58-72

Modelos Lineares com Matriz de Planejamento de Posto Completo

O modelo Linear Geral na forma matricial pode ser escrito como

$$\underset{\sim}{Y} = X \underset{\sim}{\beta} + \underset{\sim}{\epsilon}$$

$$\underset{\sim}{Y} = \underset{n\times 1}{\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}} \qquad X = \underset{n\times(K+1)}{\begin{bmatrix} 1 & x_{11} & \cdots & x_{K1} \\ 1 & x_{12} & & x_{K2} \\ \vdots & & & \\ 1 & x_{1n} & & x_{Kn} \end{bmatrix}} \qquad \underset{\sim}{\beta} = \underset{(K+1)\times 1}{\begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_K \end{bmatrix}} \qquad \underset{\sim}{\epsilon} = \underset{n\times 1}{\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix}}$$

$$\Rightarrow \begin{cases} Y_1 = \beta_0 + \beta_1 x_{11} + \beta_2 x_{21} + \dots + \beta_K x_{K1} + \epsilon_1 \\ \vdots \\ Y_n = \beta_0 + \beta_1 x_{1n} + \beta_2 x_{2n} + \dots + \beta_K x_{Kn} + \epsilon_n \end{cases}$$

$\underset{\sim}{Y}$ - vetor aleatório observável de variáveis resposta

X - matriz não aleatória de variáveis explicativas (ou preditoras) $n > K+1$

$\underset{\sim}{\beta}$ - vetor de parâmetros desconhecidos

$\underset{\sim}{\epsilon}$ - vetor aleatório não observável

$E(\underset{\sim}{\epsilon}) = \underset{\sim}{0} \implies E(\underset{\sim}{Y}) = X\underset{\sim}{\beta}$

$Var(\underset{\sim}{\epsilon}) = \Sigma$

$r(X) =$ nº de colunas linearmente independentes de X

Se $r(X) = k+1$, X é de posto completo.

Inferência para $\underset{\sim}{\epsilon} \sim N_n(\underset{\sim}{0}, \sigma^2 I_n)$

$r(X) = k+1$ $\qquad$ σ^2 desconhecido

$\underset{\sim}{Y} = X\underset{\sim}{\beta} + \underset{\sim}{\epsilon} \implies \underset{\sim}{Y} \sim N_n(X\underset{\sim}{\beta}, \sigma^2 I_n)$

$$\begin{cases} E(\underset{\sim}{Y} \mid X) = X\underset{\sim}{\beta} \\ E(Y_i \mid \underset{\sim}{X}_i) = \beta_0 + \beta_1 x_{1i} + \dots + \beta_k x_{ki} = \underset{\sim}{X}_i \underset{\sim}{\beta} \\ \underset{\sim}{X}_i \text{ - i-ésima linha da matriz } X, \quad i = 1, 2 \dots n \end{cases}$$

$\implies E(\underset{\sim}{\epsilon}) = \underset{\sim}{0} \implies E(\epsilon_i) = 0 \quad i = 1, 2 \dots n$

$Var(\underset{\sim}{\epsilon}) = \sigma^2 I_n \implies Var(\epsilon_i) = \sigma^2 \ \forall i$ (homocedasticidade)

$Cov(\epsilon_i, \epsilon_l) = 0 \quad i \neq l$

$\underset{\sim}{\epsilon} \sim N_n \implies \epsilon_i, \epsilon_l$ são independentes #

$\underset{\sim}{\epsilon} \sim N_n \implies \epsilon_i \sim N(0, \sigma^2) \quad i = 1, 2 \dots n$

Estimação de $\underset{\sim}{\beta}$

$$E(Y_i \mid \underset{\sim}{X}_i) = \beta_0 + \beta_1 x_{1i} + \dots + \beta_K x_{Ki} = \underset{\sim}{X}_i \underset{\sim}{\beta} \quad \text{(parâmetro)}$$

estimado por

$$\widehat{E(Y_i \mid \underset{\sim}{X}_i)} = \hat{\beta}_0 + \hat{\beta}_1 x_{1i} + \dots + \hat{\beta}_K x_{Ki}$$

$$= \underset{\sim}{X}_i \hat{\underset{\sim}{\beta}} = \hat{Y}_i \qquad \hat{\underset{\sim}{\beta}} = [\hat{\beta}_0 \ \hat{\beta}_1 \dots \hat{\beta}_K]$$

↳ estimador de $\underset{\sim}{\beta}$

Prever o valor de $Y_i \mid \underset{\sim}{X}_i$ por

$$\widehat{E(Y_i \mid \underset{\sim}{X}_i)} = \underset{\sim}{X}_i \hat{\underset{\sim}{\beta}} = \hat{Y}_i$$

$$\hat{Y}_i \begin{cases} \text{estimador de } E(Y_i \mid \underset{\sim}{X}_i) \\ \text{previsor de } Y \mid \underset{\sim}{X}_i \end{cases}$$

Para os elementos da amostra

Resíduo: $e_i = Y_i - \hat{Y}_i \qquad i = 1, 2 \dots n$

Vetor de Resíduos: $\underset{\sim}{e} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} = \begin{bmatrix} Y_1 - \hat{Y}_1 \\ Y_2 - \hat{Y}_2 \\ \vdots \\ Y_n - \hat{Y}_n \end{bmatrix} = \underset{\sim}{Y} - \hat{\underset{\sim}{Y}}$

$$\underset{\sim}{\hat{Y}} = \begin{bmatrix} \hat{Y}_1 \\ \hat{Y}_2 \\ \vdots \\ \hat{Y}_n \end{bmatrix} = X\underset{\sim}{\hat{\beta}}$$

Estimador de Mínimos Quadrados de β

Soma de Quadrados do Resíduo - S

$$S = \sum_{i=1}^{n} e_i^2 = \underset{\sim}{e}'\underset{\sim}{e} = (\underset{\sim}{Y} - \underset{\sim}{\hat{Y}})'(\underset{\sim}{Y} - \underset{\sim}{\hat{Y}}) =$$

$$(\underset{\sim}{Y} - X\underset{\sim}{\hat{\beta}})'(\underset{\sim}{Y} - X\underset{\sim}{\hat{\beta}}) = \underset{\sim}{Y}'\underset{\sim}{Y} - \underset{\sim}{Y}'X\underset{\sim}{\hat{\beta}} - \underset{\sim}{\hat{\beta}}'X'\underset{\sim}{Y} + \underset{\sim}{\hat{\beta}}'X'X\underset{\sim}{\hat{\beta}} =$$

$$\underbrace{1\times n \;\; n\times(k+1)\;(k+1)\times 1}_{1\times 1}$$

$$= \underset{\sim}{Y}'\underset{\sim}{Y} - 2\underset{\sim}{Y}'X\underset{\sim}{\hat{\beta}} + \underset{\sim}{\hat{\beta}}'X'X\underset{\sim}{\hat{\beta}}$$

$$f(\underset{\sim}{x}) = \underset{\sim}{a}'\underset{\sim}{x} \Rightarrow \frac{\partial f(\underset{\sim}{x})}{\partial \underset{\sim}{x}} = \underset{\sim}{a}$$

$$f(\underset{\sim}{x}) = \underset{\sim}{x}'A\underset{\sim}{x} \Rightarrow \frac{\partial f(\underset{\sim}{x})}{\partial \underset{\sim}{x}} = 2A\underset{\sim}{x}$$

$$\therefore \frac{\partial S}{\partial \underset{\sim}{\hat{\beta}}} = -2X'\underset{\sim}{Y} + 2X'X\underset{\sim}{\hat{\beta}} = \underset{\sim}{0} \qquad k+1 \text{ equações}$$

$$\Rightarrow$$

$\Rightarrow \quad X'X\hat{\beta} = X'\underset{\sim}{Y}$ Sistema de equações Normais

$r(X) = k+1 \Rightarrow r(X'X) = k+1 \Rightarrow$

$X'X$ é de posto completo $= k+1 \Rightarrow \exists\,(X'X)^{-1}$

$$\Rightarrow \underset{\sim}{\hat{\beta}} = \underbrace{\underset{(k+1)\times(k+1)}{(X'X)^{-1}}\ \underset{(k+1)\times n}{X'}\ \underset{n\times 1}{\underset{\sim}{Y}}}_{(k+1)\times 1}$$ est. min. quad. de $\underset{\sim}{\beta}$

$$\underset{\sim}{\hat{\beta}} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_k \end{bmatrix}$$

<u>Ver exemplo da pág. 65 das notas de aula</u>

Y - renda $\quad X_1$ - anos de escolaridade $\quad X_2$ - idade

$$\hat{\beta} = \begin{bmatrix} 2,3 \\ 2,1 \\ -0,2 \end{bmatrix} \qquad \hat{Y} = 2,3 + 2,1\,x_1 - 0,2\,x_2$$

$x_1 = 7 \quad x_2 = 30 \quad \hat{Y} = 10,7$ { estimativa da renda média das pessoas com 7 anos de escol/ e 30 anos

Previsão da renda de um indivíduo com 7 anos de escol/ e 30 anos

Estimadores de Máxima Verossimilhança

$$\hat{\underset{\sim}{\beta}}_{mv} = (X'X)^{-1}X'\underset{\sim}{Y}$$

$$\hat{\sigma}^2_{mv} = \frac{(\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}})'(\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}})}{n} = \frac{\sum_{i=1}^{n} e_i^2}{n}$$

$\hat{\sigma}^2_{mv}$ é viciado para σ^2

Propriedades de $\hat{\underset{\sim}{\beta}}$

$$\hat{\underset{\sim}{\beta}} = \underbrace{(X'X)^{-1}X'}_{A\ -\ (k+1)\times n}\underset{\sim}{Y} = A\underset{\sim}{Y}$$ é linear em $\underset{\sim}{Y}$

$$E(\hat{\underset{\sim}{\beta}}) = E(A\underset{\sim}{Y}) = (X'X)^{-1}X'E(\underset{\sim}{Y}) = (X'X)^{-1}X'X\underset{\sim}{\beta} = \underset{\sim}{\beta}$$

$$Var(\hat{\underset{\sim}{\beta}}) = A\,Var(\underset{\sim}{Y})\,A' = (X'X)^{-1}X'\sigma^2 I X(X'X)^{-1}$$

$$= (X'X)^{-1}\sigma^2$$

$$\underset{\sim}{Y} \sim N_n \Rightarrow A\underset{\sim}{Y} \sim N_{k+1}\left(\underset{\sim}{\beta}, (X'X)^{-1}\sigma^2\right) \Rightarrow$$

$$\Rightarrow \hat{\beta}_j \sim N(\beta_j, a_{jj}\sigma^2)$$

$$(X'X)^{-1} = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0K} \\ a_{10} & a_{11} & \cdots & a_{1K} \\ \vdots & & & \\ a_{K0} & a_{K1} & \cdots & a_{KK} \end{bmatrix}$$

$$Cov(\hat{\beta}_i, \hat{\beta}_j) = a_{ij}\,\sigma^2 \qquad i,j = 0,1,2,\dots,K$$

$$\hat{\underset{\sim}{\beta}} \sim N_{K+1} \Rightarrow \begin{bmatrix} \hat{\beta}_i \\ \hat{\beta}_j \end{bmatrix} \sim N_2 \Rightarrow$$

$\hat{\beta}_i$ e $\hat{\beta}_j$ são independentes $\Longleftrightarrow a_{ij} = 0$

Ver exemplo, pag 68 $\rightarrow$ Onde está y usar Y

Estamos utilizando o Modelo com intercepto. Neste caso, as somas de quadrados são dadas por:

SST - soma de quadrados total (corrigida pela média)

$$SST = \sum_{i=1}^{n} (Y_i - \bar{Y})^2 = \sum_{i=1}^{n} Y_i^2 - n\bar{Y}^2 =$$

na forma matricial

$$= \underset{\sim}{Y}'\underset{\sim}{Y} - \frac{\underset{\sim}{Y}'\underset{\sim}{1}\underset{\sim}{1}'\underset{\sim}{Y}}{n} = \underset{\sim}{Y}'\left[I - \frac{\underset{\sim}{1}\underset{\sim}{1}'}{n}\right]\underset{\sim}{Y}$$

↳ forma quadrática em $\underset{\sim}{Y}$

SSE - soma de quadrados do resíduo

$$SSE = \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2 = (\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}})'(\underset{\sim}{Y} - X\hat{\underset{\sim}{\beta}}) =$$

$$= \underset{\sim}{Y}'\underset{\sim}{Y} - 2\underset{\sim}{Y}'X\hat{\underset{\sim}{\beta}} + \hat{\underset{\sim}{\beta}}'\underbrace{X'X\hat{\underset{\sim}{\beta}}}_{X'\underset{\sim}{Y}} = \underset{\sim}{Y}'\underset{\sim}{Y} - \hat{\underset{\sim}{\beta}}'X'\underset{\sim}{Y}$$

Para escrever como forma quadrática em $\underset{\sim}{Y}$

$$SSE = \underset{\sim}{Y}'\underset{\sim}{Y} - \underset{\sim}{Y}'X(X'X)^{-1}X'\underset{\sim}{Y} = \underset{\sim}{Y}'\left[I - X(X'X)^{-1}X'\right]\underset{\sim}{Y}$$

SSR - Soma de Quadrados da Regressão

$$SSR = \sum_{i=1}^{n} (\hat{Y}_i - \bar{Y})^2 = \sum \hat{Y}_i^2 - 2\bar{Y}\underbrace{\sum \hat{Y}_i}_{= n\bar{Y}} + n\bar{Y}^2$$

Resultados 1, 2 ~~3~~ Pag 70

Como $\sum_{i=1}^{n} \hat{Y}_i = \sum_{i=1}^{n} Y_i$ (Resultado 2) então em todo modelo com intercepto

$$SSR = \sum_{i=1}^{n} \hat{Y}_i^2 - n\bar{Y}^2$$

Devido ao resultado 2 $\cancel{\sum_{i=1}^{n} (\hat{Y}_i - Y_i) = \sum e_i}$

$$\sum (Y_i - \hat{Y}_i) = \sum e_i = 0$$ em todo modelo com intercepto

Na forma matricial

$$SSR = \hat{\underset{\sim}{\beta}}' X' \underset{\sim}{Y} - n\bar{Y}^2$$ → Consequência 1 pag 71

fórmula mais conhecida para cálculos

SSR escrita como forma quadrática em $\underset{\sim}{Y}$
(Consequência 2)

$$SSR = \underset{\sim}{Y}' \left[X(X'X)^{-1}X' - \frac{\underset{\sim}{1}\,\underset{\sim}{1}'}{n} \right] \underset{\sim}{Y}$$

Lembrando que $SSE = \underset{\sim}{Y}' \left[I - X(X'X)^{-1}X' \right] \underset{\sim}{Y}$

e $SST = \underset{\sim}{Y}' \left[I - \frac{\underset{\sim}{1}\,\underset{\sim}{1}'}{n} \right] \underset{\sim}{Y}$

temos que $\underbrace{SST}_{\text{Variabilidade total da variável } Y} = SSR + \underbrace{SSE}_{\text{Variabilidade em } Y \text{ que a equação de regressão não consegue explicar}}$

$$R^2 = \frac{SSR}{SST}$$ coeficiente de explicação ou coeficiente de correlação múltipla do Modelo

$0 \leq R^2 \leq 1$ Proporção da variabilidade em Y que a equação de regressão consegue explicar.