Induction

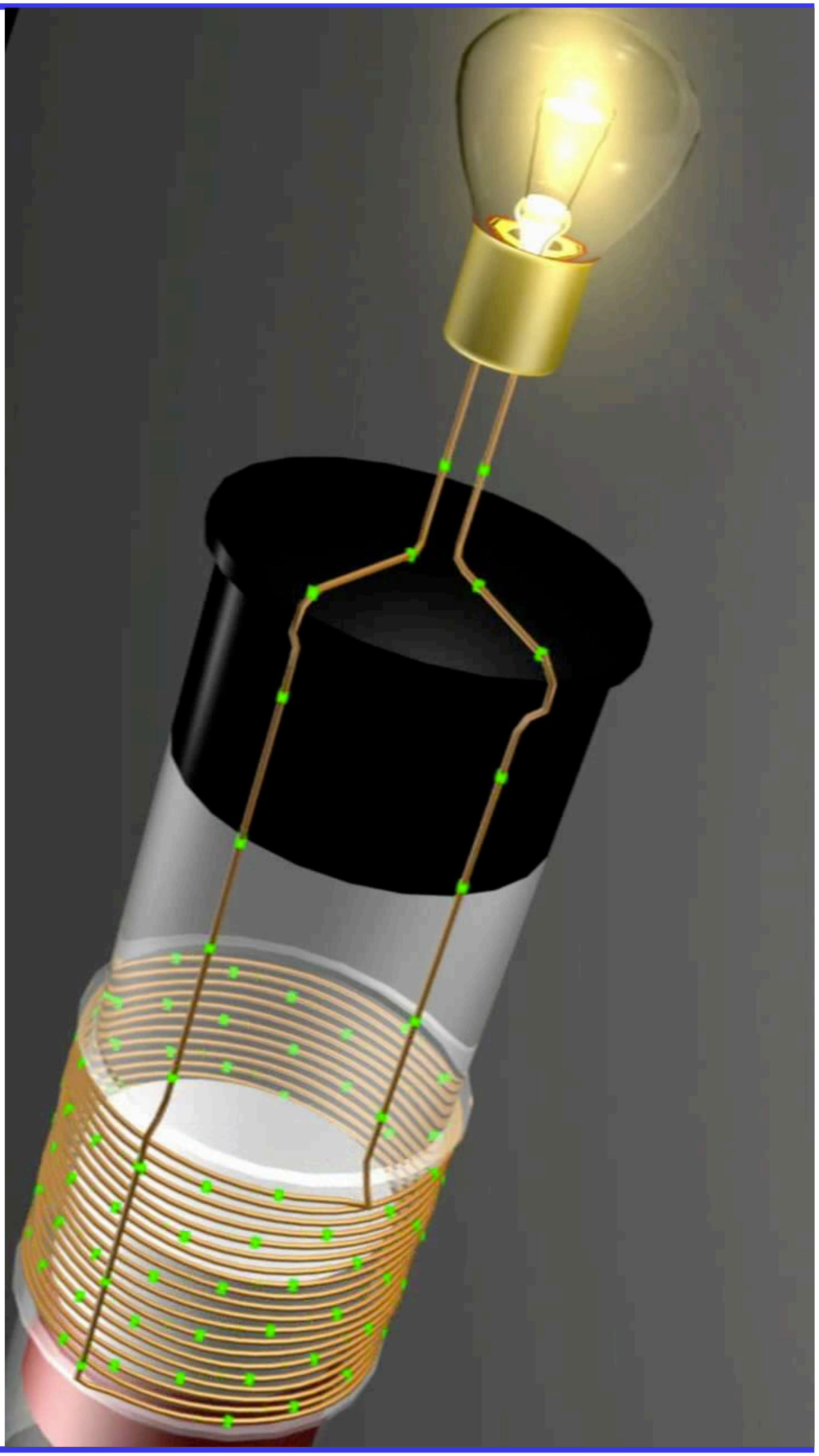
⚡ Faraday's law

⚡ Magnetic diffusion and Skin depth

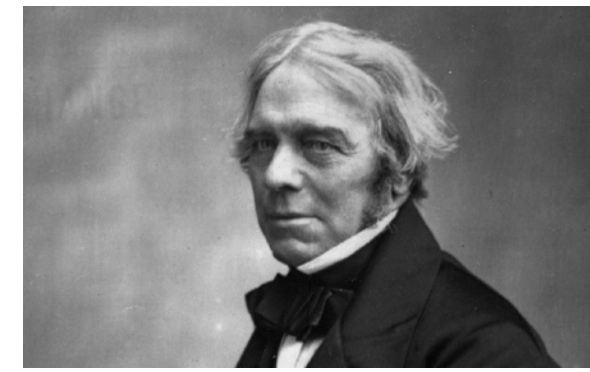
⚡ Induction

⚡ Superconductivity

⚡ Inductance and self-inductance

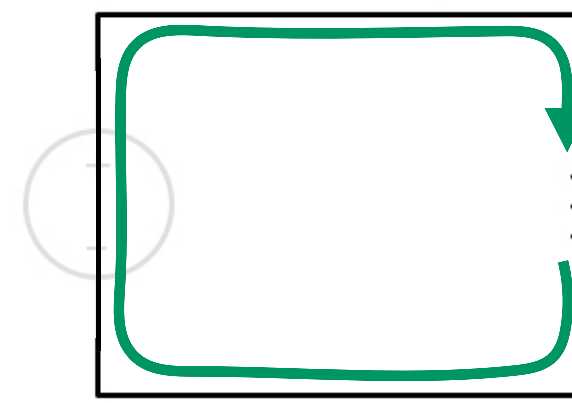


Faraday's law of induction



- In 1831 Michael Faraday described how the **time variation** of the **magnetic field flux through a closed circuit** generated a potential difference along that circuit, and therefore a **current**.
- That potential difference could not be ascribed to a specific place on the circuit — it was not like we applied a battery to a specific point in the circuit. That potential is "spread" all along the circuit, and for that reason this is often referred to as an **electromotive force**. We then define this force as in integral over the circuit:

$$\Delta\phi \sim \vec{E} \cdot \Delta\vec{x} \quad \Rightarrow \quad \mathscr{E} = \oint d\vec{l} \cdot \vec{E}$$

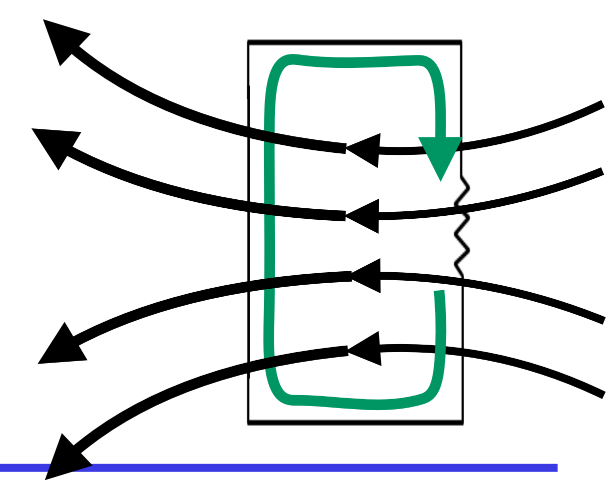


- And since we are talking about the **flux** through a given surface, it is useful to define the **magnetic flux** as:

$$\Phi_S = \int d\vec{S} \cdot \vec{B}$$

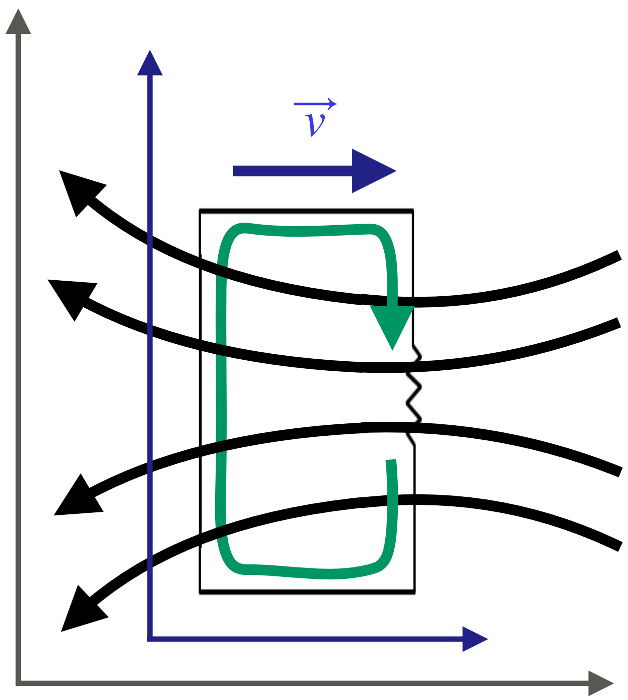
- The observation by Faraday (which, by the way relies on the notion of **field** , that he introduced himself!) was that the time variation of this flux determined the electromotive force:

$$\mathscr{E} = -\frac{d\Phi_B}{dt} \quad \Leftrightarrow \quad \oint_C d\vec{l} \cdot \vec{E} = -\frac{d}{dt}\int_{S(C)} d\vec{S} \cdot \vec{B}$$



- It is remarkable that Faraday's law introduces **two new elements** into electromagnetism: first, it **connects** the electric field with the magnetic field. And second, it introduces the notion of **time** into the game.
- The **sign** is sometimes referred to as **Lenz's law**: the magnetic field of the induced current **opposes** the change in flux of the external magnetic field through that circuit.

Faraday's law of induction



- An interesting property of Faraday's law is that a change in flux can happen in basically two ways:

  (i) we keep the **circuit fixed**, but we **increase the flux** — e.g., by increasing the magnitude of the magnetic field through the loop;

  (ii) we keep the **field configuration fixed**, but we **move/change the circuit** in such a way that the flux changes.

- In order to explore these relationships, let's consider a circuit which moves with a velocity $\vec{v}$ in a region with inhomogeneous magnetic fields, as shown in the figure, so the flux of the magnetic field inside the circuit changes as a result of that movement.

- This is obviously **identical** to a situation where we move the source of those fields in the direction of the circuit: the two configurations should generate the same current in that loop.

- However, notice that in the reference frame of the "lab" (where it is the circuit which is moving), the field is actually constant. How should we then consider Faraday's law, since it it refers to the time variation of the flux through a circuit that is now moving:

$$\oint_C d\vec{l} \cdot \vec{E} = -\frac{d}{dt}\int_{S(C)} d\vec{S} \cdot \vec{B}$$

Faraday's law of induction

- We can do this by using, in the integral of the magnetic flux, the frame of reference of the circuit, $\vec{x}_c = \vec{x} - \vec{v}t$, in which the circuit (and the surface) is fixed. We then have:

$$\oint_C d\vec{l} \cdot \vec{E} = -\int_{S(C)} d\vec{S} \cdot \left[\frac{d}{dt} \vec{B} \left(t, \vec{x}_c = \vec{x} - \vec{v}t\right) \right]$$

$$= -\int_{S(C)} d\vec{S} \cdot \left[\frac{\partial \vec{B}}{\partial t} + \left(\frac{\partial \vec{x}_c}{\partial t} \cdot \vec{\nabla}_c \right) \vec{B} \right] = -\int_{S(C)} d\vec{S} \cdot \left[\frac{\partial \vec{B}}{\partial t} - \left(\vec{v} \cdot \vec{\nabla}_c \right) \vec{B} \right]$$

- The last term can be rewritten using that $(\vec{v} \cdot \vec{\nabla})\vec{B} = \vec{v}(\vec{\nabla} \cdot \vec{B}) - \vec{\nabla} \times (\vec{v} \times \vec{B}) = -\vec{\nabla} \times (\vec{v} \times \vec{B})$, leading to:

$$\oint_C d\vec{l} \cdot \vec{E} = -\int_{S(C)} d\vec{S} \cdot \left[\frac{\partial \vec{B}}{\partial t} + \vec{\nabla}_c \times \left(\vec{v} \times \vec{B} \right) \right]$$

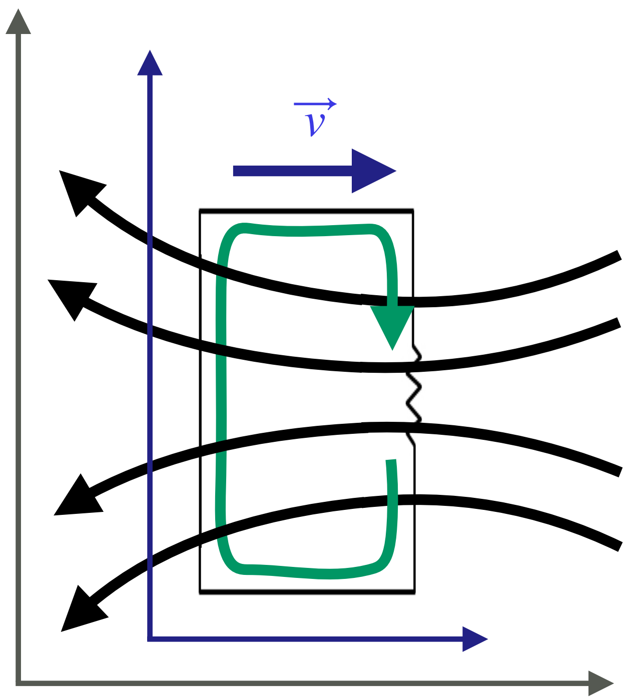
- Now using the Stokes theorem and passing the last term to the left-hand side we get:

$$\oint_C d\vec{l} \cdot \left(\vec{E} + \vec{v} \times \vec{B} \right) = -\int_{S(C)} d\vec{S} \cdot \frac{\partial \vec{B}}{\partial t}$$

- What we have done is, in effect, express Faraday's law in the reference frame of the "lab", where the magnetic field is static. So, the left-hand side must be the electric field in the reference frame of the circuit! And in fact, the expression is that for the Lorentz force!

- So, even if the magnetic field is not really changing at all ($\partial \vec{B}/\partial t = 0$), there is still an induction in the circuit, due to the fact that the circuit is moving, and therefore its charges will feel the Lorentz force from that velocity.

- In other words: in the frame where the magnetic field is static, the **circulation** of the electric and magnetic fields in the Lorentz force **cancel each other exactly**! (Notice, however, that the charges **do move** inside that circuit: it is only this "circulation" that cancels out!)



We will see later this this is a bit of an oversimplification: we really ought to use the Lorentz transformations! Nevertheless, Faraday's law remains exactly valid, as do our derivation – for the most part!

Energy of the magnetic field

- Faraday's law then tells us that:

$$\oint_C d\vec{l} \cdot \vec{E} = \int_{S(C)} d\vec{S} \cdot \left(\vec{\nabla} \times \vec{E}\right) = -\int_{S(C)} d\vec{S} \cdot \frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

- For a discussion about energy, though, it is useful to return to the notion of electromotive force, hence":

$$\mathscr{E} = \oint_C d\vec{l} \cdot \vec{E} = -\frac{d\Phi_B}{dt}$$

- This electromotive force does work on the charges that move around that circuit. In fact:

$$\oint_C (I\, d\vec{l}) \cdot \vec{E} = \oint_C (dq\, \vec{v}_q) \cdot \vec{E} = \oint_C d\vec{F}_q \cdot \vec{v}_q = \frac{d\,W}{dt}$$

- Therefore:

$$\frac{dW}{dt} = -I\frac{d\Phi_B}{dt}$$

- Let's consider what this means for a circuit that is being kept fixed, as we increase the magnetic field from zero up to a given value.
- We can express the work that is done in that process, in order to generate the electromotive force, as:

$$\frac{dW}{dt} = -I\int_{S(C)} d\vec{S} \cdot \frac{\partial \vec{B}}{\partial t} = -I\int_{S(C)} d\vec{S} \cdot \frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{A}\right) = -I\frac{\partial}{\partial t}\int_{S(C)} d\vec{S} \cdot \left(\vec{\nabla} \times \vec{A}\right)$$

$$= -I\frac{\partial}{\partial t}\oint_C d\vec{l} \cdot \vec{A}$$

Energy of the magnetic field

- We just derived an expression for the work that is done by the electromotive force for a circuit with a fixed current I, as we increase the field $\vec{A}$ (or, equivalently, $\vec{B}$):

$$\frac{dW}{dt} = -I\oint_C d\vec{l} \cdot \frac{\partial \vec{A}}{\partial t}$$

- Therefore, the energy that flowed **to the circuit, from the magnetic field,** is equal but with opposite sign:

$$\frac{dU_B}{dt} = +\oint_C \left(Id\vec{l}\right) \cdot \frac{\partial \vec{A}}{\partial t} = \int \left(\vec{J}\, dV\right) \cdot \frac{\partial \vec{A}}{\partial t}$$

- Now employ Ampère's law to express the current density in terms of the magnetic field, $\vec{\nabla} \times \vec{H} = \vec{J}$:

$$\frac{dU_B}{dt} = \int dV \left(\vec{\nabla} \times \vec{H}\right) \cdot \frac{\partial \vec{A}}{\partial t} = \int dV \left[\vec{H} \cdot \left(\vec{\nabla} \times \frac{\partial \vec{A}}{\partial t}\right) + \vec{\nabla} \cdot \left(\vec{H} \times \frac{\partial \vec{A}}{\partial t}\right)\right]$$

$$= \int dV \left[\vec{H} \cdot \left(\frac{\partial \vec{B}}{\partial t}\right)\right] + \oint_{S(V)} d\vec{S} \cdot \left[\vec{H} \times \frac{\partial \vec{A}}{\partial t}\right] \quad \to \quad \int dV \left[\vec{H} \cdot \left(\frac{\partial \vec{B}}{\partial t}\right)\right]$$

- Finally, assuming a linear constitutive relation between $\vec{B}$ and $\vec{H}$ we obtain:

$$\frac{dU_B}{dt} = \frac{\partial}{\partial t}\int dV\, \frac{1}{2}\vec{H} \cdot \vec{B} \qquad \Longrightarrow \qquad \rho_B = \frac{1}{2}\vec{H} \cdot \vec{B}$$ is the **energy density** of the magnetic field.

- Together with the result for the energy of the electric field we obtain the total energy density of the electromagnetic field:

$$\rho_{EM} = \frac{1}{2}\left(\vec{D} \cdot \vec{E} + \vec{H} \cdot \vec{B}\right) \quad ,$$ which is correct even in a relativistic sense!

The quantum Hall effect

- The previous discussion means that we the energy in the magnetic field for a collection of N current loops is something like:

  $$U_B = U_B(\Phi_1, \vec{r}_1\, ;\, \Phi_2, \vec{r}_2\, ;\, \ldots\, ;\, \Phi_N, \vec{r}_N) \quad ,$$

  where Φ_i is the magnetic flux through the loop i .

- Therefore, the energy buildup for those loops is expressed as:

  $$dU_B = \sum_{i=1}^{N} \left[\frac{\partial U_B}{\partial \Phi_i} d\Phi_i + (\vec{\nabla}_i U_B) \cdot d\vec{r}_i \right] \quad ,$$

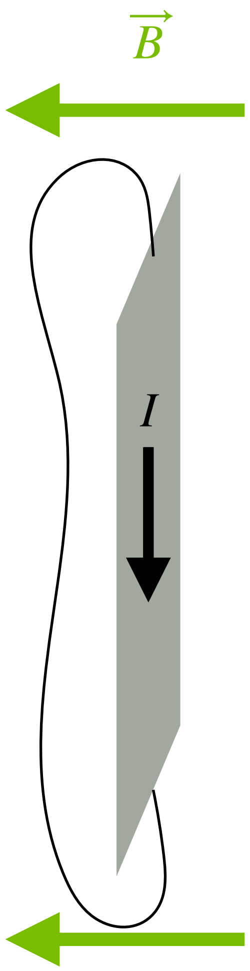
  where the partial derivatives here mean that all other variables are kept constant.

- Since $\Delta U_B = I\, \Delta\Phi$, we get that:

  $$dU_B = \sum_{i=1}^{N} \left[I_i\, d\Phi_i - \vec{F} \cdot d\vec{r}_i \right]$$

- In practice it may be very hard to keep all the other magnetic fluxes constant: only a superconductor is able to do that (as we will see later). But let's assume we can do it, in an approximate way.

- Let's now do an experiment to measure how the energy gets distributed inside a conductor and how that is related to the flux. In this experiment we have a 2D film (a strip) that carries the electrons, and a magnetic field applied perpendicular to this strip. In this setup, the electrons form something called an "electron gas".



The quantum Hall effect

- Now, let's apply a potential between the two edges of this conducting strip, in such a way that the electrons drifting through the conductor are gently pushed to one side.
- We can define a "resistivity" related to the drift of these electrons to the **side** of the strip, as a function of the current **along** the strip:

$$R_\perp = \frac{V}{I} \quad .$$

- Now, since $I = \Delta U_B / \Delta \Phi_B$, we have:

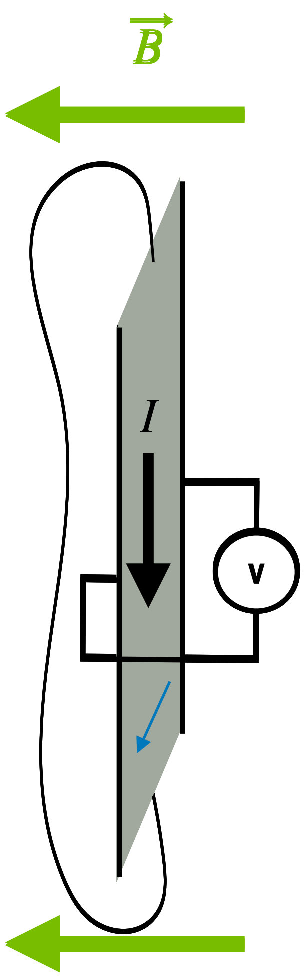
$$R_\perp = V \frac{\Delta \Phi_B}{\Delta U_B}$$

- However, now think about the tiniest possible variation in energy for this configuration: this would correspond to a single electron moving along the potential difference V, so $\Delta U_e = eV$.
- In an electron gas, some of the electronic quantum levels can be degenerate, such that when one electron moves, the others like it move as well. Let's call this degeneracy number n. Then, we have $\Delta U_n = neV$.
- Now, in 1982 Laughlin showed that the magnetic flux through a current-carrying strip like this is actually quantized, and the "quanta" of magnetic flux are given by:

$$\delta \Phi_B = h/e$$

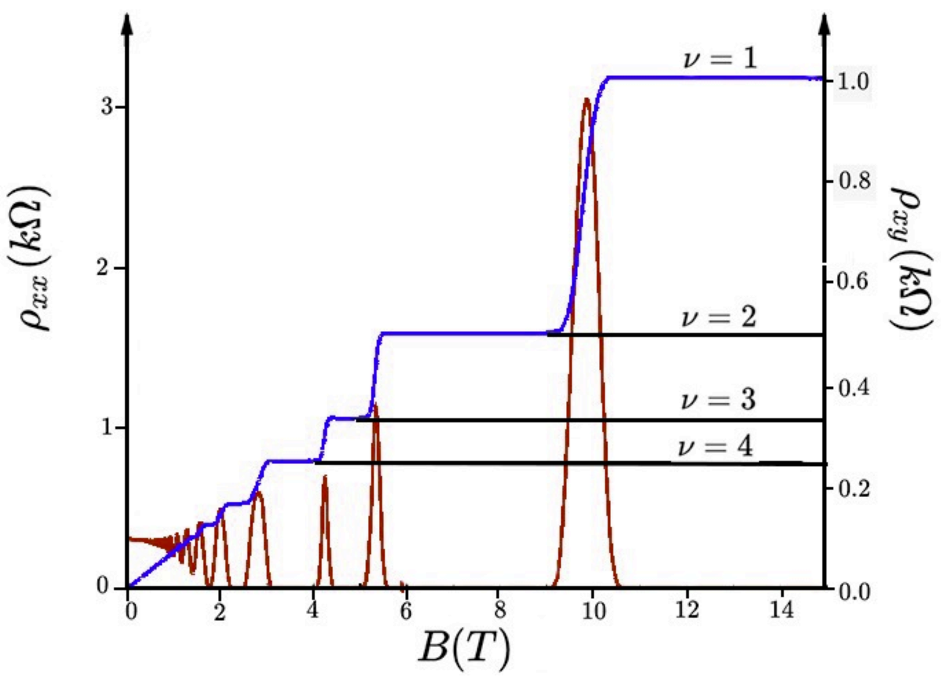
- Substituting into the resistivity we get that:

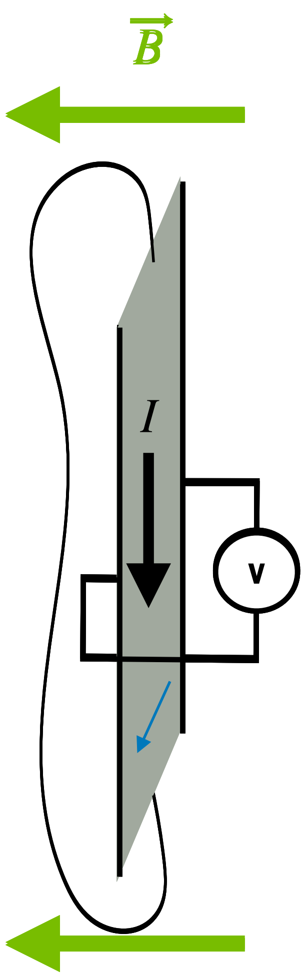
$$R_\perp = V \frac{h/e}{n\, e\, V} = \frac{h}{n\, e^2}$$



The quantum Hall effect

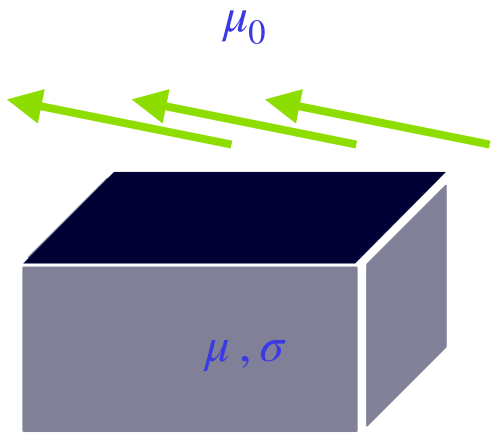
- We can now think that we keep the current constant and slowly vary the magnetic field, changing the magnetic flux. The results are as shown in the plot below, and is known as the "Quantum Hall Effect".





- The perpendicular resistivity is shown in blue, and the longitudinal resistivity in red. An **very naive** explanation for this complementarity of the longitudinal resistivity with respect to the perpendicular (Hall) is that when the field (and flux) have just the right values, it is not possible for an integer number of electrons to switch from one end of the strip to the other. As a result, they accumulate, increasing the resistivity.

Magnetic diffusion



- Let's assume that there is some slowly varying magnetic field in the vicinity of a material, in such a way that the induced electric fields and currents in that material are not too strong.
- In that case we can assume that the induced currents are proportional to the induced electric field, according to Ohm's law:

$$\vec{J} = \sigma\,\vec{E}$$

- We then get that:

$$\vec{\nabla} \times \vec{H} = \vec{J} = \sigma\vec{E} \quad \leftrightarrow \quad \vec{\nabla} \times \vec{B} = \mu\sigma\vec{E}$$

But since $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$, we get that $\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = -\dfrac{\partial(\vec{\nabla} \times \vec{B})}{\partial t} = -\mu\sigma\dfrac{\partial \vec{E}}{\partial t}$

which leads to:

$$\vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \vec{\nabla}^2\vec{E} = -\mu\sigma\frac{\partial \vec{E}}{\partial t}$$

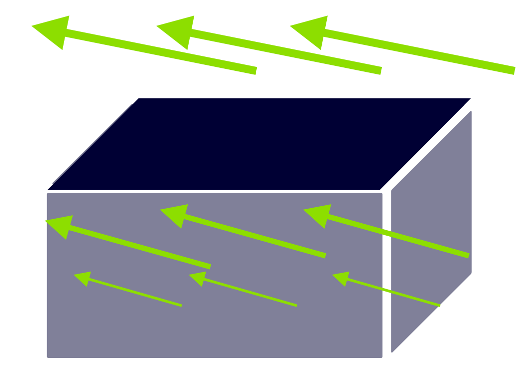
- But there are no charges in this problem (except, maybe, surface charges in the dielectric), so $\vec{\nabla} \cdot \vec{E} = 0$, hence:

$$\vec{\nabla}^2\vec{E} = \mu\sigma\frac{\partial \vec{E}}{\partial t}$$

- Similarly, we can derive an equation for the magnetic field and the vector potential:

$$\vec{\nabla}^2\vec{A} = \mu\sigma\frac{\partial \vec{A}}{\partial t} \quad , \quad \text{or} \quad \vec{\nabla}^2\vec{B} = \mu\sigma\frac{\partial \vec{B}}{\partial t} \quad ,$$ which is called the magnetic diffusion equation .

Magnetic diffusion



- The magnetic diffusion equation tells us that a magnetic field that changes with time can penetrate a medium through the induced currents, which generate more magnetic field, which generate more currents, and so on and so forth.
- As it is clear from its name, the "diffusion" in this case refers to the way in which the magnetic field penetrates a material, similar to the way in which heat is diffused in a medium. From the diffusion equation we get that:

$$\vec{\nabla}^2 \vec{A} \sim \frac{\Delta \vec{A}}{\Delta x^2} \sim \mu\sigma \frac{\partial \vec{A}}{\partial t} \sim \mu\sigma \frac{\Delta \vec{A}}{\Delta t} \quad ,$$

  that is, for a field that changes on timescales of Δt, the diffusion scale is $\Delta x \sim \sqrt{\Delta t / \mu\sigma}$.

  This is in fact called the **skin depth** of the material.

- As an example, consider a set up in which the magnetic field in the region $z > 0$ is $\vec{H}_> = H_0 \, e^{i\omega t} \, \hat{x}$, and in the region $z \leq 0$ we have a material with magnetic permeability μ and conductivity σ.
- The boundary conditions tell us that

$$\Delta \vec{H}_{||} = \vec{K} \times \hat{n} \to 0 \quad \text{(no free currents!)} \quad , \quad \text{and} \quad \Delta B_\perp = 0$$

- But in this case there is no field in the z direction and the only component is $\vec{H} \sim \hat{x}$, so we know that any induced surface currents will be in the $\hat{y}$ direction. Therefore, we try a solution of the sort:

$$\vec{H}_< = H_0 \, f(z) \, e^{i\omega t} \hat{x}$$

Magnetic diffusion

- Substituting this trial solution into the diffusion equation gives us:

$$\frac{d^2}{d\,z^2}\left[H_0\,f(z)\,e^{i\omega t}\,\hat{x}\right] = \mu\sigma\,\frac{d}{dt}\left[H_0\,f(z)\,e^{i\omega t}\,\hat{x}\right]$$

$$\Rightarrow \quad \frac{d^2 f}{d\,z^2} = i\omega\mu\sigma f(z)$$

- This is just like any other exponential/trigonometric solution, except that the frequency is not pure real nor pure imaginary, but complex:

$$f(z) = f_+\,e^{qz} + f_-\,e^{-qz} \quad , \quad \text{with} \quad q^2 = i\omega\mu\sigma$$

- This can be easily solved using $i = e^{i\pi/2}$, hence:

$$q = \left(e^{i\pi/2}\omega\mu\sigma\right)^{1/2} = e^{i\pi/4}\sqrt{\omega\mu\sigma} = \frac{1+i}{\sqrt{2}}\sqrt{\omega\mu\sigma}$$

- Clearly, in the $z \leq 0$ region we have only the solution which decays, $\sim e^{qz}$ (notice that in this step we use the "boundary condition" that the field at spatial infinity, $z \to -\infty$, goes to zero!), leaving us with the solution:

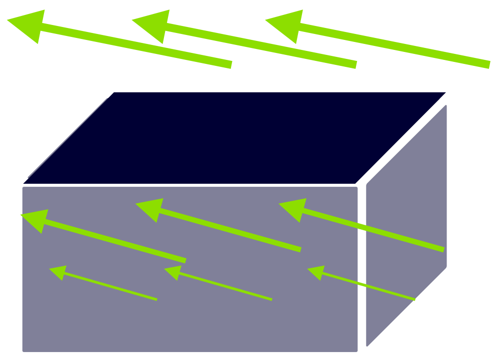
$$\vec{H}_< = H_0\,\exp\left[\frac{1+i}{2}\sqrt{\omega\mu\sigma}\,z\right]\,e^{i\omega t}\,\hat{x}$$

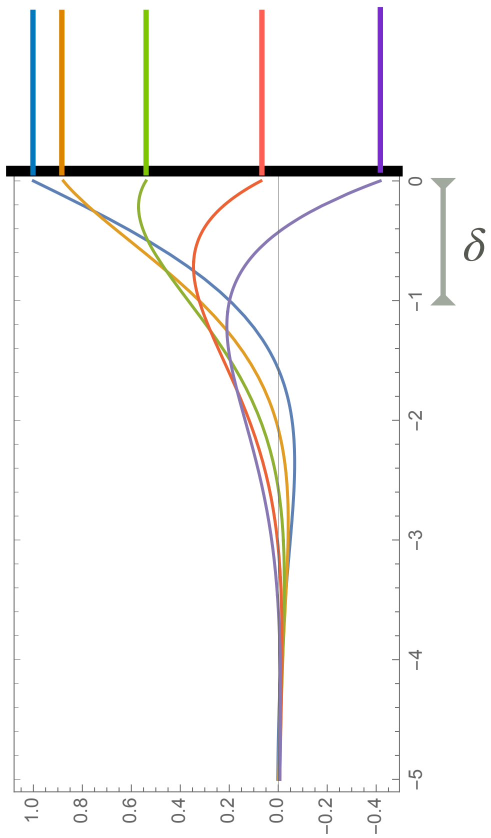
- We now define the skin depth of the material as:

$$\delta = \frac{2}{\sqrt{\omega\mu\sigma}} \quad \Rightarrow \quad \vec{H}_< = H_0\,e^{z/\delta}\cos(\omega t + z/\delta)\,\hat{x}$$

- I will leave it to you as an exercise to show that the electric field and the induced surface current are given by:

$$\vec{E}_< = \frac{\mu\omega\delta}{\sqrt{2}}\,H_0\,e^{z/\delta}\,\cos(\omega t + z/\delta - 3\pi/4)\,\hat{y} \quad , \quad \text{and that} \quad \vec{J} = \sigma\vec{E} \quad , \quad \text{so that} \quad \int_{-\infty}^{0} dz\,J_y(z) = -H_0\cos\omega t \quad !$$





Induction: examples

- Let's start by considering a very thin, conducting disk of radius R, which is made up of many thin concentric circular wire loops.
- Let's say that through this disk passes a magnetic field $\vec{B}$ that has an angle with the normal to that disk, as shown in the figure.
- The **flux** of the magnetic field through an individual loop of radius r is therefore:

$$\Phi_B(r) = \pi r^2 B \cos\theta$$

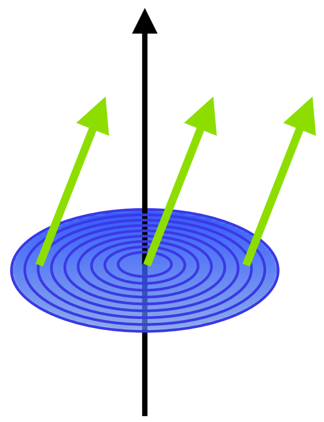
- An electromotive force will be generated if we (a) increase the value of the field, or (b) rotate the disk. Let's assume that we keep the disk rigidly in its place, but vary the field, so that:

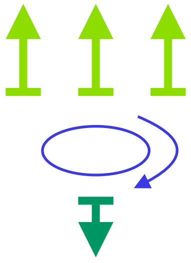
$$\mathcal{E}_r = -\frac{d\Phi_B}{dt} = -\pi r^2 \frac{dB}{dt} \cos\theta$$

- But if we have an arbitrarily large number of loops, this sum of electromotive forces could become arbitrarily large as well! So, what is going on here?
- What happens in any real situation is that, in each loop, the external field generates circulating currents which **oppose** the external field. The currents are proportional to the resistance of each loop, which is itself proportional to the length of that circuit ($2\pi r$), so:

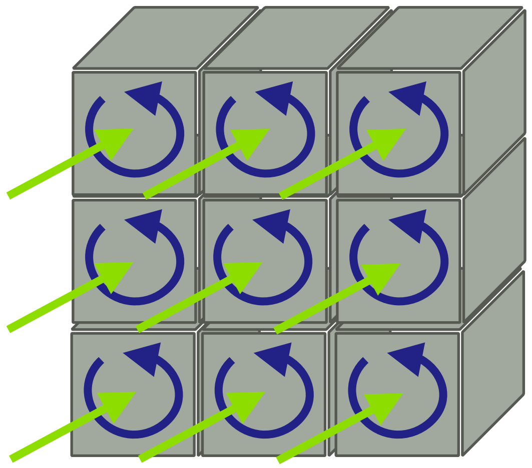
$$\vec{J}_r = \sigma \vec{E}_r \quad , \quad \mathcal{E}_r = \oint_r d\vec{l} \cdot \vec{E}_r \quad \Rightarrow \quad I_r \sim \frac{\mathcal{E}_r}{r} \sim r$$

- Therefore, the current grows from the center, and the "backreaction" of the currents induced on the loops lower the field for the next loop, and so on and so forth.
- Another way to think about this is that this is a question about the **self-inductance** of a material with some bulk volume. We will come back to this very soon.

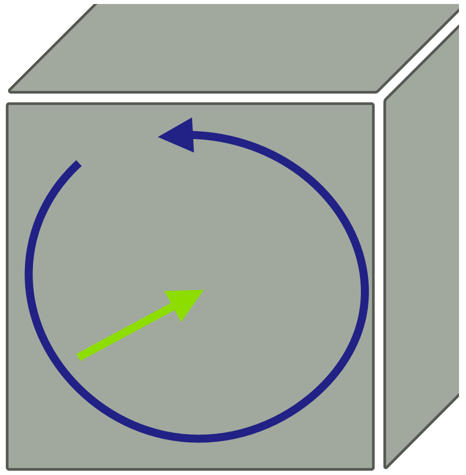




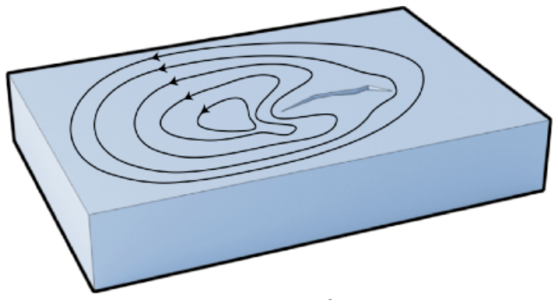
Induction: examples



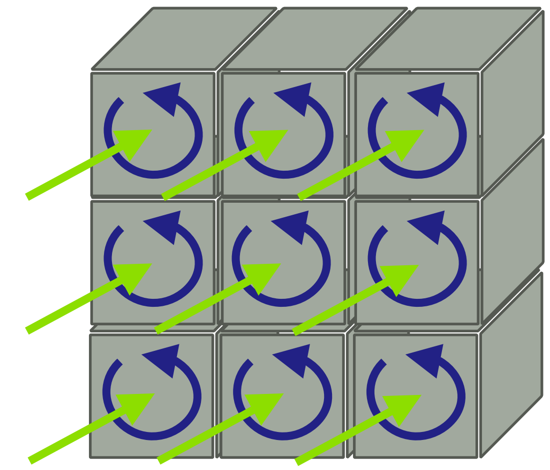
- When a time-varying magnetic field hits a conductor, it will induce currents all over that conductor.
- But **where** exactly are those currents? We give them a name, ***Eddy currents*** (also known as ***Foucault currents***).
- We should **always remember** that the laws of Electrodynamics are **local**: what is happening in a microscopic region is determined by the fields in that microscopic region.

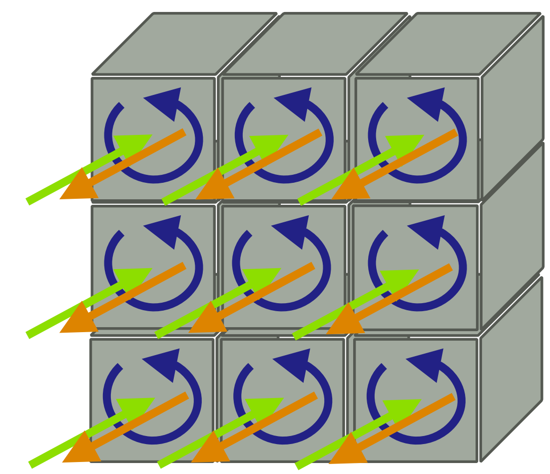


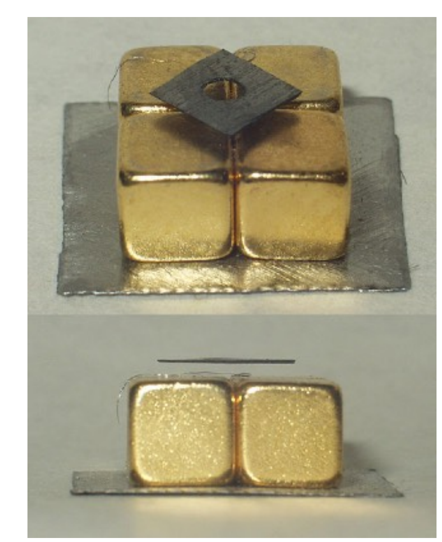
- Eddy currents are induced in a completely local manner, but the way in which the small domains combine are determined by the shape and geometry of the material, by the crystal structure, and even by small defects in the material. The overall effect are currents which “look" macroscopic, but they are fundamentally microscopic!



Induction: examples





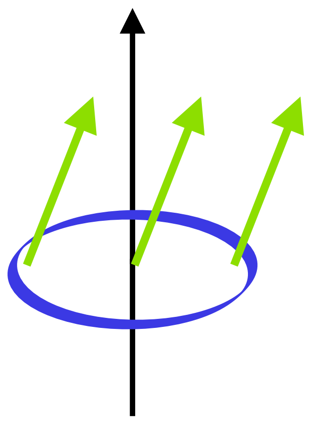


- A radical example of a material which is able to generate very large Eddy currents is a superconductor.
- In a superconductor, there is no resistivity, $R \to 0$: the charges can move freely, without any obstruction. This means that an electromotive force will induce currents which can be arbitrarily large, since $I = \mathscr{E}/R$.
- The magnetic field generated by the induced currents will grow and grow, until it cancels out the external field that caused those induced currents in the first place! At that point, the currents will cease to grow, since the total field in that circuit is zero!
- Microscopically, you can think of the Eddy currents as magnetic dipoles which cancel out the external magnetic field, in a similar way that local charges in a conductor cancel out an external electric field near a conductor.
- This means that, as you try to push a magnet closer to a superconductor, there will be a counteracting magnetic field from the induced currents, in order to cancel the field inside the superconducting material, resulting in a repulsive force. In fact, you can compute the magnetic pressure of a magnetic field on a superconductor:

$$P_B = \frac{B^2}{2\mu_0}$$

- This forms the basis for the "levitation" experiments that you are all familiar with! *[Now let's watch a nice YouTube video: https://www.youtube.com/watch?v=zPqEEZa2Gis]*

Induction: examples



- OK, now let's compute the inductance in some particular cases. Consider a single loop of radius R, with a section of area a, and let's ignore the self-inductance of the loop on itself.
- Let's ask what is the work that is done as we induce a current on that loop.
- In the first situation, let's assume a neutral wire with a conductivity σ. In that case we can think of negative charges moving in one direction, and positive charges moving in the other direction. In other words:

$$\vec{F}_+ = + q\vec{E} \quad \text{and} \quad \vec{F}_- = - q\vec{E} \quad , \quad \text{but} \quad \vec{F}_{Tot} = \vec{F}_+ + \vec{F}_- = 0$$

- However, each one of those charges move left and right, and in fact it is the negative charges that are really drifting through the conductor, so if that movement is in a stationary regime then the power dissipated is given by:

$$P = \frac{dW}{dt} = \oint d\vec{F} \cdot \vec{v} = \oint (dq\vec{v}) \cdot \vec{E} = \oint (Id\vec{l}) \cdot \vec{E} = \int_V (\vec{J} dV) \cdot \vec{E}$$

$$= \sigma \int dV \vec{E}^2$$

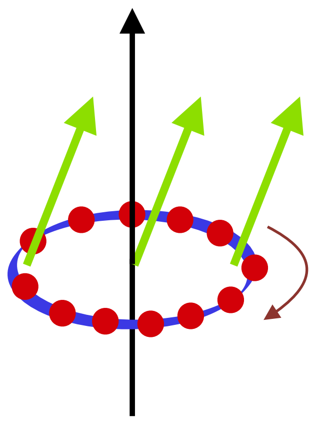
- Now, from the discussion of the last slide we have that:

$$E = -\frac{1}{2} r\dot{B} \cos\theta \, \hat{\varphi} \quad , \quad \text{therefore:}$$

$$P = \sigma(a 2\pi r) \vec{E}^2 = \frac{\sigma\pi a r^3}{2} \dot{B}^2 \cos\theta^2$$

- Ok, so this is typically dissipated by heat, and that comes from work done to maintain the external field.
- But what if we could convert some of that energy into **motion**?

Induction: examples



- Let's now say that our loop of radius R is made of a non-conducting material, in which we place some charges q at regular intervals — we can think of a linear charge density λ .

- Let's ask what happens as we induce a current on that loop by varying the magnetic flux. The power is given, again, by:

$$P = \frac{dW}{dt} = \oint d\vec{F} \cdot \vec{v} = \oint (dq\vec{v}) \cdot \vec{E} = \oint (\lambda dl\vec{v}) \cdot \vec{E}$$

- If the ring starts to rotate, it does so with some angular velocity ω, so that $v = \omega r$, and this calculation then leads to:

$$P = \lambda\,(2\pi r)\,(\omega r)\,\left(-\frac{1}{2}r\dot{B}\,\cos\theta\right) = -\,\lambda\,\omega\,\pi r^2\,\dot{B}\,\cos\theta$$

- Therefore, this power is being converted into angular momentum of the ring.

- Sure, the power itself comes from the source of the external field, and the magnetic field carries that power from the source to the wire.

- But this calculation shows that the magnetic field is also able to **carry angular momentum**, and transfer it to the wire!

Inductance and self-inductance

- In order to start the discussion about inductance let's start with the simplest scenario: two coaxial solenoids, both of the same length (h).
- The inner solenoid, or radius R_1 and cross-section $A_1 = \pi R_1^2$, has N_1 loops, while the outer solenoid, of radius R_2, has N_2 loops.
- The magnetic field of the outer solenoid is:

$$B_2 = \frac{N_2}{h} \mu_0 I_2 \quad ,$$

leading to a total flux inside the inner solenoid of:

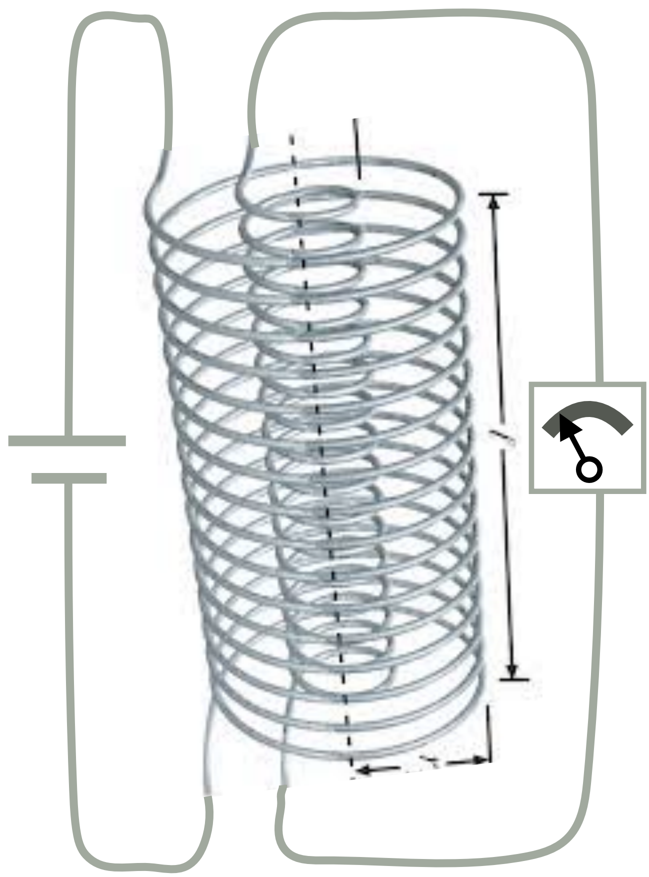
$$\Phi_{1\leftarrow 2} = N_1 A_1 \, \mu_0 \frac{N_2}{h} I_2 \quad ,$$ and the proportionality is the **inductance**: $$M_{1\leftarrow 2} = \mu_0 \, A_1 \frac{N_1 N_2}{h}$$

- When we invert the situation, we have a magnetic field in the inner solenoid of:

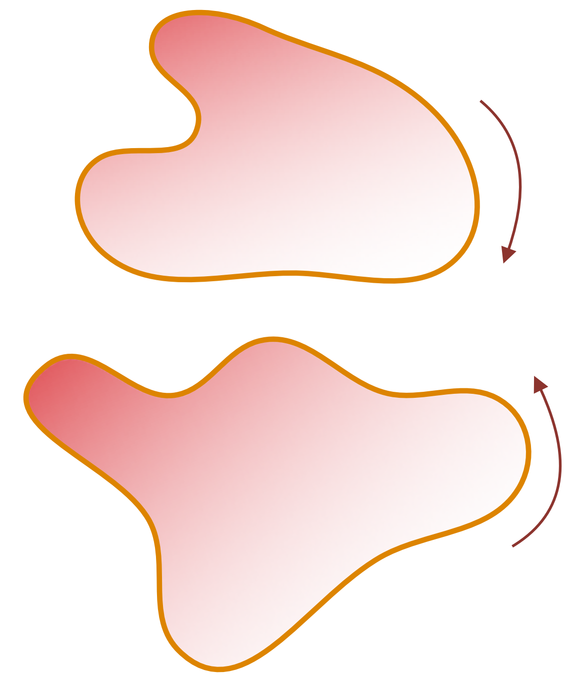
$$B_1 = \frac{N_1}{h} \mu_0 I_1 \quad ,$$

but since the area occupied by this magnetic field is still A_1 , the flux inside the outer solenoid is:

$$\Phi_{2\leftarrow 1} = N_2 A_1 \, \mu_0 \frac{N_1}{h} I_1 \quad ,$$ so the **mutual inductances are identical**: $$M_{2\leftarrow 1} = M_{1\leftarrow 2} = \mu_0 \, A_1 \frac{N_1 N_2}{h}$$



Inductance and self-inductance



- Inductance can also be framed as a statement in terms of the vector potential
- The field generated by one loop is given by:

$$\vec{B}_1(\vec{r}) = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\vec{l}_1 \times (\vec{r} - \vec{r}_1)}{|\vec{r} - \vec{r}_1|^3} \quad , \quad \text{or} \quad \vec{A}_1(\vec{r}) = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\vec{l}_1}{|\vec{r} - \vec{r}_1|}$$

- The flux of the first circuit through the second is given by:

$$\Phi_{2\leftarrow 1} = \int d\vec{S}_2 \cdot \vec{B}_1 = \int d\vec{S}_2 \cdot \left(\vec{\nabla} \times \vec{A}_1 \right) = \oint d\vec{l}_2 \cdot \vec{A}_1$$

$$= \frac{\mu_0}{4\pi} I_1 \oint \oint \frac{d\vec{l}_2 \cdot d\vec{l}_1}{|\vec{r}_2 - \vec{r}_1|}$$

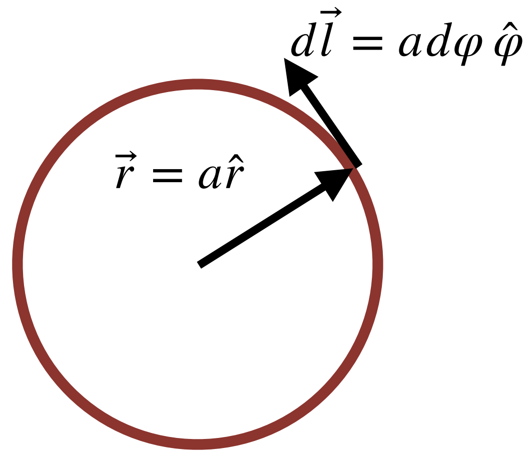
- The flux of the second circuit through the first is given by the same expression:

$$\Phi_{1\leftarrow 2} = = \frac{\mu_0}{4\pi} I_2 \oint \oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{|\vec{r}_1 - \vec{r}_2|}$$

- Is it obvious, then, that the mutual inductance $L_{12} = \Phi_{1\leftarrow 2}/I_2 = \Phi_{2\leftarrow 2}/I_1 = L_{21}$. This is called the "Neumann formula".
- Although the inductance may be hard to compute in practice, it is in fact very easy to **measure**!

Inductance and self-inductance

- It is interesting to compute the self-inductance that a system has with itself. Let's consider a circular loop of radius a that we consider, for now, to be made of a very thin wire.

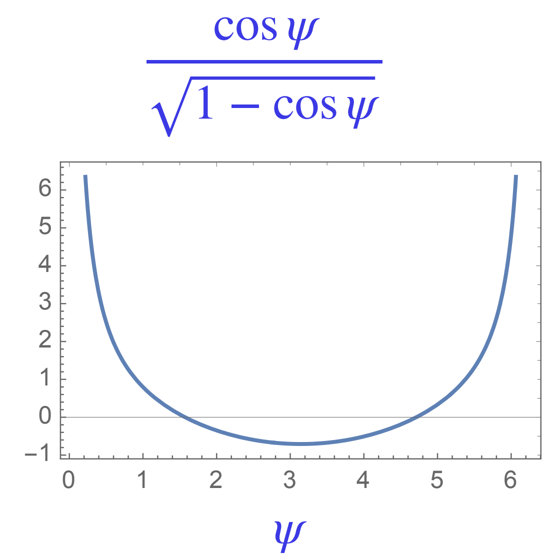


- The self-inductance is then expressed as the sum of the inductances of each piece of that wire with every other piece of the wire, as the same double integral:

$$L = \frac{\mu_0}{4\pi} \oint \oint \frac{d\vec{l}_2 \cdot d\vec{l}_1}{|\vec{r}_2 - \vec{r}_1|}$$

$$= \frac{\mu_0}{4\pi} \oint \oint \frac{(a d\varphi_1 \hat{\varphi}_1) \cdot (a d\varphi_2 \hat{\varphi}_2)}{|a\hat{r}_2 - a\hat{r}_1|}$$

$$= \frac{\mu_0}{4\pi} \int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_2 \frac{a^2 \cos(\varphi_1 - \varphi_2)}{a\sqrt{2 - 2\cos(\varphi_1 - \varphi_2)}}$$



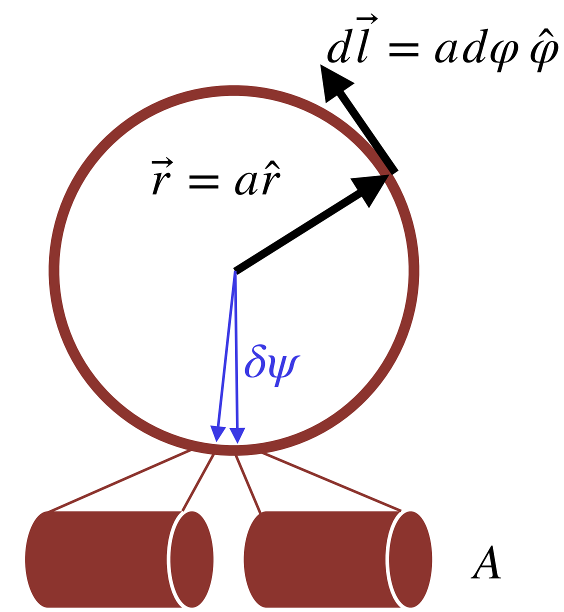
- We can change variables and use $\psi = \varphi_2 - \varphi_1$, $\alpha = \varphi_2 + \varphi_1$, and after taking into account the Jacobian for this transformation we obtain:

$$L = \frac{\mu_0}{4} a \int_0^{2\pi} d\psi \frac{\cos\psi}{\sqrt{1 - \cos\psi}}$$, but in that case this expression diverges!

Inductance and self-inductance

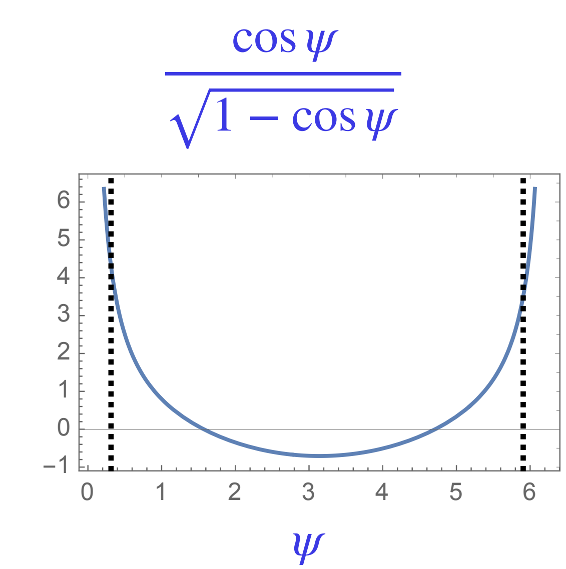
- The problem here is that we assumed that the wire is infinitely thin, instead of assuming it has a cross section with a finite area, where, for simplicity, we can assume that $A \ll a^2$.
- If we want to make a rigorous calculation, we end up with an elliptical integral (!!).
- However, we can obtain the same qualitative result if we "regularize" the limits $\psi \to 0$ and $\psi \to 2\pi$ in the previous integral. In fact, looking at a wire with a finite cross section we can even estimate the size of this angle:

$$\delta\psi \sim \frac{\sqrt{A}}{a}$$

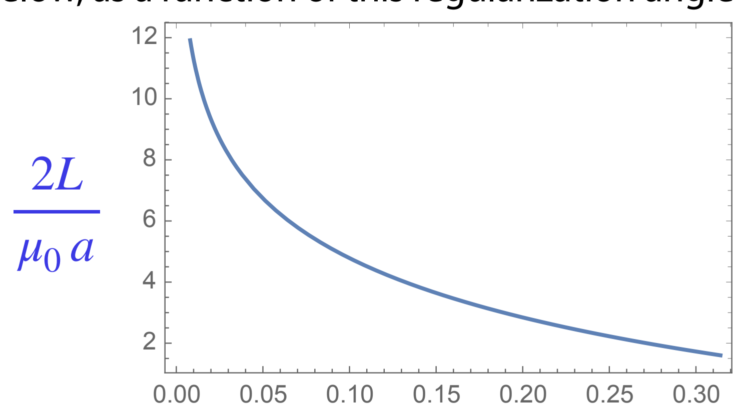


- Using these finite intervals, the integral can actually be evaluated analytically:

$$L = \frac{\mu_0}{4} a \int_{\delta\psi}^{2\pi-\delta\psi} d\psi \, \frac{\cos\psi}{\sqrt{1-\cos\psi}} = \frac{\mu_0 \, a}{2} \left[\frac{\sin\psi}{\sqrt{1-\cos\psi}} \left(2\cos\frac{\psi}{2} + \log\tan\frac{\psi}{4} \right) \right]_{\delta\psi}^{2\pi-\delta\psi}$$



The result is shown below, as a function of this regularization angle $\delta\psi$:



Next class:

- Maxwell's equations
- The Maxwell "displacement current"
- Introduction to Electrodynamics
- Jackson, Ch. 6