Generating random variables II (Exercises)

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Exercise 1.* Simulate random bi-dimensional vector (ξ, η) with density $p(x, y) = 6x$ on the triangle

$$T = \{(x, y) : x > 0, y > 0, x + y \leq 1\}.$$

*[S], p.56, chapter 2

Exercise 1. Simulate random bi-dimensional vector (ξ, η) with density $p(x,y) = 6x$ on the triangle $T = \{(x,y) \colon x, y > 0, x + y \leq 1\}$.

The decomposition

$$p_{\xi\eta}(x,y) = p_\xi(x) p_{\eta|\xi}(y \mid x) = p_\eta(x) p_{\xi|\eta}(x \mid y),$$

provides two cases: (a) generate first ξ second η; (b) first η second ξ.

(a) Consider the first case:

$$\begin{aligned} p_\xi(x) &= \int_0^{1-x} p_{\xi\eta}(x,y)dy = 6x(1-x),\ 0 < x < 1, \\ p_{\eta|\xi}(y \mid x) &= p_{\xi\eta}(x,y)/p_\xi(x) = (1-x)^{-1},\ 0 < y < 1 - x \end{aligned}$$

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$p_\xi(x) = 6x(1-x), x \in [0,1], p_{\eta|\xi}(y \mid x) = (1-x)^{-1},\ y \in [0, 1-x]$.

$$\begin{aligned} F_\xi(x) &= \int_0^x p_\xi(u)du = 3x^2 - 2x^3,\ x \in [0,1], \\ F_{\eta|\xi}(y \mid x) &= \int_0^y p_{\eta|\xi}(u \mid x)du = y(1-x)^{-1},\ y \in [0, 1-x], \end{aligned}$$

Thus, let $U_1, U_2 \sim U[0,1]$ then ξ, η is the solution of

$$\begin{cases} 3\xi^2 + 2\xi^3 = U_1 \\ \eta = U_2(1-\xi) \end{cases}$$

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(b) first η, second ξ

$$\begin{aligned} p_\eta(y) &= \int_0^{1-y} p_{\xi\eta}(x, y)dx = 3(1-y)^2, \ y \in (0, 1) \\ p_{\xi|\eta}(x \mid y) &= p_{\xi\eta}(x, y)/p_\eta(y) = 2x(1-y)^{-2}, \ x \in (0, 1-y) \end{aligned}$$

$$\begin{aligned} F_\eta(y) &= \int_0^{y} p_\eta(v)dv = 1 - (1-y)^3, \ y \in (0, 1) \\ F_{\xi|\eta}(x \mid y) &= \int_0^{x} p_{\xi|\eta}(v \mid y)dv = x^2(1-y)^{-2}, \ x \in (0, 1-y) \end{aligned}$$

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$$\begin{array}{rcl} F_\eta(y) & = & 1 - (1-y)^3,\ y \in (0,1) \\ F_{\xi|\eta}(x \mid y) & = & x^2(1-y)^{-2},\ x \in (0, 1-y) \end{array}$$

Thus

$$\left\{ \begin{array}{rcl} U_1 & = & 1 - (1-\eta)^3 \\ U_2 & = & \xi^2(1-\eta)^{-2} \end{array} \right.$$

Exercise 1. Simulate random bi-dimensional vector (ξ, η) with density $p(x, y) = 6x$ on the triangle $T = \{(x, y) : x > 0, y > 0, x + y \leq 1\}$.

$$\begin{cases} U_1 = 1 - (1-\eta)^3 \\ U_2 = \xi^2 (1-\eta)^{-2} \end{cases} \Longrightarrow \begin{cases} U_1 = (1-\eta)^3 \\ U_2 = \xi^2 (1-\eta)^{-2} \end{cases}$$

Finally

$$\Longrightarrow \begin{cases} \eta = 1 - (U_1)^{1/3} \\ \xi = (U_2)^{1/2} (U_1)^{1/3} \end{cases}$$

the second variant is better, because we need to solve the equation of 3rd order in the first variant.

Exercise 2. Simulate random point Q uniformly distributed in the ball

$$B = \{x^2 + y^2 + z^2 < R^2\}.$$

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Let $Q = (\xi, \eta, \zeta)$. The joint density is constant in the ball

$$p_Q(x, y, z) = \frac{1}{(4/3)\pi R^3}, \ (x, y, z) \in B.$$

Direct calculation of conditional densities is boring. Let us try to change coordinates.

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In polar coordinates

$$x = r \sin\theta \cos\phi, y = r \sin\theta \sin\phi, z = r \cos\theta.$$

Jacobian

$$\left| \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} \right| = r^2 \sin\theta$$

and

$$p_Q(r, \theta, \phi) = \frac{1}{(4/3)\pi R^3} r^2 \sin\theta$$

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$$B = \{x^2 + y^2 + z^2 < R^2\}.$$

$$\begin{aligned} p_Q(r,\theta,\phi) &= \frac{1}{(4/3)\pi R^3} r^2 \sin\theta \\ &= \frac{3r^2}{R^3} \times \frac{\sin\theta}{2} \times \frac{1}{2\pi} \end{aligned}$$

It means that coordinates r_Q, θ_Q, ϕ_Q of point Q are independents.

Exercise 2. Simulate random point Q uniformly distributed in the ball

$$B = \{x^2 + y^2 + z^2 < R^2\}.$$

Equations for inverse cdf are

$$\int_0^{r_Q} \frac{3r^2}{R^3} dr = U_1, \int_0^{\theta_Q} \frac{\sin\theta}{2} d\theta = 1 - U_2, \int_0^{\phi_Q} \frac{1}{2\pi} d\phi.$$

Thus we obtain

$$r_Q = R(U_1)^{1/3}, \cos\theta_Q = 2U_2 - 1, \phi_Q = 2\pi U_3.$$

Integral superposition method.

Generalization of the superposition method where the target distribution $F(x)$ can be represented as

$$F(x) = \sum_k c_k F_k(x).$$

Integral superposition method based on

$$p_\xi(x) = \int_{-\infty}^{\infty} p(x, y) dy.$$

It is rarely used. It is useful when a density represented as an integral.

Integral superposition method.

Example. Let

$$p_\xi(x) = n\int_1^\infty y^{-n} e^{-xy}\,dy$$

use $p(x,y) = p_\eta(y)p_\xi(x \mid y)$.

$$\begin{aligned} p_\eta(y) &= \int_0^\infty p(x,y)\,dx = ny^{-n-1},\ 1 < y < \infty, \\ p_\xi(x \mid y) &= p(x,y)/p_\eta(y) = ye^{-xy},\ 0 < x < \infty, \end{aligned}$$

$$\Longrightarrow F_\eta(y) = 1 - y^{-n},\ F_\xi(x \mid y) = 1 - e^{-xy}$$

$$\Longrightarrow \eta = (U_1)^{-1/n},\ \xi = -\frac{\ln U_2}{\eta}$$

$$\Longrightarrow \xi = -(U_1)^{1/n}\ln U_2.$$

Exercise 3. Let $X \sim F(x) = x^n, x \in [0,1]$. By inverse method $X = (U)^{1/n}$. Prove that X can be simulated by $X = \max\{U_1, \dots, U_n\}$.

Exercise 4. Let $X \sim \Gamma(n)$: the density $p_n(x) = (\Gamma(n))^{-1} x^{n-1} e^{-x}$, where $n \geq 1$ is integer. Prove that for any n r.v. X can be generated as $X = -\ln(U_1 \cdots U_n)$.

Exercise 5. Let $X \sim B(n,p)$. Prove $X = \sum_{i=1}^n [p - U_i]_+$, where $[x]_+ = 1$ if $x \geq 0$ and $[x]_+ = 0$ if $x < 0$.

Exercise 6. Simulate $X \sim F(x) = 1 - \frac{1}{3}(2e^{-x} + e^{-5x}), x \in (0, \infty)$ using inverse method.

Exercise 7. Simulate point $P = (X, Y)$ uniformly distributed in anel $R_1^2 < x^2 + y^2 < R_2^2$.

Exercise 8*. Energy of a neutron is considered as a r.v. $X \in (0, \infty)$ with density

$$p(x) = ce^{-x/T}\mathrm{sh}\big(b\sqrt{2x/T}\big),$$

where b, T are parameters, and c is normalising constant $c = \sqrt{2/\pi}e^{-b^2/2}(bT)^{-1}$. Prove that ξ can be represented as

$$\xi = T\left(\frac{(\zeta + b)^2}{2} - \ln U\right),$$

where $\zeta \sim N(0,1)$ and ζ, U are independent.

Exercise 9. Suggest the simulation for the r.v.

$$X \sim F(t) = \exp\left\{-\frac{cF}{\kappa}t\right\}\exp\left\{\frac{cV}{(\gamma\kappa)^2 n}(1 - e^{-\gamma\kappa n t})\right\}$$
$$\times\, c\left(\frac{F}{\kappa} - \frac{V}{\gamma\kappa}e^{-\gamma\kappa n t}\right).$$

where c, F, V, κ, γ are parameters.

References:

[RC] Cristian P. Robert and George Casella. *Introducing Monte Carlo Methods with R.* Series "Use R!". Springer

[S] Sobol, I.M. *Monte-Carlo numerical methods.* Nauka, Moscow, 1973. (In Russian)