Pasta 16

Pasta / Nº cópias

Gabarito - 2ª lista de Modelos Lineares

1

S/F

1) $(\underset{\sim}{X} - \underset{\sim}{\mu}) \sim N_n(\underset{\sim}{0}, V)$

$\underset{\sim}{X}' A \underset{\sim}{X} \sim \chi^2_{R, \lambda} \Rightarrow \begin{cases} AV \text{ é idempotente } (*) \\ R = r(A) \end{cases}$

Como $(\underset{\sim}{X} - \underset{\sim}{\mu}) \sim N_n(\underset{\sim}{0}, V)$, $(\underset{\sim}{X} - \underset{\sim}{\mu})' A (X - \underset{\sim}{\mu}) \sim \chi^2$

$\Longleftrightarrow$ AV é idempotente, o que vale $(*)$.

nº de graus de liberdade : $r(A) = R$

Parâmetro de Não centralidade : $\frac{1}{2} \underset{\sim}{\mu}' A \underset{\sim}{\mu} =$

$= \frac{1}{2} \underset{\sim}{0}' A \underset{\sim}{0} = 0$

$\therefore (\underset{\sim}{X} - \underset{\sim}{\mu})' A (\underset{\sim}{X} - \underset{\sim}{\mu}) \sim \chi^2_R$.

2) Como $Var(\underset{\sim}{X}) = I$ $Var(\underset{\sim}{Y}) = I$ então $Cov(\underset{\sim}{X}, \underset{\sim}{Y}) = R$

$\therefore E(\underset{\sim}{X}' A \underset{\sim}{Y}) = tr(AR') + \underset{\sim}{\mu}_1' A \underset{\sim}{\mu}_2$

3) $X_1 \sim N(0, 2)$

4) $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N_2 \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \right)$

5) $\begin{bmatrix} X_0 \\ \hline X_1 \\ X_2 \end{bmatrix} \quad \begin{bmatrix} 0 \\ \hline 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \underset{\sim}{\mu}_1 \\ \hline \underset{\sim}{\mu}_2 \end{bmatrix} \quad \left[\begin{array}{c|cc} 4 & 1 & 0 \\ \hline 1 & 2 & 1 \\ 0 & 1 & 3 \end{array}\right] = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22} \end{bmatrix}$

a distribuição condicional de X_0 dado $X_1 = x_1$ e $X_2 = x_2$

é $N_1(\underset{\sim}{\mu}_c, \Sigma_c)$

$$\underset{\sim}{\mu}_c = \underset{\sim}{\mu}_1 + \Sigma_{12} \Sigma_{22}^{-1} \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \underset{\sim}{\mu}_2 \right)$$

$$= 0 + [1 \;\; 0] \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \frac{1}{5} [1 \;\; 0] \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{5}(3x_1 - x_2)$$

$$\Sigma_c = \Sigma_{11} - \Sigma_{12}\, \Sigma_{22}^{-1}\, \Sigma_{21}$$

$$= 4 - \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{17}{5}$$

$$\therefore\ X_0 \mid X_1 = x_1, X_2 = x_2 \quad \sim N\left(\frac{3x_1 - x_2}{5}, \frac{17}{5} \right)$$

6) $$\rho_{12} = \frac{Cov(X_1, X_2)}{\sqrt{Var(X_1)}\,\sqrt{Var(X_2)}} = \frac{1}{\sqrt{2}\,\sqrt{3}} = \frac{1}{\sqrt{6}}$$

$$\rho_{01} = \frac{Cov(X_0, X_1)}{\sqrt{Var(X_0)}\,\sqrt{Var(X_1)}} = \frac{1}{2\sqrt{2}}$$

$$\rho_{02} = 0$$

7) $\rho_{01|2}$ - coeficiente de correlação parcial entre X_0 e X_1 fixado X_2

$$\left[\begin{array}{cc|c} 4 & 1 & 0 \\ 1 & 2 & 1 \\ \hline 0 & 1 & 3 \end{array}\right] \qquad \Sigma_{11} = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \qquad \Sigma_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Sigma_{22} = 3$$

$$\Sigma_{11} - \Sigma_{12}\,\Sigma_{22}^{-1}\,\Sigma_{21} = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & 5/3 \end{bmatrix}$$

$$\rho_{01|2} = \frac{\sigma_{01\cdot 2}}{\sqrt{\sigma_{00\cdot 2}\,\sigma_{11\cdot 2}}} = \frac{1}{\sqrt{4}\,\sqrt{5/3}} = 0{,}387$$

8) $Z = 4x_0 - 6X_1 + X_2 = \begin{bmatrix} 4 & -6 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = A\,\underset{\sim}{X}$

$\therefore Z \sim N(\mu, \sigma^2)$ $\quad \mu = \begin{bmatrix} 4 & -6 & 1 \end{bmatrix} \underset{\sim}{0} = 0$

$$\sigma^2 = \begin{bmatrix} 4 & -6 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ -6 \\ 1 \end{bmatrix} = 79$$

$\therefore Z \sim N(0, 79)$

9) $\sigma_{11|2} = \dfrac{5}{3}$

10) $Z_1 = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}$ $\qquad Z_2 = \begin{bmatrix} 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}$

$$\mathrm{Cov}(Z_1, Z_2) = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \Sigma \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = 2$$

11) $\underset{\sim}{Y} \sim N_n(\underset{\sim}{\mu}, \Sigma) \quad r(\Sigma) = n$

Basta provar que $A\Sigma$ é idempotente.

Pós multiplicando por Σ

Σ in. gen. $A \Rightarrow A\Sigma A = A \Rightarrow A\Sigma A\Sigma = A\Sigma \Rightarrow$

$A\Sigma$ é idempotente.

12) Tomando $A = \Sigma^{-1}$, no Teorema 1,

$A\Sigma = \Sigma^{-1}\Sigma = I$, que é idempotente.

$r(A) = r(\Sigma^{-1}) = r(\Sigma) = n$

$\lambda = \frac{1}{2}\underset{\sim}{\mu}' A \underset{\sim}{\mu} = \frac{1}{2}\underset{\sim}{\mu}'\Sigma^{-1}\underset{\sim}{\mu}$

$\therefore \underset{\sim}{Y}'\Sigma^{-1}\underset{\sim}{Y} \sim \chi^2_{n,\ \frac{1}{2}\underset{\sim}{\mu}'\Sigma^{-1}\underset{\sim}{\mu}}$

13) $\underset{\sim}{\mu} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

$$\frac{1}{2}\left[t_1^2\sigma_{11} + t_2^2\sigma_{22} + t_3^2\sigma_{33} + 2t_1t_2\sigma_{12} + 2t_1t_3\sigma_{13} + 2t_2t_3\sigma_{23}\right] =$$

$$t_1^2 + \frac{1}{2}t_2^2 + 2t_3^2 - \frac{1}{2}t_1t_2 - t_1t_3$$

$\sigma_{11} = 2 \quad \sigma_{22} = 1 \quad \sigma_{33} = 4 \quad \sigma_{12} = -\frac{1}{2} \quad \sigma_{13} = -1 \quad \sigma_{23} = 0$

$$\therefore \Sigma = \begin{bmatrix} 2 & -1/2 & -1 \\ -1/2 & 1 & 0 \\ -1 & 0 & 4 \end{bmatrix}$$

$$2X_1 - 3X_2 + X_3 \sim N(7, 23)$$

$$P(2X_1 - 3X_2 + X_3 > c) = P\left(Z > \frac{c-7}{\sqrt{23}}\right) = 0{,}95$$

$Z \sim N(0,1)$

$$\frac{c-7}{\sqrt{23}} = -1{,}64 \qquad c = 0{,}865$$

14) Vamos inicialmente obter a distribuição conjunta de $(X_1, \bar{X})' = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} X_1 \\ \bar{X} \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 & -1/2 & -1 \\ -1/2 & 1 & 0 \\ -1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1/3 \\ 0 & 1/3 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 2 & 1/6 \\ 1/6 & 4/9 \end{bmatrix}$$

$$\begin{pmatrix} X_1 \\ \bar{X} \end{pmatrix} \sim N_2\left(\begin{bmatrix} 1 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2 & 1/6 \\ 1/6 & 4/9 \end{bmatrix} \right)$$

$$X_1 \mid \bar{X} = 6 \sim N_1(\mu, \sigma^2)$$

$$\mu = \underset{\sim}{\mu}_1 + \Sigma_{12} \Sigma_{22}^{-1} (6 - \underset{\sim}{\mu}_2)$$

$$= 1 + \frac{1}{6} \cdot \frac{9}{4} \left(6 - \frac{2}{3}\right) = 3$$

$$\sigma^2 = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} = 2 - \frac{1}{6} \cdot \frac{9}{4} \cdot \frac{1}{6} = \frac{31}{16}$$

15) $E(\underset{\sim}{Y}' A \underset{\sim}{Y}) = tr(\sigma^2 A) + \underset{\sim}{\mu}' A \underset{\sim}{\mu} =$

$= \sigma^2 tr(A) + \underset{\sim}{\mu}' A \underset{\sim}{\mu}$.

$Var(\underset{\sim}{Y}' A \underset{\sim}{Y}) = 2\ tr\left((\sigma^2 A)^2 \right) + 4 \underset{\sim}{\mu}' A \sigma^2 I A \underset{\sim}{\mu}$

$= 2\sigma^4 tr(A^2) + 4\sigma^2 \underset{\sim}{\mu}' A^2 \underset{\sim}{\mu}$

16) $m_{\underset{\sim}{Y}}(\underset{\sim}{t}) = E\left(e^{\underset{\sim}{t}'\underset{\sim}{Y}}\right) = E\left(e^{\underset{\sim}{t}' B \underset{\sim}{X} + \underset{\sim}{t}'\underset{\sim}{b}}\right)$

$= e^{\underset{\sim}{t}'\underset{\sim}{b}}\, E\left(e^{\underset{1\times q}{\underset{\sim}{t}'} \underset{q\times p}{B} \underset{\sim}{X}}\right) = e^{\underset{\sim}{t}'\underset{\sim}{b}} \exp\left[\underbrace{\underset{\sim}{t}' B}_{1\times p} \mu + \frac{1}{2}\underset{\sim}{t}' B \right.$

$\left. \Sigma B' \underset{\sim}{t} \right] = \exp\left[\underset{\sim}{t}' \left[\underbrace{B \underset{\sim}{\mu}}_{\underbrace{q\times p\ p\times 1}_{q\times 1}} + \underbrace{\underset{\sim}{b}}_{q\times 1}\right] + \frac{1}{2}\underset{\sim}{t}' B \Sigma B' \underset{\sim}{t}\right]$

$\therefore \underset{\sim}{Y} \sim N_q\left(B\underset{\sim}{\mu} + \underset{\sim}{b},\ B\Sigma B'\right)$

17) $E(\cancel{y}|x) = -1 + \frac{1}{2}x$

$$= \mu_y - \frac{\sigma_{XY}}{\sigma_{XX}}\mu_x + \frac{\sigma_{XY}}{\sigma_{XX}}x$$

$$= 3 - \frac{\sigma_{XY}}{4} \cdot 8 + \frac{\sigma_{XY}}{4}x$$

$$\frac{\sigma_{XY}}{4} = \frac{1}{2} \Rightarrow \sigma_{XY} = 2$$

$$\mathrm{Var}(Y|x) = \sigma_{YY} - \frac{(\sigma_{XY})^2}{\sigma_{XX}} = 2 - \frac{2^2}{4} = 1$$

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2}$$

18) Dist de $\begin{bmatrix} X \\ U+V \end{bmatrix}$ $\qquad X = A\underset{\sim}{Y} \quad U = B\underset{\sim}{Y} \quad V = C\underset{\sim}{Y}$

$$\begin{bmatrix} X \\ U+V \end{bmatrix} = \begin{bmatrix} A_{r\times n} \\ \underbrace{B+C}_{r\times n} \end{bmatrix} \underset{\sim}{Y}_{n\times 1} = \begin{bmatrix} A\underset{\sim}{Y} \\ (B+C)\underset{\sim}{Y} \end{bmatrix}_{2r\times 1} = \underset{\sim}{Z}$$

$$m_{\underset{\sim}{Z}}(\underset{\sim}{t}) = E\left[e^{(\underset{\sim}{t}_1' \; \underset{\sim}{t}_2') \begin{bmatrix} A\underset{\sim}{Y} \\ (B+C)\underset{\sim}{Y} \end{bmatrix}} \right]$$

$$\downarrow$$

$$\underset{\sim}{t}' = (\underset{1\times r}{\underset{\sim}{t}_1'} \quad \underset{1\times r}{\underset{\sim}{t}_2'})$$

$$= E\left[e^{\underset{\sim}{t}_1' A \underset{\sim}{Y} + \underset{\sim}{t}_2'(B+C)\underset{\sim}{Y}} \right] = E\left[e^{[\underset{\sim}{t}_1' A + \underset{\sim}{t}_2'(B+C)]\underset{\sim}{Y}} \right]$$

$$= \exp\left[[\underset{\sim}{t}_1' A + \underset{\sim}{t}_2'(B+C)]\underset{\sim}{\mu} + \frac{1}{2}[\underset{\sim}{t}_1' A + \underset{\sim}{t}_2'(B+C)] I [A'\underset{\sim}{t}_1 + (B+C)'\underset{\sim}{t}_2] \right]$$

$$\cancel{\therefore \underset{\sim}{Z} \sim N_{2r}(\underset{\sim}{\mu}^*, \Sigma^*)}$$

$$\cancel{\mu^* / A} \quad = \exp\left[(\underset{\sim}{t}_1' \; \underset{\sim}{t}_2') \begin{bmatrix} A\underset{\sim}{\mu} \\ (B+C)\underset{\sim}{\mu} \end{bmatrix} + \frac{1}{2} (\underset{\sim}{t}_1' \; \underset{\sim}{t}_2') \begin{bmatrix} A \\ B+C \end{bmatrix} [A' \;\; B'+C'] \begin{bmatrix} \underset{\sim}{t}_1 \\ \underset{\sim}{t}_2 \end{bmatrix} \right]$$

$$\therefore \underset{\sim}{Z} \sim N_{2r}(\underset{\sim}{\mu}^*, \Sigma^*), \quad \underset{\sim}{\mu}^* = \begin{bmatrix} A\underset{\sim}{\mu} \\ (B+C)\underset{\sim}{\mu} \end{bmatrix} \qquad \Sigma^* = \begin{bmatrix} AA' & AB'+AC' \\ BA'+CA' & (B+C)(B'+C') \end{bmatrix}$$

$$\therefore \begin{bmatrix} \underset{\sim}{X} \\ \\ \underset{\sim}{U} + \underset{\sim}{V} \end{bmatrix} \sim N_{2r}\left(\begin{bmatrix} A\underset{\sim}{\mu} \\ \\ (B+C)\underset{\sim}{\mu} \end{bmatrix} \begin{bmatrix} AA' & AB' + AC' \\ \\ BA' + CA' & (B+C)(B'+C') \end{bmatrix} \right)$$

$$Cov(X, U) = 0$$

$$Cov(A\underset{\sim}{Y}, B\underset{\sim}{Y}) = AIB' = AB' = 0$$

$$Cov(X, V) = 0$$

$$Cov(AY, CY) = AIC' = AC' = 0$$

19) a) Pelo resultado 6 $\left[\underset{\sim}{V}_i \sim N_p(\underset{\sim}{\mu}_i, \Sigma_i),\ i=1,2\ldots n \text{ indep} \right.$

$$\left. \Rightarrow \sum_{i=1}^{n} a_i \underset{\sim}{V}_i \sim N_p\left(\sum_1^n a_i \underset{\sim}{\mu}_i, \sum_1^n a_i^2 \Sigma_i \right) \right]$$

$$\underset{\sim}{V}_1 \sim N_p\left(\underset{\sim}{0}, \frac{4}{16}\Sigma\right)$$

$$\underset{\sim}{V}_2 \sim N_p\left(\frac{1}{4}\underset{\sim}{\mu}, \frac{7}{16}\Sigma\right)$$

b) Pelo resultado 10

$$\begin{bmatrix} \underset{\sim}{V}_1 \\ \underset{\sim}{V}_2 \end{bmatrix} \sim N_{2p}\left(\begin{pmatrix} \underset{\sim}{0} \\ \frac{1}{4}\underset{\sim}{\mu} \end{pmatrix}, \begin{bmatrix} \frac{4}{16}\Sigma & -\frac{1}{16}\Sigma \\ \frac{-1}{16}\Sigma & \frac{7}{16}\Sigma \end{bmatrix} \right)$$

20) a) Resultado 6 $\Rightarrow \underset{\sim}{X} - \underset{\sim}{Y} \sim N\left(\begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}, \Sigma_X + \Sigma_Y \right)$

b) Pelo resultado 9, $\begin{bmatrix} \underset{\sim}{X} \\ \underset{\sim}{Y} \end{bmatrix} \sim N_6\left(\begin{bmatrix} \underset{\sim}{\mu}_X \\ \underset{\sim}{\mu}_Y \end{bmatrix}, \begin{bmatrix} \Sigma_X & 0 \\ 0 & \Sigma_Y \end{bmatrix} \right)$

[este é o item c)]

Seja $A = \begin{bmatrix} I_3 & -I_3 \\ I_3 & I_3 \end{bmatrix}$.

$$A \begin{pmatrix} \underset{\sim}{X} \\ \underset{\sim}{Y} \end{pmatrix} = \begin{bmatrix} \underset{\sim}{X} - \underset{\sim}{Y} \\ \underset{\sim}{X} + \underset{\sim}{Y} \end{bmatrix} \sim N_6\left(\underset{\sim}{\mu}^*, \Sigma^* \right)$$

$$\underset{\sim}{\mu}^* = A \begin{bmatrix} \underset{\sim}{\mu}_X \\ \underset{\sim}{\mu}_Y \end{bmatrix} = \begin{bmatrix} \underset{\sim}{\mu}_X - \underset{\sim}{\mu}_Y \\ \underset{\sim}{\mu}_X + \underset{\sim}{\mu}_Y \end{bmatrix}$$

$$\Sigma^* = A \begin{bmatrix} \Sigma_X & 0 \\ 0 & \Sigma_Y \end{bmatrix} A' = \begin{bmatrix} \Sigma_X & -\Sigma_Y \\ \Sigma_X & \Sigma_Y \end{bmatrix} \begin{bmatrix} I_3 & I_3 \\ -I_3 & I_3 \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma_X + \Sigma_Y & \Sigma_X - \Sigma_Y \\ \Sigma_X - \Sigma_Y & \Sigma_X + \Sigma_Y \end{bmatrix}$$

21) $Q = (X_1 - X_2)^2 + (X_2 - X_3)^2 + \dots + (X_{n-1} - X_n)^2 =$

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & -1 & \dots & 0 & 0 \\ \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & -1 \end{bmatrix}_{(n-1)\times n}}_{B} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} = B \underset{\sim}{X} = \begin{bmatrix} X_1 - X_2 \\ X_2 - X_3 \\ \vdots \\ X_{n-1} - X_n \end{bmatrix}$$

$$Q = (BX)' BX = X' B' B X = X' A X$$

$$A = B'B = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ \vdots \\ 0 & 0 & 0 & \dots & -1 & 1 \\ 0 & 0 & 0 & \dots & 0 & -1 \end{bmatrix}_{n\times(n-1)} \begin{bmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots \\ 0 & 0 & 0 & 0 & & 1 & -1 \end{bmatrix}_{(n-1)\times n} =$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ \vdots \\ 0 & 0 & 0 & 0 & & 2 & -1 \\ 0 & 0 & 0 & 0 & & -1 & 1 \end{bmatrix}_{n\times n}$$

$$E(\underset{\sim}{X}' A \underset{\sim}{X}) = tr(A\Sigma) + \underset{\sim}{\mu}' A \underset{\sim}{\mu} \qquad \underset{\sim}{\mu} = \begin{bmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{bmatrix}_{n\times 1}$$

$$\underset{\sim}{\mu}' A = [\mu - \mu \quad 2\mu - \mu - \mu \quad -\mu + 2\mu - \mu \ldots \mu - \mu] = \underset{\sim}{0}$$

$$\therefore E(\underset{\sim}{X}' A \underset{\sim}{X}) = tr(\sigma^2 A) = \sigma^2 [2(n-2)+2] =$$

$$= \sigma^2 [2n - 2]$$

22) $$\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} - \begin{bmatrix} \bar{X} \\ \bar{X} \\ \vdots \\ \bar{X} \end{bmatrix} = I_n \underset{\sim}{X} - \frac{1}{n} \begin{bmatrix} 1 & 1 & \ldots & 1 & 1 \\ 1 & 1 & \ldots & 1 & 1 \\ \vdots & & & & \\ 1 & 1 & \ldots & 1 & 1 \end{bmatrix} \underset{\sim}{X} \qquad \underset{\sim}{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$

$$= \left[I_n - \frac{1}{n} \underset{\sim}{1} \underset{\sim}{1}' \right] \underset{\sim}{X} = B \underset{\sim}{X}$$

$$\sum_{i=1}^{n} (X_i - \bar{X})^2 = \underset{\sim}{X}' B' B \underset{\sim}{X} = \underset{\sim}{X}' A \underset{\sim}{X}$$

$$A = \left[I - \frac{1}{n} \underset{\sim}{1} \underset{\sim}{1}' \right]' \left[I - \frac{1}{n} \underset{\sim}{1} \underset{\sim}{1}' \right] = \left[I - \frac{1}{n} \underset{\sim}{1} \underset{\sim}{1}' \right] \left[I - \frac{1}{n} \underset{\sim}{1} \underset{\sim}{1}' \right] =$$

$$= I - \frac{1}{n} \underset{\sim}{1} \underset{\sim}{1}' - \frac{1}{n} \underset{\sim}{1} \underset{\sim}{1}' + \frac{1}{n^2} \underset{\sim}{1} \underbrace{\underset{\sim}{1}' \underset{\sim}{1}}_{n} \underset{\sim}{1}' = I - \frac{1}{n} \underset{\sim}{1} \underset{\sim}{1}'$$

$$\underset{\sim}{\mu}' A \underset{\sim}{\mu} = \underbrace{[\mu \ \mu \ldots \mu]}_{\mu \underset{\sim}{1}'} \left[I - \frac{1}{n} \underset{\sim}{1} \underset{\sim}{1}' \right] \underbrace{\begin{bmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{bmatrix}}_{\underset{\sim}{1} \mu} =$$

$$\mu \underset{\sim}{1}' \underset{\sim}{1} \mu - \frac{1}{n} \mu \underset{\sim}{1}' \underset{\sim}{1} \underset{\sim}{1}' \underset{\sim}{1} \mu = 0$$

$$\Sigma = \begin{bmatrix} \sigma^2 & \rho\sigma^2 & \cdots & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 & \cdots & \rho\sigma^2 \\ & & & \\ \rho\sigma^2 & \rho\sigma^2 & \cdots & \sigma^2 \end{bmatrix}$$

$$A\Sigma = \left[I - \frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}'\right]\Sigma = \left[\Sigma - \frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}'\Sigma\right]$$

$$E\left(\sum_{i=1}^{n}(x_i-\bar{x})^2\right) = tr(A\Sigma) + \underset{\sim}{\mu}'A\underset{\sim}{\mu} = tr(\Sigma) - tr\left(\frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}'\Sigma\right)$$

$$= n\sigma^2 - tr\left(\begin{bmatrix} 1/n & 1/n & \cdots & 1/n \\ 1/n & 1/n & \cdots & 1/n \\ & & & \\ 1/n & 1/n & \cdots & 1/n \end{bmatrix}\Sigma\right) =$$

$$= n\sigma^2 - n\left[\frac{n-1}{n}\rho\sigma^2 + \frac{\sigma^2}{n}\right] = \sigma^2(n-1)(1-\rho)$$

$$\therefore E(Q) = E\left(\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1}\right) = \sigma^2(1-\rho)$$

23) $$\underset{\sim}{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} \sim N_n\left(\begin{bmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{bmatrix}, \sigma^2 I_n \right)$$

$$\bar{Y} = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix} \underset{\sim}{Y} = B \underset{\sim}{Y}$$

$S^2 = \frac{1}{n-1} \underset{\sim}{Y}' A \underset{\sim}{Y}$ para $A = \begin{bmatrix} \frac{n-1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & \frac{n-1}{n} & & -\frac{1}{n} \\ \vdots & & & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & & \frac{n-1}{n} \end{bmatrix}$

$\bar{Y}$ e S^2 são independentes $\iff \underset{1\times n}{B}\ \underset{n\times n}{\Sigma}\ \underset{n\times n}{A} = O_{1\times n}$

$$B \Sigma A = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix} \sigma^2 I_n \begin{bmatrix} \frac{n-1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & \frac{n-1}{n} & & -\frac{1}{n} \\ \vdots & & & \\ -\frac{1}{n} & -\frac{1}{n} & & \frac{n-1}{n} \end{bmatrix}$$

$$= \sigma^2 \begin{bmatrix} \frac{n-1-(n-1)}{n^2} & \frac{n-1-(n-1)}{n^2} & \cdots & \frac{n-1-(n-1)}{n^2} \end{bmatrix} = \sigma^2 \underset{\sim}{0}' = \underset{\sim}{0}'$$

24) $\underset{\sim}{Y} \sim N_n(\underset{\sim}{0}, I_n)$

$$(Y_1 - Y_2)^2 + (Y_2 - Y_3)^2 + \dots + (Y_{n-1} - Y_n)^2 = \underset{\sim}{Y}' A \underset{\sim}{Y}$$

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ & & & & & & \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & 0 & & -1 & 1 \end{bmatrix} \qquad \begin{matrix} \Sigma = I_n \\ \underset{\sim}{\mu} = \underset{\sim}{0} \end{matrix}$$

$$Var(\underset{\sim}{Y}' A \underset{\sim}{Y}) = 2 tr (A\Sigma)^2 + 4 \underset{\sim}{\mu}' A \Sigma A \underset{\sim}{\mu}$$

$$(A\Sigma)^2 = A^2$$

$$tr (A\Sigma)^2 = tr(A^2) = (2 + \underbrace{6 + 6 \dots + 6}_{n-2} + 2) =$$

$$= [6(n-2) + 4] = [6n - 8]$$

Obs: A é simétrica $\Rightarrow tr(A^2) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2$

$$\therefore Var(\underset{\sim}{Y}' A \underset{\sim}{Y}) = 2 \times [6n - 8]$$

25) $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

Se $\mu = \mu_1$, $\bar{X} - \mu_0 \sim N(\mu_1 - \mu_0, \sigma^2/n)$ e $\dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(\mu_1 - \mu_0, 1)$

Sabe-se que

$\bar{X}$ é independente de S^2 e $\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

$$\therefore \; Z = \frac{\dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}}{\sqrt{\dfrac{S^2(n-1)}{\sigma^2(n-1)}}} \sim t \text{ não central com parâmetro}$$

de não centralidade $\mu_1 - \mu_0$ e $n-1$ g.l.

$$Z = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$$

Utilidade do resultado: Cálculo do Poder do Teste sob a alternativa $\mu = \mu_1$, $\mu_1 > \mu_0$

$$\Pi(\mu_1) = P(\text{rejeitar } H_0 \mid \mu = \mu_1) = P\left(\frac{\bar{X} - \mu_0}{S/\sqrt{n}} \geq t_c \;\middle|\; \mu = \mu_1\right)$$

Se $\mu = \mu_1$, $\dfrac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t_{n-1, \lambda}$ $\quad \lambda = \mu_1 - \mu_0$

Num teste de hipóteses μ_0 é especificado, μ_1 é o valor de μ fixado na hipótese alternativa, t_c foi obtido a partir de α e $\Pi(\mu_1)$ é calculado através da tabela da distribuição $t_{n-1, \lambda}$, com $\lambda = \mu_1 - \mu_0$.

26) $\underset{\sim}{X} \sim N(\underset{\sim}{\mu}, \Sigma)$

$\underset{\sim}{X}' A \underset{\sim}{X} \sim \chi^2_{r(A), \frac{1}{2}\underset{\sim}{\mu}' A \underset{\sim}{\mu}} \iff A\Sigma$ é idempotente

$Q_1 \sim \chi^2_a \Rightarrow AI_n = A$ é idempotente

$Q_2 = \underset{\sim}{Y}'\underset{\sim}{Y} - \underset{\sim}{Y}' A \underset{\sim}{Y} = \underset{\sim}{Y}'[I - A]\underset{\sim}{Y}$

$[I-A]I_n[I-A]I_n = I - A - A + A^2 = I - A - A + A = [I-A]I_n$

$\therefore [I-A]I_n$ é idempotente $\Rightarrow Q_2 \sim \chi^2$

$\frac{1}{2}\underset{\sim}{\mu}' A \underset{\sim}{\mu} = \frac{1}{2}\underset{\sim}{0}' A \underset{\sim}{0} = 0$

Como $I - A$ é idempotente $r(I-A) = tr(I - A) =$

$= tr(I) - \underbrace{tr(A)}_{r(A) = a} = n - a$

$r(A) = a$ pq. A é idempotente

$\therefore Q_2 \sim \chi^2_{n-a}$

27) $$\begin{bmatrix} Y_1 + Y_2 + Y_3 \\ Y_1 - Y_2 - Y_3 \end{bmatrix} = A \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

para $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$, portanto $\begin{bmatrix} Y_1 + Y_2 + Y_3 \\ Y_1 - Y_2 - Y_3 \end{bmatrix} \sim$

$N_2\left(A\underset{\sim}{\mu},\ A\Sigma A'\right)$.

$$A\Sigma A' = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & a & 0 \\ a & 1 & a \\ 0 & a & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix} =$$

$$= \begin{bmatrix} 3+4a & -2a-1 \\ -2a-1 & 3+2a \end{bmatrix}.$$

Como $A\underset{\sim}{Y} \sim N_2$, $Y_1 + Y_2 + Y_3$ e $Y_1 - Y_2 - Y_3$ serão independentes se $-2a-1 = 0 \Rightarrow a = -\frac{1}{2}$.

28) $\bar{X} = \left[\frac{1}{n} \; \frac{1}{n} \cdots \frac{1}{n}\right] = B \underset{\sim}{X}$

$B = \underset{\sim}{1}' \frac{1}{n}$

$\sum_{i=1}^{n} (X_i - \bar{X})^2 = \underset{\sim}{X}' A \underset{\sim}{X}, \quad A = \left[I - \frac{1}{n} \underset{\sim}{1} \underset{\sim}{1}' \right]$

$$\Sigma = \begin{bmatrix} \sigma^2 & (1-\rho)\sigma^2 & \cdots & (1-\rho)\sigma^2 \\ (1-\rho)\sigma^2 & \sigma^2 & & (1-\rho)\sigma^2 \\ \vdots & & & \\ (1-\rho)\sigma^2 & (1-\rho)\sigma^2 & & \sigma^2 \end{bmatrix}$$

$$B\Sigma = \left[\frac{\sigma^2}{n} + \frac{(n-1)(1-\rho)\sigma^2}{n} \quad \frac{\sigma^2}{n} + \frac{(n-1)(1-\rho)\sigma^2}{n} \quad \cdots \quad \frac{\sigma^2}{n} + \frac{(n-1)(1-\rho)\sigma^2}{n} \right]$$

$$B\Sigma A = \frac{\sigma^2[1 + (1-\rho)(n-1)]}{n} \underset{\sim}{1}' \left[I - \frac{1}{n} \underset{\sim}{1} \underset{\sim}{1}' \right] =$$

$$= \frac{\sigma^2[1 + (1-\rho)(n-1)]}{n} \left[\underset{\sim}{1}' - \frac{1}{n} n \underset{\sim}{1}' \right] = \underset{\sim}{0}$$

$\therefore \bar{X}$ e $\sum_{i=1}^{n} (X_i - \bar{X})^2$ são independentes

29) $\underset{\sim}{X} = \begin{bmatrix} \underset{\sim}{X}_1 \\ \underset{\sim}{X}_2 \end{bmatrix} \sim N(\underset{\sim}{\mu}, \Sigma) \Rightarrow$

$$\begin{bmatrix} \underset{\sim}{X}_1 - \underset{\sim}{\mu}_1 \\ \underset{\sim}{X}_2 - \underset{\sim}{\mu}_2 \end{bmatrix} \sim N(\underset{\sim}{0}, \Sigma)$$

$$\underset{\sim}{Y} = I_p(\underset{\sim}{X}_1 - \underset{\sim}{\mu}_1) - \Sigma_{12}\Sigma_{22}^{-1}(\underset{\sim}{X}_2 - \underset{\sim}{\mu}_2) =$$

$$= \begin{bmatrix} I_p & -\underset{p\times q}{\Sigma_{12}}\,\underset{q\times q}{\Sigma_{22}^{-1}} \end{bmatrix} \begin{bmatrix} \underset{\sim}{X}_1 - \underset{\sim}{\mu}_1 \\ \underset{\sim}{X}_2 - \underset{\sim}{\mu}_2 \end{bmatrix}$$

$$\begin{bmatrix} \underset{\sim}{Y} \\ \underset{\sim}{X}_2 - \underset{\sim}{\mu}_2 \end{bmatrix} = \begin{bmatrix} I_p & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I_q \end{bmatrix} \begin{bmatrix} \underset{\sim}{X}_1 - \underset{\sim}{\mu}_1 \\ \underset{\sim}{X}_2 - \underset{\sim}{\mu}_2 \end{bmatrix} \sim N_{p+q}$$

$$= A \begin{bmatrix} \underset{\sim}{X}_1 - \underset{\sim}{\mu}_1 \\ \underset{\sim}{X}_2 - \underset{\sim}{\mu}_2 \end{bmatrix}$$

$$\operatorname{Var}\begin{pmatrix} \underset{\sim}{Y} \\ \underset{\sim}{X}_2 \end{pmatrix} = A\,\Sigma\,A' =$$

$$= \begin{bmatrix} I_p & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I_q \end{bmatrix} \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22} \end{bmatrix} \begin{bmatrix} I_p & 0 \\ -\Sigma_{22}^{-1}\Sigma_{12}' & I_q \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}' & \overbrace{\Sigma_{12} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{22}}^{0} \\ \Sigma_{12}' & \Sigma_{22} \end{bmatrix} \begin{bmatrix} I_p & 0 \\ -\Sigma_{22}^{-1}\Sigma_{12}' & I_q \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}' & 0 \\ \underbrace{\Sigma_{12}' - \Sigma_{22}\Sigma_{22}^{-1}\Sigma_{12}'}_{0} & \Sigma_{22} \end{bmatrix}$$

$\therefore$ $\operatorname{Cov}(\underset{\sim}{Y}, \underset{\sim}{X}_2) = 0$ e como $\begin{pmatrix} \underset{\sim}{Y} \\ \underset{\sim}{X}_2 - \underset{\sim}{\mu}_2 \end{pmatrix} \sim N_{p+q}$ e portanto

$$\begin{pmatrix} \underset{\sim}{Y} \\ \underset{\sim}{X}_2 \end{pmatrix} \sim N_{p+q}\,,$$ então $\underset{\sim}{Y}$ e $\underset{\sim}{X}_2$ são independentes.