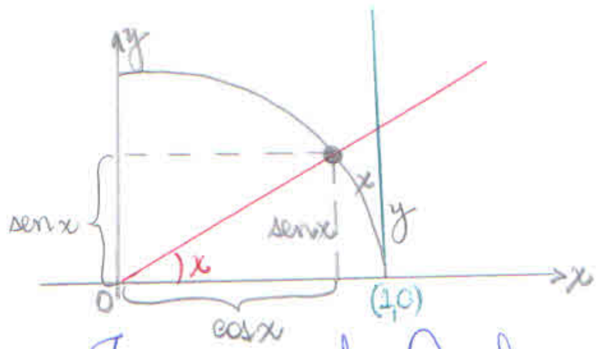
Limites Fundamentais

1) Limite Trigonométrico Fundamental

$$\boxed{\lim_{x \to 0} \frac{\operatorname{sen} x}{x} = 1}$$

Considere:

a) $0 < x < \pi/2$



De acordo com o Teorema do Confronto e aplicando o limite a expressão quando $x \to 0$ tem-se:

$$\frac{y}{\operatorname{sen} x} = \frac{1}{\cos x}$$

$$y = \frac{\operatorname{sen} x}{\cos x} \quad \therefore \quad y = \operatorname{tg} x$$

$$\operatorname{sen} x < x < \operatorname{tg} x$$

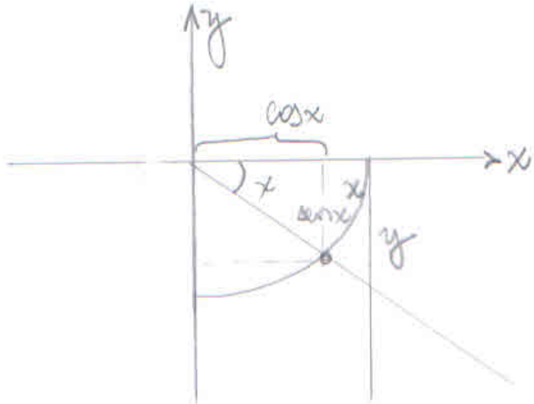
$$\frac{1}{\operatorname{sen} x} > \frac{1}{x} > \frac{1}{\operatorname{tg} x} \quad \times (\operatorname{sen} x > 0)$$

$$\frac{\operatorname{sen} x}{\operatorname{sen} x} > \frac{\operatorname{sen} x}{x} > \frac{\operatorname{sen} x}{\operatorname{tg} x}$$

$$1 > \frac{\operatorname{sen} x}{x} > \cos x \quad (1)$$

$$\underbrace{\lim_{x \to 0} 1}_{=1} > \lim_{x \to 0} \frac{\operatorname{sen} x}{x} > \underbrace{\lim_{x \to 0} \cos x}_{=1}$$

Analisando a região que os arcos são negativos:



b) $\frac{-\pi}{2} < x < 0$

$$\operatorname{sen} x > x > \operatorname{tg} x$$

$$\frac{1}{\operatorname{sen} x} < \frac{1}{x} < \frac{1}{\operatorname{tg} x} \quad \times (\operatorname{sen} x < 0)$$

$$1 > \frac{\operatorname{sen} x}{x} > \frac{\operatorname{sen} x}{\operatorname{tg} x}.$$

$$1 > \frac{\operatorname{sen} x}{x} > \cos x \quad (2)$$

De acordo com (1) e (2) as funções estão relacionadas

$$1 > \frac{\operatorname{sen} x}{x} > \cos x$$

Aplicando o Teorema do Confronto quando $x \to 0$ tem-se:

$$\underbrace{1}_{\lim_{x \to 0} 1 = 1} > \frac{\operatorname{sen} x}{x} > \underbrace{\cos x}_{\lim_{x \to 0} \cos x = 1}$$

$$\downarrow$$

$$\lim_{x \to 0} \frac{\operatorname{sen} x}{x}$$

Portanto $\lim_{x \to 0} \frac{\operatorname{sen} x}{x} = 1$

Exemplos de Aplicação

Resolver os limites.

a) $\lim\limits_{x\to 0} \dfrac{\text{sen } 13x}{x} = \dfrac{0}{0}$

$= \lim\limits_{x\to 0} \dfrac{13.\text{sen } 13x}{13.x}$

$= \lim\limits_{x\to 0} 13 . \underbrace{\left(\lim\limits_{x\to 0} \dfrac{\text{sen } 13x}{13x}\right)}_{=1} = 13.1 = 13$

b) $\lim\limits_{x\to 0} \dfrac{\text{sen } 5x}{\text{sen } 7x} = \dfrac{0}{0}$

$= \lim\limits_{x\to 0} \dfrac{5x\,\text{sen } 5x . 7}{5x\,\text{sen } 7x . 7}$

$= \underbrace{\lim\limits_{x\to 0} \dfrac{\text{sen } 5x}{5x}}_{1} . \lim\limits_{x\to 0} \dfrac{7x}{\text{sen } 7x} . \lim\limits_{x\to 0} \dfrac{7}{5}$

$\underbrace{\lim\limits_{x\to 0} \left(\dfrac{\text{sen } 7x}{7x}\right)^{-1}}_{1} . \dfrac{7}{5}$

$1 . 1 . \dfrac{7}{5} = \dfrac{7}{5}$

c) $\lim\limits_{x\to 0} \dfrac{1-\cos 3x}{7x^2} = \dfrac{0}{0}$

$= \lim\limits_{x\to 0} \dfrac{(1-\cos 3x).(1+\cos 3x)}{7x^2.(1+\cos 3x)} = \dfrac{1 + \cancel{\cos 3x} - \cancel{\cos 3x} - \cos^2 3x}{7x^2.(1+\cos 3x)}$

$= \lim\limits_{x\to 0} \dfrac{1-\cos^2 3x}{7x^2.(1+\cos 3x)} = \lim\limits_{x\to 0} \dfrac{\text{sen}^2 3x}{7x^2(1+\cos 3x)}$

$= \lim\limits_{x\to 0} \dfrac{3x^* \,\text{sen } 3x}{3x\; 7x^2} \dfrac{\text{sen } 3x . 3x^*}{(1+\cos 3x) . 3x} = \dfrac{9\cancel{x^2}\,\text{sen } 3x . \text{sen } 3x}{7\cancel{x^2} . 3x . 3x . \underbrace{(1+\cos 3x)}}$

$(1+\cos 3x)$

$$= \lim_{x\to 0} \frac{9}{7} \cdot \underbrace{\lim_{x\to 0} \frac{\operatorname{sen} 3x}{3x}}_{1} \cdot \underbrace{\lim_{x\to 0} \frac{\operatorname{sen} 3x}{3x}}_{1} \cdot \lim_{x\to 0} \frac{1}{(1+\cos 3x)}$$

$$\frac{9}{7} \cdot \frac{1}{2} = \frac{9}{14}$$

d) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$ Resp. $\frac{1}{2}$

e) $\lim_{x\to 0} \frac{\operatorname{tg} 17x}{13x}$ Resp. $\frac{17}{13}$

2) Limite Fundamental Exponencial

$$\boxed{\lim_{x\to \pm\infty} \left(1+\frac{1}{x}\right)^x = e}$$

$e \rightarrow$ número de Euler
$e = 2,7182818$

$$\left(1+\frac{1}{x}\right)^x \longrightarrow \left(1+\frac{1}{x \to \infty}\right)^{x\to\infty} = (1+0)^\infty = 1^\infty$$

No Limite Exponencial será considerado um processo de indução para utilização das séries.

$x \to \infty$

$x = 1 \Rightarrow \left(1+\frac{1}{1}\right)^1 = 2$

$x = 2 \Rightarrow \left(1+\frac{1}{2}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4} = 2,25$

$x = 3 \Rightarrow \left(1+\frac{1}{3}\right)^3 = \left(\frac{4}{3}\right)^3 = 2,370$

$x = 4 \Rightarrow \left(1+\frac{1}{4}\right)^4 = \left(\frac{5}{4}\right)^4 = 2,441$

$x = 10 \Rightarrow \left(1+\frac{1}{10}\right)^{10} = \left(\frac{11}{10}\right)^{10} = 2,59$

$$x = 100 \Rightarrow \left(1 + \frac{1}{100}\right)^{100} = \left(\frac{101}{100}\right)^{100} = 2{,}704$$

$$x = 1000 \Rightarrow \left(1 + \frac{1}{1000}\right)^{1000} = \left(\frac{1001}{1000}\right)^{1000} = 2{,}716$$

$$x = 10000 \Rightarrow \left(1 + \frac{1}{10000}\right)^{10000} = \left(\frac{10001}{10000}\right)^{10000} = 2{,}7181$$

$$x = 100000 \Rightarrow \left(1 + \frac{1}{100.000}\right)^{100.000} = \left(\frac{100001}{100000}\right)^{100000} = 2{,}7183$$

$$\therefore \boxed{\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^{x} = e}$$

Considerando $\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^{x} = e$, tem-se:

Seja $x = -(1+y)$, $y > 0$

$$\left(1 + \frac{1}{x}\right)^{x} = \left(1 + \frac{1}{-(1+y)}\right)^{-(1+y)} = \left(1 - \frac{1}{(1+y)}\right)^{-(1+y)}$$

$$\left(\frac{\cancel{1} + y \cancel{- 1}}{1+y}\right)^{-(1+y)} = \left(\frac{y}{1+y}\right)^{-(1+y)} = \left(\frac{1+y}{y}\right)^{1+y} = \left(1 + \frac{1}{y}\right)^{1+y}$$

Com $x \to -\infty \Rightarrow y \to \infty$

Segue que

$$\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^{x} = \lim_{y \to \infty} \left(1 + \frac{1}{y}\right)^{1+y}$$

Esta adição veio de uma multiplicação

$$\underbrace{\lim_{y \to \infty} \left(1 + \frac{1}{y}\right)}_{=1} \cdot \lim_{y \to \infty} \left(1 + \frac{1}{y}\right)^{y}$$

$$\therefore \lim_{y \to \infty} \left(1 + \frac{1}{y}\right)^{y} = e$$

Exemplos de Aplicação

Resolver os limites.

a) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{2x}$

$= \underbrace{\lim_{x \to \infty} \left[\left(1 + \frac{1}{x}\right)^{x}\right.}_{e} \left.\right]^{2}$

$= e^{2}$

b) $\lim_{x \to 0} (1 + x)^{1/x}$ $\quad x = \frac{1}{y}$, $x \to 0$, $y \to \infty$ $\quad \boxed{\frac{1}{x} = \frac{1}{\frac{1}{y}} = y}$

$= \underbrace{\lim_{y \to \infty} \left(1 + \frac{1}{y}\right)^{y}}_{} = e$

c) $\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^{(x+5)}$ $\quad y = x+5$

$= \underbrace{\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^{x}}_{e} \cdot \underbrace{\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^{5}}_{1} = e$ $\quad \frac{1}{-\infty} = 0$

d) $\lim_{x \to \infty} \left(1 + \frac{1}{3x}\right)^{7x} = \lim_{x \to \infty} \left(1 + \frac{1}{3x}\right)^{\frac{7 \cdot 3x}{3}}$

$= \underbrace{\lim_{x \to \infty} \left[\left(1 + \frac{1}{3x}\right)^{3x}\right]}_{e}^{\frac{7}{3}} = e^{\frac{7}{3}}$