PGF5003: Classical Electrodynamics I
Problem Set 1

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1 Question

Given the following vector field $\mathbf{F} = f\vec{\nabla}g$, *with* f *and* g *scalar functions, prove the first Green identity:*

$$\int_V dV \left(f\nabla^2 g + \vec{\nabla} f \cdot \vec{\nabla} g \right) = \oint_{S(V)} d\vec{S} \cdot \left(f\vec{\nabla} g \right). \tag{1}$$

What do we need to suppose about f *and* g *to deduce this identity?*

1.1 Solution

Considering the vector field $\mathbf{F} = f\vec{\nabla}g$, its divergence is given by

$$\vec{\nabla} \cdot \mathbf{F} = \vec{\nabla} \cdot (f\vec{\nabla}g) = \vec{\nabla} f \cdot \vec{\nabla} g + f\nabla^2 g. \tag{2}$$

Integrating the above relation in the volume V we get

$$\int_V dV \left[\vec{\nabla} f \cdot \vec{\nabla} g + f\nabla^2 g \right] = \int_V dV \vec{\nabla} \cdot \mathbf{F} = \int_V dV \vec{\nabla} \cdot (f\vec{\nabla} g). \tag{3}$$

Then, using the **divergence theorem** for the field $\mathbf{F}$

$$\int_V dV \vec{\nabla} \cdot \mathbf{F} = \oint_{S(V)} d\mathbf{S} \cdot \mathbf{F} \tag{4}$$

we have

$$\int_V dV \left(f\nabla^2 g + \vec{\nabla} f \cdot \vec{\nabla} g \right) = \oint_{S(V)} d\mathbf{S} \cdot \left(f\vec{\nabla} g \right) \square. \tag{5}$$

Basically, f needs to be once continuously differentiable (C^1) and g needs to be twice continuously differentiable (C^2).

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2 Question

Show that

$$\nabla^2 \left(\frac{1}{r} \right) = -4\pi\delta(\mathbf{r}). \tag{6}$$

2.1 Solution

If we take the gradient of $1/r$ we have

$$\nabla \frac{1}{r} = -\frac{\hat{r}}{r^2} = -\frac{\mathbf{r}}{r^3}, \tag{7}$$

for $r \neq 0$. Then, we can write

$$\nabla^2 \left(\frac{1}{r}\right) = \nabla \cdot \left(-\frac{\mathbf{r}}{r^3}\right) = 0, \tag{8}$$

when $r \neq 0$. However, integrating the above quantity in all the space (in spherical coordinates) and using the **divergence theorem**, we get

$$\begin{aligned} \int_V dV \nabla^2 \left(\frac{1}{r}\right) &= \int_V dV \nabla \cdot \left(-\frac{\mathbf{r}}{r^3}\right) = -\oint_{S(V)} d\mathbf{S} \cdot \frac{\mathbf{r}}{r^3} \\ &= -\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta r^2 \hat{r} \cdot \frac{\mathbf{r}}{r^3} = -4\pi. \end{aligned} \tag{9}$$

Therefore, the value of the Laplacian is zero everywhere except zero and the integral over any volume containing the origin is equal to -4π. In this way,

$$\nabla^2 \left(\frac{1}{r}\right) = -4\pi\delta(\mathbf{r})\Box. \tag{10}$$

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3 Question

Using the second Green identity:

$$\int_V dV \left(f\nabla^2 g - g\nabla^2 f\right) = \oint_{S(V)} d\mathbf{S} \cdot \left(f\vec{\nabla} g - g\vec{\nabla} f\right), \tag{11}$$

taking $f = \phi(\mathbf{x}')$ (for the electrostatic potential $\mathbf{E} = \vec{\nabla}\phi$) and $g = 1/R = 1/|\mathbf{x} - \mathbf{x}'|$, show that

$$\phi(\mathbf{x}) = \int_V d^3x' \frac{\rho(\mathbf{x}')}{R} + \frac{1}{4\pi}\oint_{S(V)} d\mathbf{S}' \cdot \left[\frac{1}{R}\vec{\nabla}'\phi(\mathbf{x}') - \phi(\mathbf{x}')\vec{\nabla}'\frac{1}{R}\right], \tag{12}$$

where $\vec{\nabla}'$ corresponds to differential operation related to $\mathbf{x}'$.

3.1 Solution

We can substitute f and g into the second Green identity as

$$\int_V dV' \left[\phi(\mathbf{x}')\nabla'^2\frac{1}{R} - \frac{1}{R}\nabla'^2\phi(\mathbf{x}')\right] = \oint_{S(V)} d\mathbf{S} \cdot \left[\phi(\mathbf{x}')\vec{\nabla}'\frac{1}{R} - \frac{1}{R}\vec{\nabla}'\phi(\mathbf{x}')\right]. \tag{13}$$

Here we need to use that

$$\nabla'^2\frac{1}{R} = \nabla'^2\frac{1}{|\mathbf{x} - \mathbf{x}'|} = -4\pi\delta(\mathbf{x} - \mathbf{x}') \tag{14}$$

and the first Maxwell equation

$$\vec{\nabla} \cdot \mathbf{E} = -\nabla^2 \phi(\mathbf{x}) = 4\pi\rho(\mathbf{x}) \tag{15}$$

then,

$$\int_V dV' \left[-4\pi\phi(\mathbf{x}')\delta(\mathbf{x} - \mathbf{x}') + \frac{4\pi\rho(\mathbf{x}')}{R} \right] = \oint_{S(V)} d\mathbf{S} \cdot \left[\phi(\mathbf{x}')\vec{\nabla}' \frac{1}{R} - \frac{1}{R}\vec{\nabla}'\phi(\mathbf{x}') \right]. \tag{16}$$

Rearranging the above expression, we arrive that

$$\phi(\mathbf{x}) = \int_V d^3x' \frac{\rho(\mathbf{x}')}{R} + \frac{1}{4\pi} \oint_{S(V)} d\mathbf{S}' \cdot \left[\frac{1}{R}\vec{\nabla}'\phi(\mathbf{x}') - \phi(\mathbf{x}')\vec{\nabla}' \frac{1}{R} \right] \Box. \tag{17}$$

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4 Question

Consider the electric field

$$\mathbf{E} = \frac{A e^{r/r_0}}{r} \hat{r}. \tag{18}$$

a) *Determine the density of charge.*

b) *Determine the total charge into a radius R.*

4.1 Solution

a) Using the first Maxwell equation $\vec{\nabla} \cdot \mathbf{E} = 4\pi\rho$, we have

$$\rho = \frac{\vec{\nabla} \cdot \mathbf{E}}{4\pi} = \frac{1}{4\pi} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{A e^{r/r_0}}{r} \right) \right] = \frac{1}{4\pi} \frac{A e^{r/r_0}}{r} \left(\frac{1}{r} + \frac{1}{r_0} \right). \tag{19}$$

b) The total charge is given just integrating the previous item in spherical coordinates as

$$\begin{aligned} Q &= \int_{V(R)} dq = \int_{V(R)} dV \rho \\ &= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^R dr r^2 \frac{1}{4\pi} \frac{A e^{r/r_0}}{r} \left(\frac{1}{r} + \frac{1}{r_0} \right) \\ &= A \left[\int_0^R dr e^{r/r_0} + \int_0^R dr \frac{r}{r_0} e^{r/r_0} \right] \\ &= A \left\{ r_0 \left(e^{R/r_0} - 1 \right) + \left[(R - r_0) e^{R/r_0} + r_0 \right] \right\} \\ &= A R e^{R/r_0}. \end{aligned} \tag{20}$$

5 Question

Consider two infinite plates (with zero thickness) with the distributions of charge σ and $-\sigma$, respectively. The plates are orthogonal to each other.

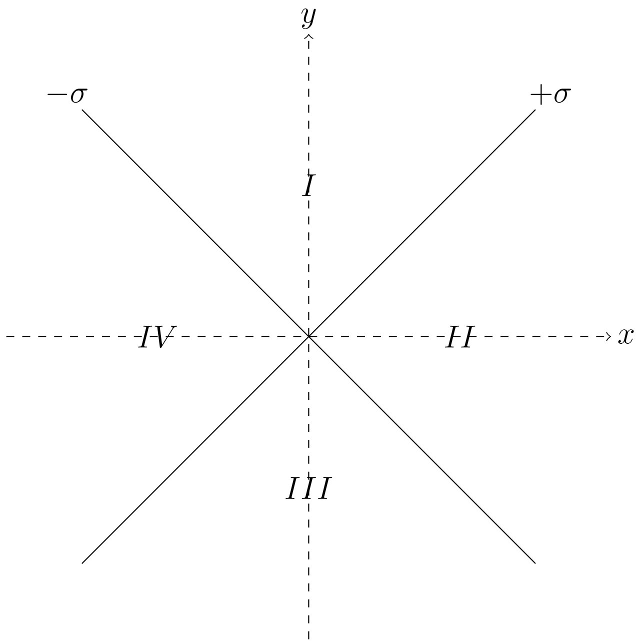

Figure 1: Figure for the question 5.

a) *Find the electric field in all the regions (I, II, III, IV and total space).*

b) *Draw a figure, representing the electrical field.*

5.1 Solution

a) Considering the Gauss theorem together with the first Maxwell's equation we have

$$\int_V dV \vec{\nabla} \cdot \mathbf{E} = \int_V dV 4\pi\rho = \oint_{S(V)} d\mathbf{S} \cdot \mathbf{E}. \tag{21}$$

Then, applying this to a cylinder with superficial density σ and superior and inferior area A

$$\begin{aligned} 4\pi\sigma A &= 2A|\mathbf{E}| \\ \mathbf{E} &= 2\pi\sigma\hat{n}. \end{aligned} \tag{22}$$

This means that the plate with $-\sigma$ will have the electrical field entering on itself and the other, with σ, have it pointing out of the plane. Considering a vector sum, and the normals written in terms of the angles with the planes and the axes, we just end up with:

$$\mathbf{E}_I = -2\pi\sigma\left(\sin\theta + \cos\theta, \sin\theta - \cos\theta, 0\right)_{\theta=45^\circ} = -2\pi\sigma\sqrt{2}\hat{x} \tag{23}$$
$$\mathbf{E}_{II} = -2\pi\sigma\left(\cos\theta - \sin\theta, \sin\theta + \cos\theta, 0\right)_{\theta=45^\circ} = -2\pi\sigma\sqrt{2}\hat{y} \tag{24}$$
$$\mathbf{E}_{III} = 2\pi\sigma\left(\cos\theta + \sin\theta, \sin\theta - \cos\theta, 0\right)_{\theta=45^\circ} = 2\pi\sigma\sqrt{2}\hat{x} \tag{25}$$
$$\mathbf{E}_{IV} = 2\pi\sigma\left(\cos\theta - \sin\theta, \sin\theta + \cos\theta, 0\right)_{\theta=45^\circ} = 2\pi\sigma\sqrt{2}\hat{y} \tag{26}$$
$$\mathbf{E}_{total} = 0. \tag{27}$$

b)

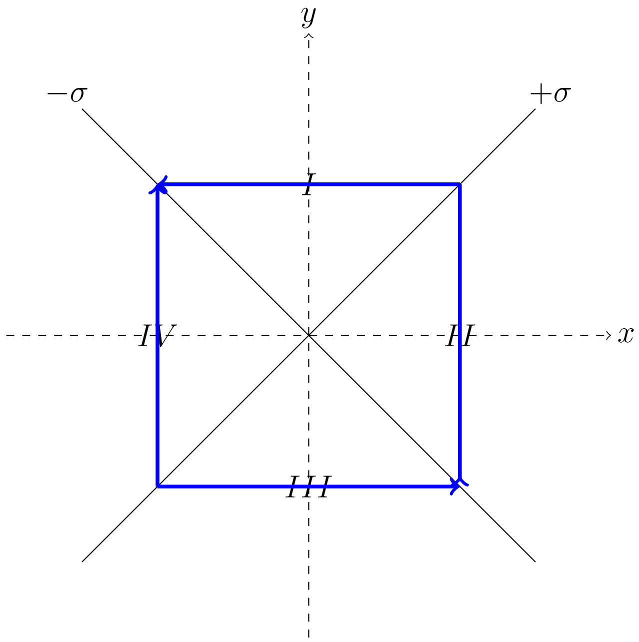

Figure 2: Figure for the solution of item b, question 5.

6 Question

A spherical shell of radius R is made with isolating material and has a surface density of charge σ (that, in principle, we do not know). The electric potential outside the sphere is $V_{out}(r) = V_0\left(\frac{R}{r}\right)^2\cos\theta$, where V_0 is a constant. The electric field $\mathbf{E}_{ins}(\mathbf{r}) = \frac{-V_0}{R}\hat{z}$. Compute:

a) *the electric field outside the sphere $\mathbf{E}_{out}(\mathbf{r})$ and the electric potential inside the sphere V_{ins};*

b) *the superficial density of charge σ;*

c) *the force per unity of area $\mathbf{f}$ over the surface of the sphere;*

6.1 Solution

a) To compute the electrical field outside, we can follow its definition according to the gradient of the potential outside the sphere as

$$\mathbf{E}_{out}(\mathbf{r}) = -\vec{\nabla} V_{out} = -\frac{\partial}{\partial r}\left[V_0 \left(\frac{R}{r}\right)^2 \cos\theta\right]\hat{r} - \frac{1}{r}\frac{\partial}{\partial\theta}\left[V_0 \left(\frac{R}{r}\right)^2 \cos\theta\right]\hat{\theta}$$
$$= 2V_0\cos\theta\frac{R^2}{r^3}\hat{r} + V_0\sin\theta\frac{R^2}{r^3}\hat{\theta}. \tag{28}$$

The electric potential inside the sphere is giving performing a line integral, "inverting" the previous relation in the region inside the sphere

$$V_{out}(R) - V_{ins}(r) = -\int_r^R d\mathbf{r}'\cdot\mathbf{E}_{ins}(\mathbf{r}') = -\int_r^R d\mathbf{r}'\cdot\frac{-V_0}{R}\hat{z}$$
$$V_0\cos\theta - V_{ins}(r) = \int_r^R dr'\frac{V_0\cos\theta}{R} = \frac{V_0\cos\theta}{R}(R-r)$$
$$V_{ins}(r) = V_0\cos\theta\frac{r}{R}. \tag{29}$$

Notice that we use $V(r=R)$ to match with the exterior potential on $r=R$.

b) We know that, if we consider the continuity equation of the electric field we get the superficial density of charge, then,

$$\sigma = \frac{1}{4\pi}\left[\mathbf{E}_{out}(\mathbf{r}) - \mathbf{E}_{ins}(\mathbf{r})\right]_{r=R}\cdot\hat{n}$$
$$= \frac{1}{4\pi}\left(2V_0\cos\theta\frac{R^2}{r^3}\hat{r} + V_0\sin\theta\frac{R^2}{r^3}\hat{\theta} + \frac{V_0}{R}\hat{z}\right)\cdot\hat{r}$$
$$= \frac{1}{4\pi}\left(2V_0\cos\theta\frac{1}{R} + 0 + \frac{V_0}{R}\cos\theta\right) = \frac{1}{4\pi}\frac{3V_0\cos\theta}{R}. \tag{30}$$

c) The force could be computed as

$$\mathbf{f} = \left.\frac{\sigma\left[\mathbf{E}_{out}(\mathbf{r}) + \mathbf{E}_{ins}(\mathbf{r})\right]}{2}\right|_{r=R}$$
$$= \frac{3V_0\cos\theta}{8\pi R}\left[2V_0\cos\theta\frac{1}{R}\hat{r} + V_0\sin\theta\frac{1}{R}\hat{\theta} - \frac{V_0}{R}\hat{z}\right]$$
$$= \frac{3V_0^2\cos\theta}{8\pi R^2}\left[2\cos\theta\hat{r} + \sin\theta\hat{\theta} - \hat{z}\right] \tag{31}$$

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7 Question

Consider a potential problem in the half-space defined by $z \geq 0$, with Dirichlet boundary conditions on the plane $z = 0$ (and at infinity).

a) *Write down the appropriate Green's function $G(\mathbf{r}, \mathbf{r}')$.*

b) *If the potential on the plane $z = 0$ is specified to be $\Phi = V$ inside a circle of radius a centered at the origin, and $\Phi = 0$ outside that circle, find an integral expression for the potential at a point P specified in terms of cylindrical coordinates ρ, ϕ, z.*

c) *Show that, along the axis of the circle ($\rho = 0$), the potential is given by*

$$\Phi = V\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right). \tag{32}$$

d) *Show that at large distances ($\rho^2 + z^2 \gg a^2$) the potential can be expanded in a power series in $(\rho^2+z^2)^{-1}$, and that the leading terms are*

$$\Phi = \frac{Va^2}{2}\frac{z}{(\rho^2+z^2)^{3/2}}\left[1 - \frac{3a^2}{4(\rho^2+z^2)} + \frac{5(3\rho^2a^2+a^4)}{8(\rho^2+z^2)^2} + \dots\right]. \tag{33}$$

Verify that the result of (c) is consistent with this results.

7.1 Solution

a) We need to find a Green function of the form

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} + H(\mathbf{r}-\mathbf{r}') \tag{34}$$

that follows $\nabla^2\Phi = -4\pi\rho$ with Dirichlet boundary conditions $\Phi_{z=0} = \alpha$ and $\Phi_{r=\infty} = \beta$.

This problem could be solved by the **method of the images**, and be thought as a situation where we have a point of charge q at $\mathbf{r}'$, which creates a potential of the form $1/|\mathbf{r}-\mathbf{r}'|$ in the presence of a flat conductor in the plane $z = 0$. Then, putting the image charge at $(x', y', -z')$ (such as the plane was a "mirror"), the potential Φ could be written as

$$\Phi = \frac{q}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} + \frac{q'}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}} \tag{35}$$

Taking $\Phi_{z=0} = 0$, leads to the conclusion that $q' = -q$.

In this way, due to the Green method the appropriate Green function is

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}. \tag{36}$$

b) Now, the potential on the plane, i.e., $z = 0$ could be written as

$$\Phi_{z=0} = \begin{cases} V, x^2+y^2 \le a \\ 0, x^2+y^2 > a \end{cases} \tag{37}$$

and, we are going to use the **Green's theorem**

$$\Phi(\mathbf{r}) = \int_V dV'\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') + \frac{1}{4\pi}\left[\int_{S(V)} dS'\left(\Phi\frac{\partial G}{\partial n'} - G\frac{\partial \Phi}{\partial n'}\right)\right]. \tag{38}$$

Here the surface is composed by the plane (at $z = 0$) and the infinity. It has zero potential at infinity and all other regions outside the circle. Then, it only remains the circle in the integration. It follows that the normal vector points to $-\hat{z}$ and we have

$$\frac{\partial G}{\partial n'}_{z'=0} = -\frac{\partial G}{\partial z'}_{z'=0} = \frac{2z}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}. \tag{39}$$

Replacing on Φ (remembering that $\rho(\mathbf{r}')=0$ and $\frac{\partial\Phi}{\partial n'}=0$) we obtain

$$\Phi(\mathbf{r})=\frac{1}{4\pi}\int_{S(V)}dS'\frac{2Vz}{[(x-x')^2+(y-y')^2+z^2]^{3/2}}. \tag{40}$$

This integral could be changed to cylindrical coordinates: $[(x,y,z),(x',y',z')]\Rightarrow[(\rho,\phi,z),(\rho',\phi',z')]$, using

$$(x-x')^2+(y-y')^2=\rho^2+(\rho')^2-2\rho\rho'\left[\cos\phi\cos\phi'+\sin\phi\sin\phi'\right]=\rho^2+(\rho')^2-2\rho\rho'\cos(\phi-\phi') \tag{41}$$

and we get

$$\Phi(\rho,\phi,z)=\frac{zV}{2\pi}\int_0^a d\rho'\rho'\int_0^{2\pi}\frac{d\phi'}{[\rho^2+(\rho')^2-2\rho\rho'\cos(\phi-\phi')+z^2]^{3/2}}. \tag{42}$$

c) Here, we can solve the previous integral for $\rho=0$. In this way this stays

$$\begin{aligned}\Phi(z)&=\frac{zV}{2\pi}\int_0^a d\rho'\rho'\int_0^{2\pi}\frac{d\phi'}{[(\rho')^2+z^2]^{3/2}}\\&=\frac{zV}{2}\int_{\rho'=0}^{\rho'=a}\frac{du}{u^{3/2}}=V\left(1-\frac{z}{\sqrt{a^2+z^2}}\right)\end{aligned} \tag{43}$$

where we have changed the variables using $u=[(\rho')^2+z^2]$.

d) Once again we use the integral solution, but now we want to do an expansion for power series in $(\rho^2+z^2)^{-1}$. One way to do this is as follows

$$\begin{aligned}\Phi(\rho,\phi,z)&=\frac{zV}{2\pi}\int_0^a d\rho'\rho'\int_0^{2\pi}\frac{d\phi'}{[\rho^2+(\rho')^2-2\rho\rho'\cos(\phi-\phi')+z^2]^{3/2}}\frac{(\rho^2+z^2)^{3/2}}{(\rho^2+z^2)^{3/2}}\\&=\frac{zV}{2\pi(\rho^2+z^2)^{3/2}}\int_0^a d\rho'\rho'\int_0^{2\pi}d\phi'\left[\frac{\rho^2+(\rho')^2-2\rho\rho'\cos(\phi-\phi')+z^2}{(\rho^2+z^2)}\right]^{-3/2}\\&=\frac{zV}{2\pi(\rho^2+z^2)^{3/2}}\int_0^a d\rho'\rho'\int_0^{2\pi}d\phi'\left[1+\frac{(\rho')^2-2\rho\rho'\cos(\phi-\phi')}{(\rho^2+z^2)}\right]^{-3/2}.\end{aligned} \tag{44}$$

Taking the expansion

$$(1+x)^n=(1+nx)+\frac{n(n-1)x^2}{2}+\ldots, \tag{45}$$

with $x=\frac{(\rho')^2-2\rho\rho'\cos(\phi-\phi')}{(\rho^2+z^2)}$, $n=-3/2$ and $n(n-1)=15/4$ we have

$$\begin{aligned}\Phi(\rho,\phi,z)&=\frac{zV}{2\pi(\rho^2+z^2)^{3/2}}\int_0^a d\rho'\rho'\int_0^{2\pi}d\phi'\left\{1-\frac{3}{2}\left[\frac{(\rho')^2-2\rho\rho'\cos(\phi-\phi')}{(\rho^2+z^2)}\right]\right.\\&\qquad\left.+\frac{15}{8}\left[\frac{(\rho')^2-2\rho\rho'\cos(\phi-\phi')}{(\rho^2+z^2)}\right]^2\right\}\\&=\frac{Va^2}{2}\frac{z}{(\rho^2+z^2)^{3/2}}\left[1-\frac{3a^2}{4(\rho^2+z^2)}+\frac{5(3\rho^2a^2+a^4)}{8(\rho^2+z^2)^2}+\ldots\right].\end{aligned} \tag{46}$$

Notice that, the above integral is easier if we notice that there is azimuthal symmetry and you could simply use $(\phi-\phi'\Rightarrow\phi')$.

Verifying what happens for $\rho=0$, we have (expanding for $(1+x)^{-1/2}=1-x/2+3x^3/8-5x^3/16+\ldots$, with $x=(a/z)^2$)

$$\begin{aligned}\Phi(\rho,\phi,z)&=\frac{Va^2}{2}\frac{z}{(z^2)^{3/2}}\left[1-\frac{3a^2}{4(z^2)}+\frac{5(a^4)}{8(z^2)^2}+\ldots\right]\\&=V\left[1-\frac{z}{\sqrt{a^2+z^2}}\right],\end{aligned} \tag{47}$$

which is a result equivalent to item (c) because the expansion $(1+x)^{-1/2}$ for $x = a^2/z^2$.

8 Question

An infinite metallic plate has a spherical overhang of radius a. This plate is grounded. A charge $+q$ is placed over the hemisphere of the overhang, with a distance d of the center of the sphere. Show that the induced charge on the overhang is

$$q' = -q\left[1 - \frac{(d^2 - a^2)}{d\sqrt{d^2 + a^2}}\right]. \tag{48}$$

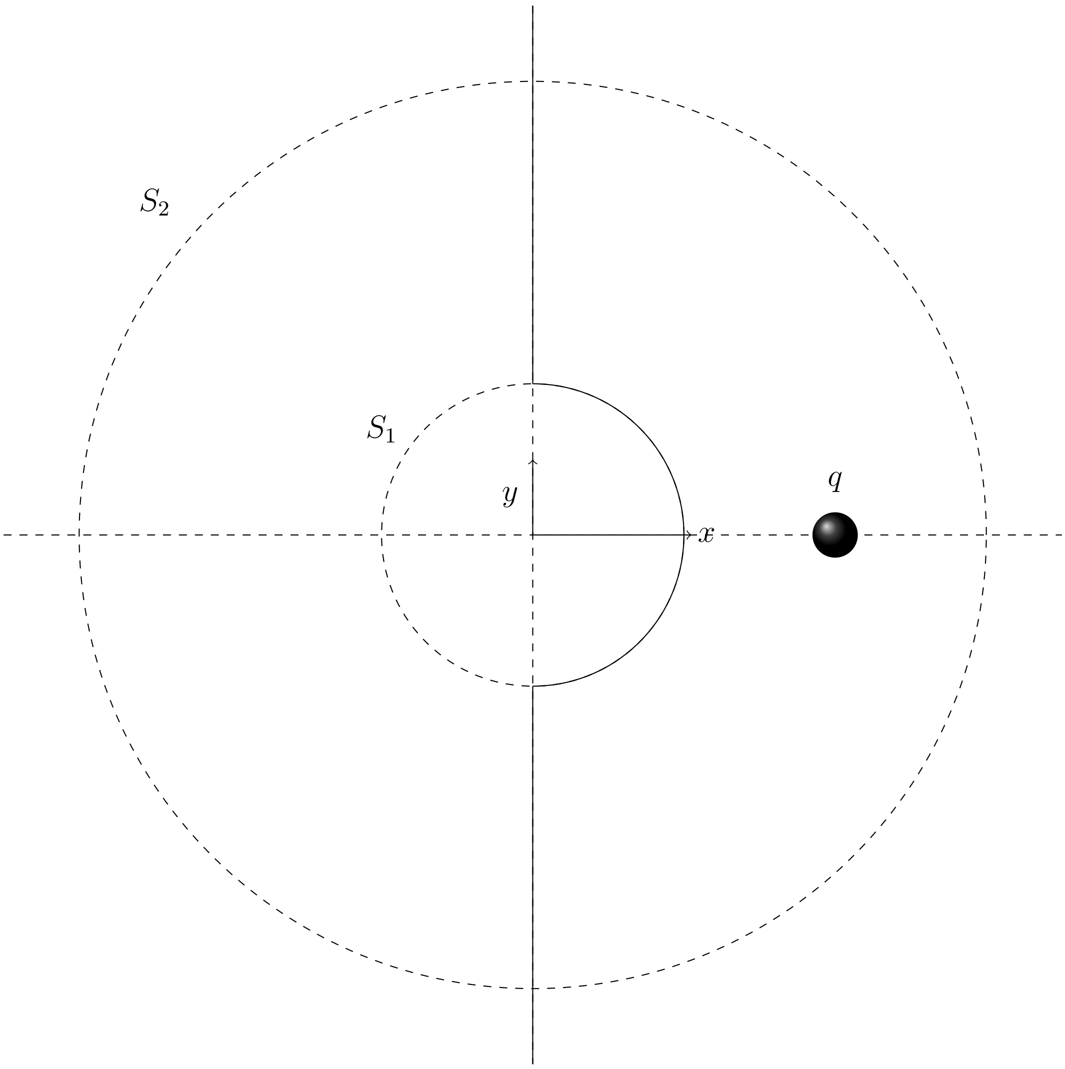

Figure 3: Figure for the question 5.

8.1 Solution

This problem could be solved by the **Method of the images**, as we have the superposition of a infinite plane and a sphere of radius a, as indicated in the figure.

Solving the Green function to the plane, considering the charge q on $\mathbf{r}'$ and the image q' on $(-x', y', z')$, we can write the potential

$$\Phi = \frac{q}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} + \frac{q'}{\sqrt{(x + x')^2 + (y - y')^2 + (z - z')^2}}. \tag{49}$$

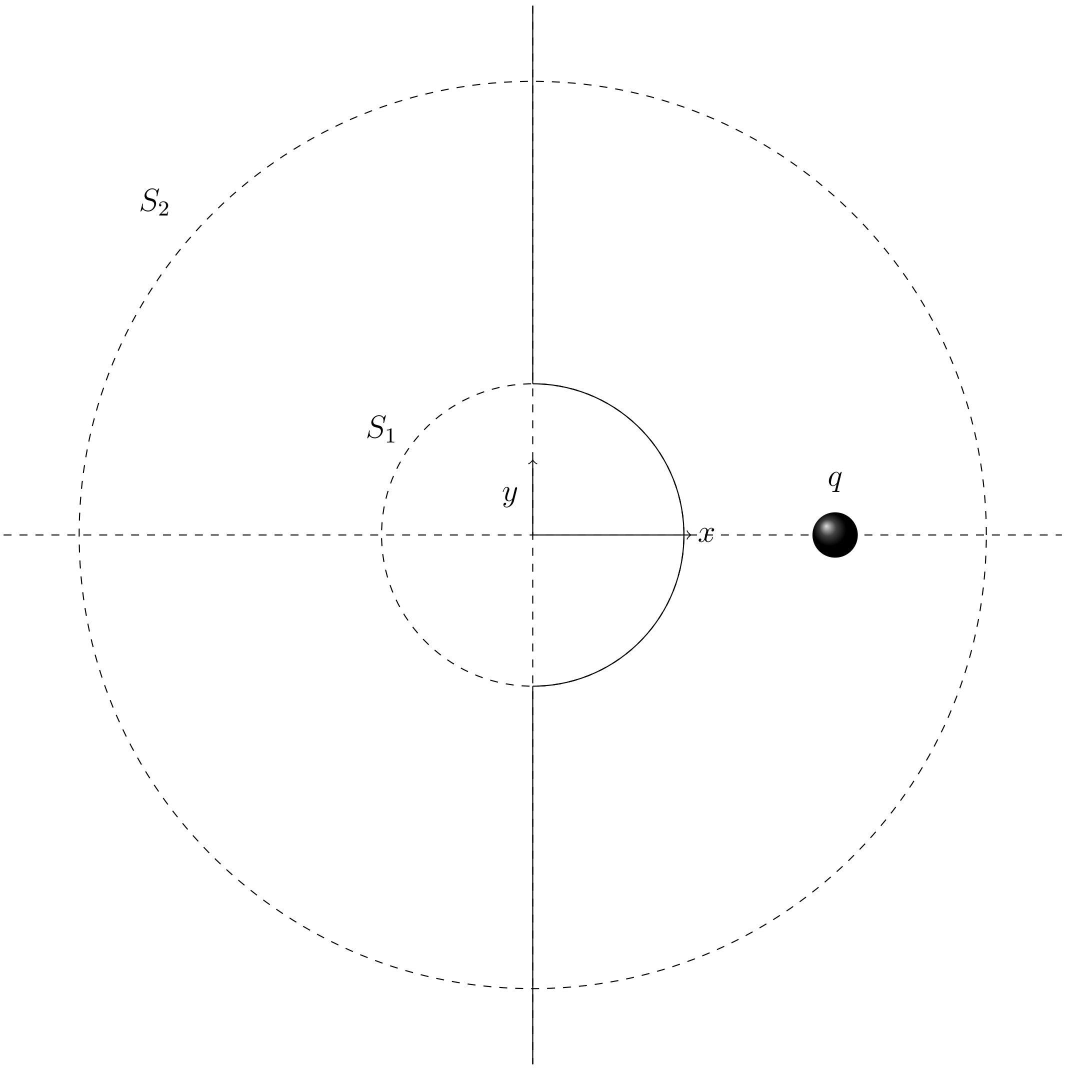

Figure 4: Infinite plane with the spherical overhang of radius a and a charge q on $d\hat{x}$.

Taking $\Phi_{x=0} = 0$, we can say that $q' = -q$.

Now you can think that, the sphere "see" two sources, given by the solution of the plane. Thus, we have to create two other image charges: one for the real charge: q'_R, on $\mathbf{r}'_{RR}$ and other for the first image q'_I, on $\mathbf{r}'_{II}$. Because it is easier, we can write the potential already in the spherical coordinates as

$$\begin{aligned}\Phi = &\frac{q}{\sqrt{r^2 + (r'_R)^2 - 2rr'_R \cos\gamma_R}} - \frac{q}{\sqrt{r^2 + (r'_I)^2 + 2rr'_I \cos\gamma_I}} \\ &+ \frac{q'_R}{\sqrt{r^2 + (r'_{RR})^2 - 2rr'_{RR} \cos\gamma_R}} + \frac{q'_I}{\sqrt{r^2 + (r'_{II})^2 - 2rr'_{II} \cos\gamma_I}} \end{aligned} \tag{50}$$

and take it on $r = a$, such that $\Phi(r = a) = 0$. Then, one terms equals the other (remembering that all the charges will be in the same line, then the angles are the same between: (r'_R, r) and (r'_{RR}, r); (r'_I, r) and (r'_{II}, r)), we have:

$$[a^2 + (r'_R)^2 - 2a(r'_R)\cos\gamma_R](q'_R)^2 = q^2[a^2 + (r'_{RR})^2 - 2ar'_{RR}\cos\gamma_R] \tag{51}$$

$$[a^2 + (r'_I)^2 + 2ar'_I\cos\gamma_I](q'_I)^2 = q^2[a^2 + (r'_{II})^2 - 2ar'_{II}\cos\gamma_I]. \tag{52}$$

We need to solve the following systems

$$[a^2 + (r'_R)^2](q'_R)^2 = q^2[a^2 + (r'_{RR})^2] \tag{53}$$
$$[a^2 + (r'_I)^2](q'_I)^2 = q^2[a^2 + (r'_{II})^2] \tag{54}$$
$$-2a(r'_R)\cos\gamma_R(q'_R)^2 = -q^2 2a(r'_{RR})\cos\gamma_R \tag{55}$$
$$2a(r'_I)\cos\gamma_I(q'_I)^2 = -q^2 2a(r'_{II})\cos\gamma_I \tag{56}$$

finding that

$$q'_R = -aq/r'_R,\ \ r'_{RR} = a^2/r'_R \text{ and } \ q'_I = aq/r'_I,\ \ r'_{II} = -a^2/r'_I. \tag{57}$$

Finally, taking the position of the real charge as $d\hat{x}$, the potential is written as

$$\begin{aligned}\Phi(r) = {} & \frac{q}{\sqrt{r^2 + d^2 - 2rd\cos\gamma}} - \frac{q}{\sqrt{r^2 + d^2 + 2rd\cos\gamma}} \\ & - \frac{qa/d}{\sqrt{r^2 + \left(\frac{a^2}{d}\right)^2 - 2r\left(\frac{a^2}{d}\right)\cos\gamma}} + \frac{qa/d}{\sqrt{r^2 + \left(\frac{a^2}{d}\right)^2 + 2r\left(\frac{a^2}{d}\right)\cos\gamma}}.\end{aligned} \tag{58}$$

Having the electric potential, σ is given, on $r = a$, due

$$\begin{aligned}\sigma &= -\frac{1}{4\pi}\nabla\Phi_{S_1} = -\frac{1}{4\pi}\frac{\partial\Phi}{\partial r}_{r=a} \\ &= \left[\frac{q(r - d\cos\gamma)}{(r^2 + d^2 - 2rd\cos\gamma)^{3/2}} - \frac{q(r + d\cos\gamma)}{(r^2 + d^2 + 2rd\cos\gamma)^{3/2}}\right. \\ &\quad \left. - \frac{qa/d(r - a^2/d\cos\gamma)}{\left[r^2 + \left(\frac{a^2}{d}\right)^2 - 2r\left(\frac{a^2}{d}\right)\cos\gamma\right]^{3/2}} + \frac{qa/d(r + a^2/d\cos\gamma)}{\left[r^2 + \left(\frac{a^2}{d}\right)^2 + 2r\left(\frac{a^2}{d}\right)\cos\gamma\right]^{3/2}}\right]_{r=a} \\ &= -\frac{1}{4\pi}\left[\frac{\frac{q}{a}(a^2 - d^2)}{(d^2 + a^2 - 2ad\cos\gamma)^{3/2}} - \frac{\frac{q}{a}(a^2 - d^2)}{(d^2 + a^2 + 2ad\cos\gamma)^{3/2}}\right].\end{aligned} \tag{59}$$

Thus, we can find the total charge integrating over the surface

$$\begin{aligned}Q &= \int dS\sigma = -\int_0^{2\pi} d\phi \int_0^{\pi} d\gamma \sin\gamma \frac{\frac{q}{a}(a^2 - d^2)}{4\pi}\left[\frac{1}{(d^2 + a^2 + 2ad\cos\gamma)^{3/2}}\right. \\ &\qquad\qquad \left. - \frac{1}{(d^2 + a^2 - 2ad\cos\gamma)^{3/2}}\right] \\ &= -q\left[1 - \frac{(d^2 - a^2)}{d\sqrt{d^2 + a^2}}\right].\end{aligned} \tag{60}$$

of course after change the variables $u = a^2 + d^2 + 2ad\cos\gamma$.