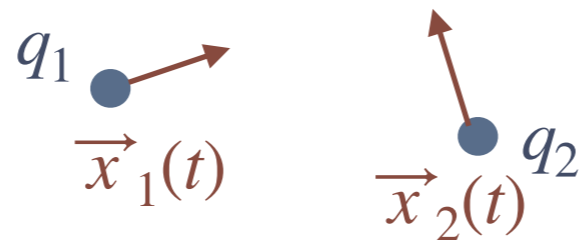


Density:  $\rho_n = q_n \delta^3[\vec{x} - \vec{x}_n(t)] = q_n \frac{dt}{dt} \delta^3[\vec{x} - \vec{x}_n(t)]$

Current:  $\vec{J}_n = q_n \vec{v}_n \delta^3[\vec{x} - \vec{x}_n(t)] = q_n \frac{d\vec{x}_n}{dt} \delta^3[\vec{x} - \vec{x}_n(t)]$

# Currents and densities

- A charge distribution can be expressed in such a way that the kinematic properties are manifestly relativistic:



$$\rho_n = q_n \frac{dt}{dt} \delta^3[\vec{x} - \vec{x}_n(t)] \quad , \quad \vec{J}_n = q_n \frac{d\vec{x}_n}{dt} \delta^3[\vec{x} - \vec{x}_n(t)]$$

$$\Rightarrow J_n^\mu = q_n \frac{dx_n^\mu}{dt} \delta^3[\vec{x} - \vec{x}_n(t)]$$

- Exercise 1:** show that we can write the 4-current as:

$$J_n^\mu(x) = \int d\tau q_n U_n^\mu \delta^4[x - x_n(\tau)] \quad , \quad \text{where for simplicity } x \rightarrow x^\alpha \quad , \quad x_n \rightarrow x_n^\alpha$$

---

# Back to Electrodynamics

- OK, now let's try to work our way back to Electrodynamics, but using our acquired knowledge about relativity, and how to work with its objects (vectors, tensors etc.)
- In vacuum, the Maxwell equations are (in *Gauss units* 😊):

$$\vec{\nabla} \times \vec{B} - \frac{1}{c} \partial_t \vec{E} = \frac{4\pi}{c} \vec{J}_q$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi \rho_q$$

$$\vec{\nabla} \times \vec{E} + \frac{1}{c} \partial_t \vec{B} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

# Back to Electrodynamics

- Let's consider the problem of an infinite wire with a linear charge density, and its electric field:

S: observer at rest

$$\rho_q = \lambda \delta(x) \delta(y) \quad \lambda = \frac{dq}{dz}$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi \rho_q \Rightarrow \int_{S(V)} d\vec{S} \cdot \vec{E} = 4\pi q_V$$

$$\vec{E} = E_\rho \hat{\rho}$$

$$d\vec{S} = 2\pi \rho dz \hat{\rho}$$

$$\Rightarrow 2\pi \rho L E_\rho = 4\pi \lambda L \quad \Rightarrow \vec{E} = \frac{2\lambda}{\rho} \hat{\rho}$$

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# Back to Electrodynamics

- This same wire, as seen by an observer in motion parallel to the wire ( $S'$ ):

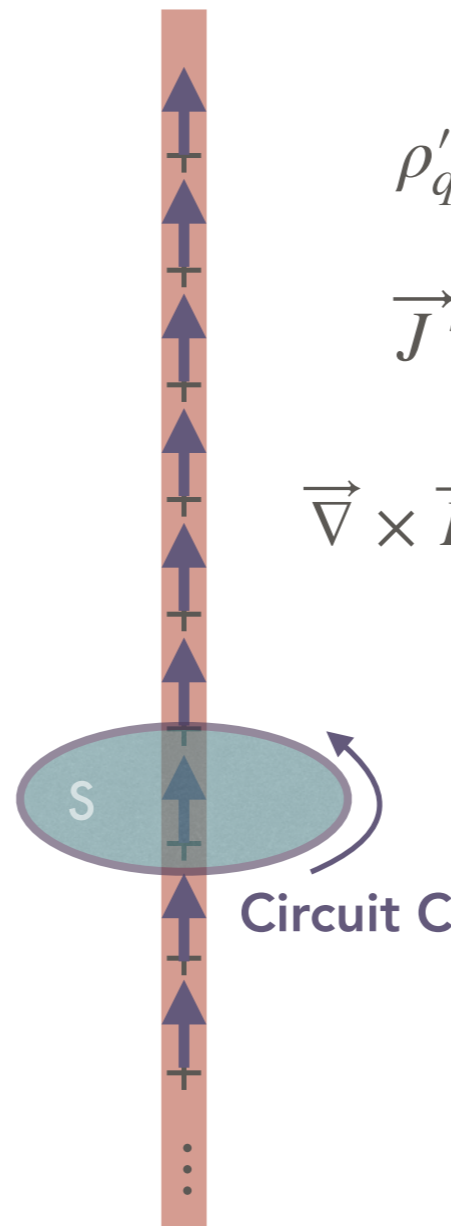
$S'$ : observer  
in motion



# Back to Electrodynamics

- This same wire, as seen by an observer in motion parallel to the wire ( $S'$ ):

$S'$ : observer  
in motion



$$\rho'_q = \lambda' \delta(x) \delta(y) \quad \lambda' = \frac{dq}{dz'}$$

$$\vec{J}' = \lambda' v \delta(x) \delta(y) \hat{z} \quad I' = \lambda' v$$

$$\vec{\nabla} \times \vec{B}' = \frac{4\pi}{c} \vec{J}' \quad \Rightarrow \quad \oint_{C(S)} d\vec{l} \cdot \vec{B}' = \frac{4\pi}{c} I'$$


$$\Rightarrow \vec{B}' = \frac{2I'}{c\rho} \hat{\phi}$$


$$\Rightarrow \vec{E}' = \frac{2\lambda'}{\rho} \hat{\rho}$$



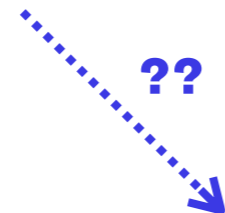
# Back to Electrodynamics

- But the two frames are just related by a Lorentz transformation — so, the fields should also be related by something like a coordinate transformation...

S   $\vec{E} \neq 0, \vec{B} = 0$

S'   $\vec{E}' \neq 0, \vec{B}' \neq 0$

$v$



The fields  $\vec{E}$  and  $\vec{B}$  are 3D vectors, and so are  $\vec{E}'$  and  $\vec{B}'$ .

But they are *different fields!*

How can we **connect the two** in the same transformation??

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# Next class:

- Relativistic electrodynamics
- The Faraday tensor
- Jackson, Ch. 11; Zangwill, Ch. 22; your favorite Special Relativity book.
- (See also Bo Thidé's book: <http://docente.unife.it/guido.zavattini/allegati/251023059-electromagnetic-field-theory-bo-thide.pdf> , Ch. 5)