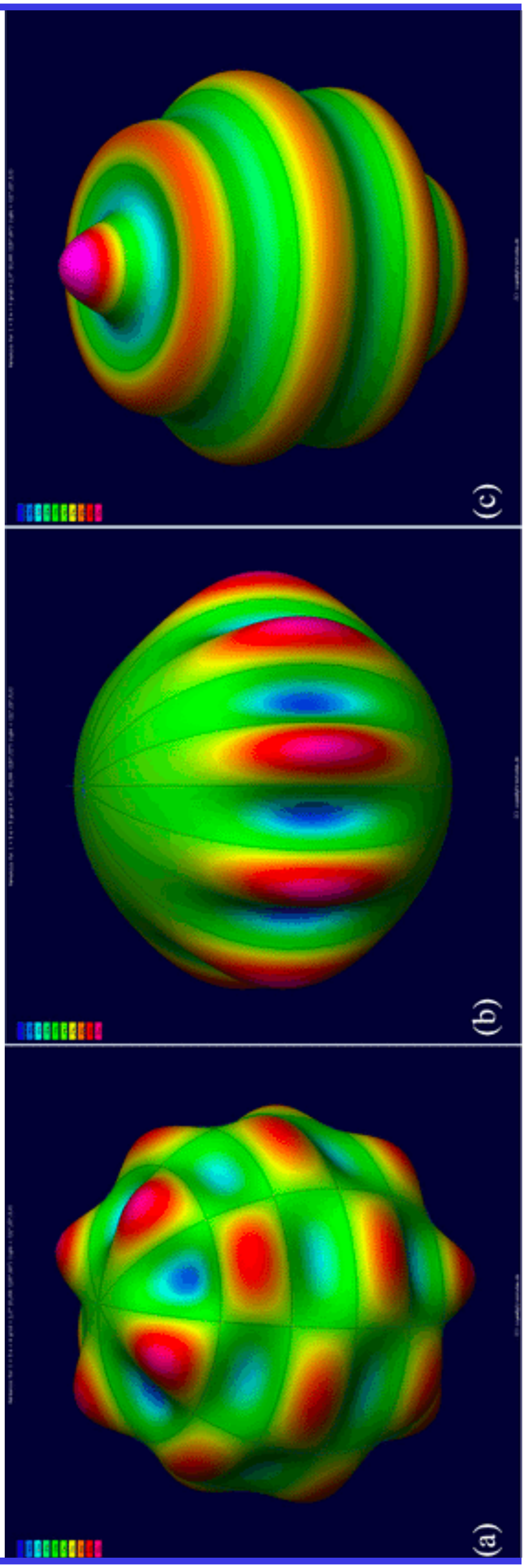
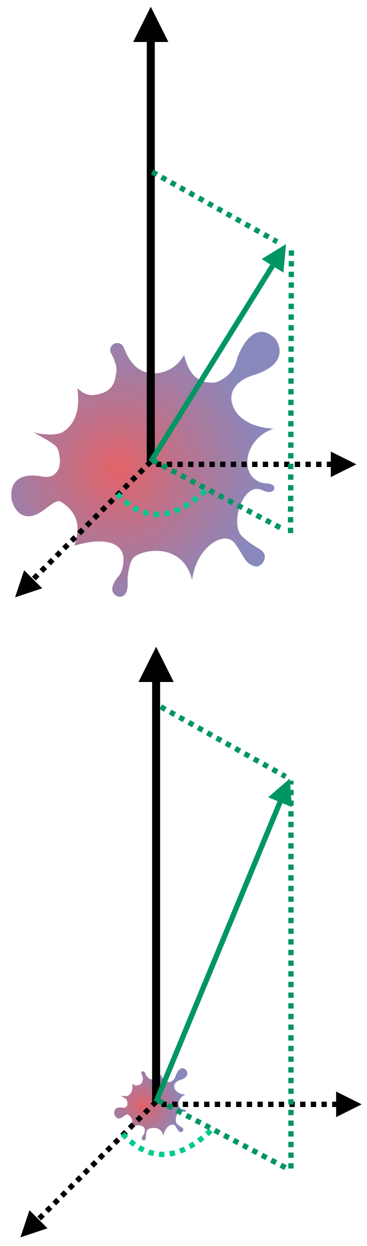
Electrodynamics

⚡Spherical coordinates
⚡Basis functions
⚡Spherical harmonics
⚡Multipole expansion

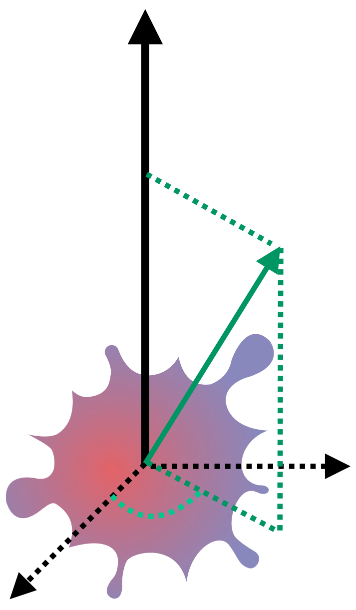


Spherical coordinates

- Many problems involve a **localized distribution** of charges (and currents), which we typically place at the **origin** of our coordinate system.
- As we move **away** from those sources, their *features* start to disappear.
- What does that *mean*? What features are we talking about, and how exactly they disappear?
- As we will see, the **multipolar expansion** is a natural, intuitive way to describe those features, and it also tells us **how fast** they decay with the distance from the source.



Spherical coordinates



- Let's start with the differential operators in Spherical Coordinates. For instance, the ***divergence*** is given by:

$$\vec{\nabla} \cdot \vec{F} = \frac{1}{r^2}\frac{\partial (r^2 F_r)}{\partial r} + \frac{1}{r \sin\theta}\left[\frac{\partial \sin\theta F_\theta}{\partial\theta} + \frac{\partial F_\varphi}{\partial\varphi}\right]$$

- The ***Laplacian*** on a *scalar function* is given by:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$

- It is also worth recall the unit vectors, line element, area and surface in spherical coordinates:

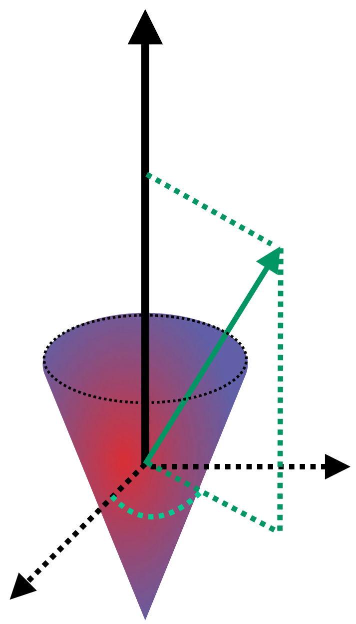
$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \cos\theta = \dfrac{z}{r} \\ \tan\varphi = \dfrac{y}{x} \end{cases}$$

$$\begin{cases} \hat{r} = \dfrac{x\hat{x} + y\hat{y} + z\hat{z}}{r} \\ \hat{\rho} = \dfrac{x\hat{x} + y\hat{y}}{\sqrt{x^2+y^2}} = \dfrac{\vec{\rho}}{\rho} \\ \hat{\theta} = \dfrac{z\,\hat{\rho} - \rho\,\hat{z}}{r} \\ \hat{\varphi} = \dfrac{-y\,\hat{x} + x\,\hat{y}}{\rho} \end{cases}$$

$$\begin{cases} d\vec{l} = dr\,\hat{r} + rd\theta\,\hat{\theta} + r\sin\theta\,d\varphi\,\hat{\varphi} \\ d\vec{S} = r^2\sin\theta\,d\theta d\varphi\,\hat{r} + r\sin\theta\,d\varphi dr\,\hat{\theta} + r\,drd\theta\,\hat{\varphi} \\ dV = r^2\sin\theta\,drd\theta d\varphi \end{cases}$$

Spherical coordinates + azimuthal symmetry



- It is often the case that in many situations the problem does not depend on the azimuthal angle φ, in which case the Laplacian operator simplifies to:

$$\nabla^2 f(r,\theta) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right)$$

- Let's now take a function that obeys the Laplace equation, so $\nabla^2 f = 0$. We can try a solution using the separation of variables:

$$f(r,\theta) = R(r)\,P(\theta) \quad ,$$

so the equation becomes:

$$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{1}{P}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) = 0 \quad , \quad \text{hence}$$

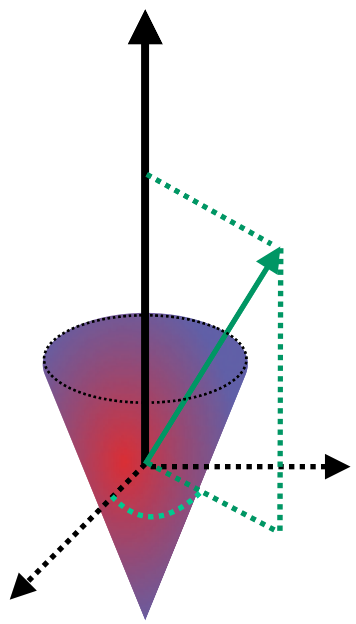
$$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = -\frac{1}{P}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) = \text{const}\,.$$

- If this separation constant is zero, the non-trivial solutions are $R \to 1/r$ and $P \to \log(\tan\theta/2)$. But these have *diverges* which rule them out from basically any relevant problems.

- Before we proceed any further, let's rewrite the equations above once again, using the change of variable:

$$\mu = \cos\theta \quad , \quad \mu : [-1,1] \quad , \quad \text{which leads to} \quad \frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{1}{P}\frac{d}{d\mu}\left((1-\mu^2)\frac{dP}{d\mu}\right) = 0$$

Spherical coordinates + azimuthal symmetry



- We now "call" the separation constant some name — which is, of course, informed by the knowledge of the solutions.

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \ell(\ell+1) = -\frac{1}{P}\frac{d}{d\mu}\left((1-\mu^2)\frac{dP}{d\mu}\right)$$

- The solutions to the radial equation are very simple *power laws*. We have:

$$R_\ell = A\,r^\ell + B\,r^{-1-\ell}$$

- The solutions for the angular function $P(\mu)$ are well-known to us: they are the Legendre Polynomials:

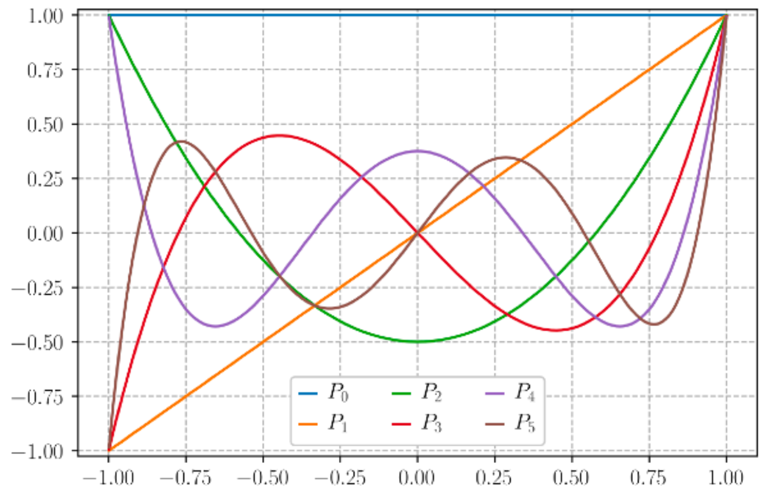
$$P_\ell = \frac{1}{2^\ell\,\ell!}\frac{d^\ell}{d\mu^\ell}\left[(\mu^2-1)^\ell\right] \quad , \quad \text{where} \quad \ell = 0\,,\,1\,,\,2\,,\,\ldots$$

- The Legendre polynomial of order ℓ is in fact a polynomial of μ, or order at most μ^ℓ . Some examples:

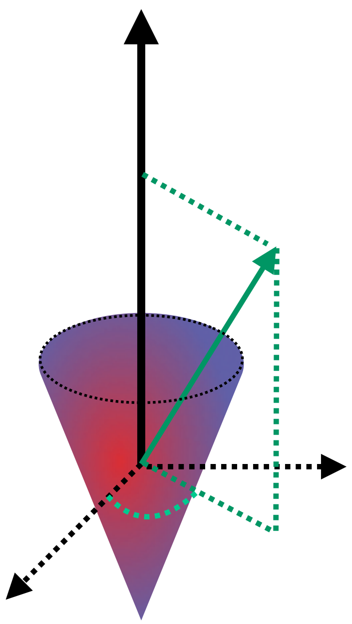
$$P_0 = 1$$
$$P_1 = \mu$$
$$P_2 = \frac{1}{2}(3\mu^2-1)$$
$$P_3 = \frac{1}{2}(5\mu^3-3\mu)$$

$$P_\ell(\mu=1) = 1$$
$$P_\ell(\mu=-1) = (-1)^\ell$$

$$\left.\frac{dP_\ell}{d\mu}\right|_{\mu=1} = \frac{1}{2}\ell(\ell+1)$$



Spherical coordinates + azimuthal symmetry



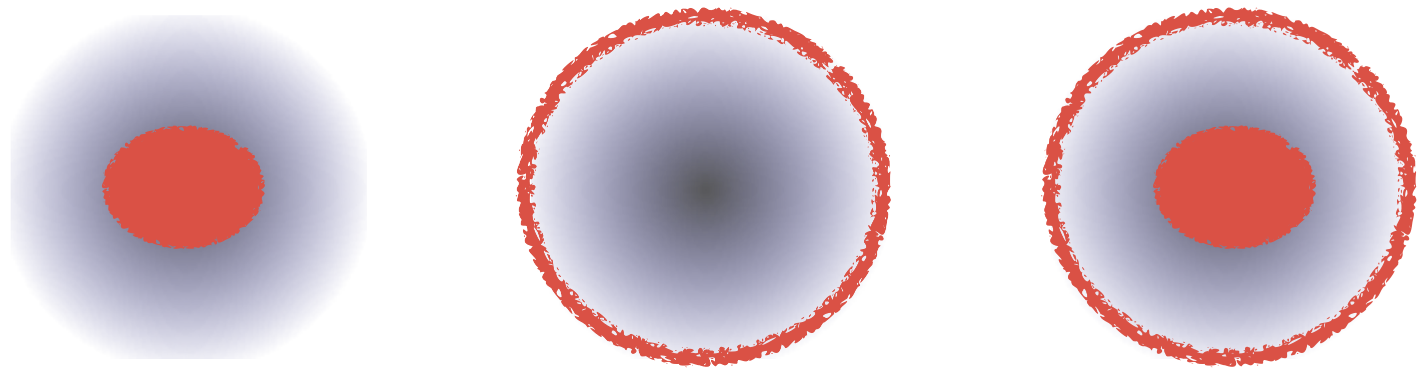
- The Legendre polynomials form a ***complete basis*** for a variable in the interval $\mu \in [-1,1]$, and is a complete set to describe functions of θ when there is azimuthal symmetry.
- These polynomials are *orthonormalized*, in the sense:

$$\int_{-1}^{1} d\mu \, P_\ell(\mu) \, P_{\ell'}(\mu) = \frac{2}{2\ell + 1} \delta_{\ell \ell'}$$

- Therefore, any solution to the Laplace equation with azimuthal symmetry can be written as:

$$f(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell \, r^\ell + B_\ell \, r^{-1-\ell} \right) P_\ell(\cos\theta)$$

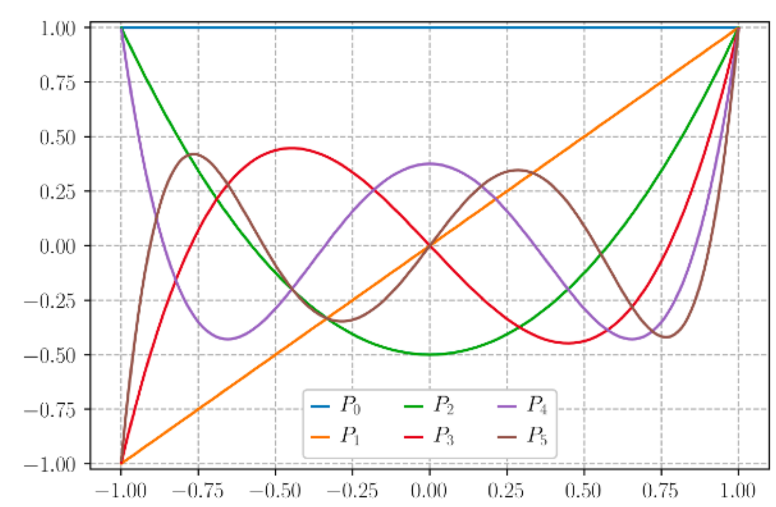
- We will tackle three general types of problems :



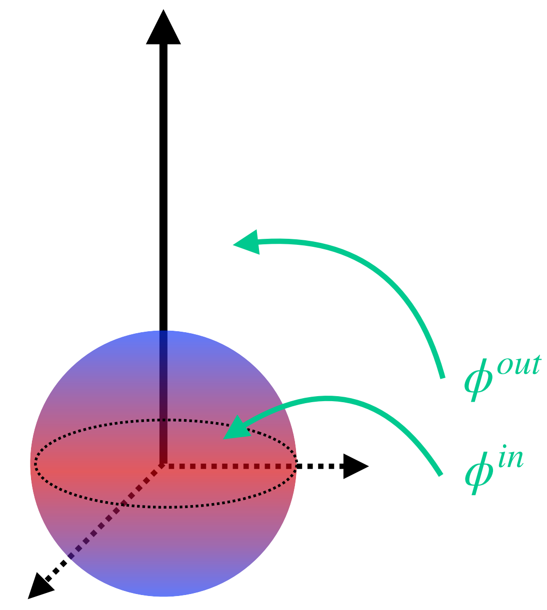
$\lim_{r\to\infty} r^\ell \to \infty$ $\Rightarrow$ Only $r^{-\ell-1}$

$\lim_{r\to 0} r^{-\ell-1} \to \infty$ $\Rightarrow$ Only r^ℓ

$\Rightarrow$ Both r^ℓ and $r^{-\ell-1}$



Spherical coordinates + azimuthal symmetry



- Let's start with a simple example: a sphere of radius R_0 with potential $\phi(r = R_0) = \phi_0 \cos^2\theta$. Find the potential everywhere — inside **and** outside the sphere.
- Let's start computing the mean potential on the sphere (also called the **monopole** of the potential):

$$\bar{\phi} = \frac{1}{4\pi}\int d^2\Omega\,\phi = \frac{1}{4\pi}\int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi\,\phi(\theta,\varphi)$$

$$= \frac{1}{4\pi}2\pi\int_{-1}^{1} d\mu\,\phi(\mu) = \frac{\phi_0}{2}\int_{-1}^{1} d\mu\,\mu^2 = \frac{\phi_0}{2}\frac{\mu^3}{3}\Bigg|_{-1}^{1} = \frac{\phi_0}{3}$$

- In fact, we can also write the potential itself in terms of its *Legendre expansion*. Notice that we can write:

$$\phi(r = R, \mu) = \frac{\phi_0}{3} + \frac{2\phi_0}{3}\times\frac{1}{2}(3\mu^2 - 1) = \frac{\phi_0}{3}P_0(\mu) + \frac{2\phi_0}{3}P_2(\mu)$$

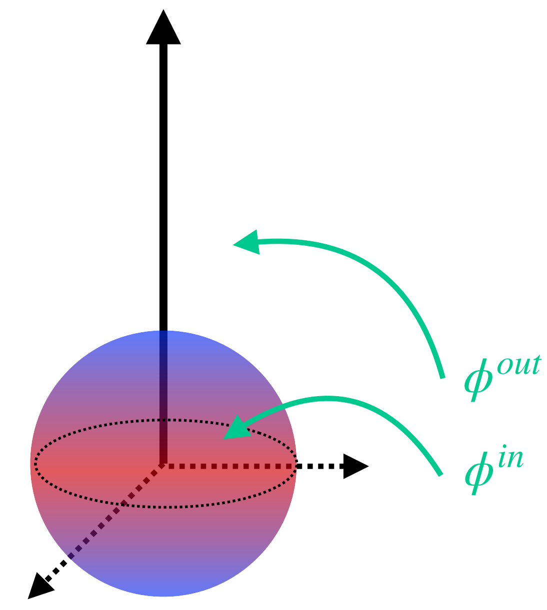
- Therefore, the boundary conditions in this example involve only the ***multipoles*** $\ell = 0$ (**monopole**) and $\ell = 2$ (**quadrupole**).
- For the field outside the sphere, we take only the $r^{-\ell-1}$ radial functions. Obviously, then, matching the boundary conditions is already done by the expansion of ϕ and we get:

$$\phi^{out}(r,\mu) = \left(\frac{r}{R_0}\right)^{-1}\frac{\phi_0}{3}P_0(\mu) + \left(\frac{r}{R_0}\right)^{-3}\frac{2\phi_0}{3}P_2(\mu)$$

- For the field inside the sphere, we take only the r^ℓ radial function, with the result:

$$\phi^{in}(r,\mu) = \left(\frac{r}{R_0}\right)^{0}\frac{\phi_0}{3}P_0(\mu) + \left(\frac{r}{R_0}\right)^{2}\frac{2\phi_0}{3}P_2(\mu)$$

Spherical coordinates + azimuthal symmetry



- I will **leave as an exercise** for you to compute the electric field. **Outside** the sphere is given by:

$$\vec{E}^{out} = \left[\frac{\phi_0}{3}\frac{R_0}{r^2} + 2\phi_0\frac{R_0^3}{r^4}P_2(\cos\theta)\right]\hat{r} + 2\phi_0\frac{R_0^3}{r^4}\sin\theta\cos\theta\,\hat{\theta}$$

- And **inside** the sphere it is:

$$\vec{E}^{in} = \left[\frac{4\phi_0}{3}\frac{r}{R_0^2}P_2(\cos\theta)\right]\hat{r} + 2\phi_0\frac{r}{R_0^2}\sin\theta\cos\theta\,\hat{\theta}$$

- In particular, notice that the difference between the field outside and inside the sphere is:

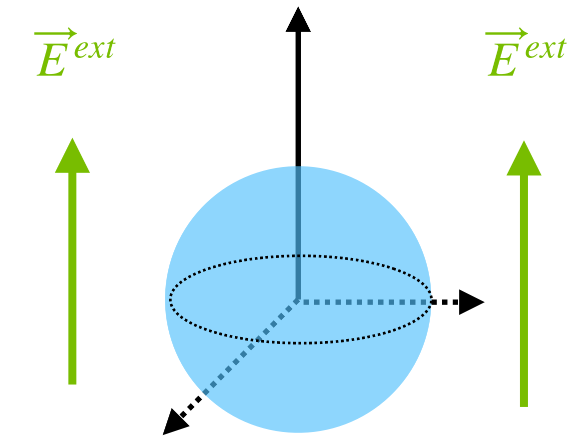
$$\Delta\vec{E}_{||} = 0$$

$$\Delta\vec{E}_{\perp} = \frac{\sigma}{\epsilon_0} = \frac{\phi_0}{3R_0} + \frac{10\,\phi_0}{3\,R_0}P_2(\cos\theta)$$

- Therefore, the ***charge*** on the sphere is $Q_0 = 4\pi R_0^2\,\sigma = 4\pi\epsilon_0\,\dfrac{\phi_0 R_0}{3}$, so that very far from the sphere we have:

$$\vec{E}^{out} \to \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2} \quad !!!$$

Spherical coordinates + azimuthal symmetry



- Another interesting “standard" exercise with axial symmetry is the **conducting sphere** with zero net charge which is placed in the presence of an external **uniform** electric field.
- We should, of course, choose the z axis along the direction of the field, hence:

$$\vec{E}^{ext} \; = \; E_0 \, \hat{z} \qquad \Longleftrightarrow \qquad \phi^{ext} = - \, E_0 \, z + \text{const} \, .$$

  Clearly, the conducting sphere will change the field near it, but far away all that remains is the external field.
- Since the sphere is a conductor, there should be no potential difference between different points in the sphere, so:

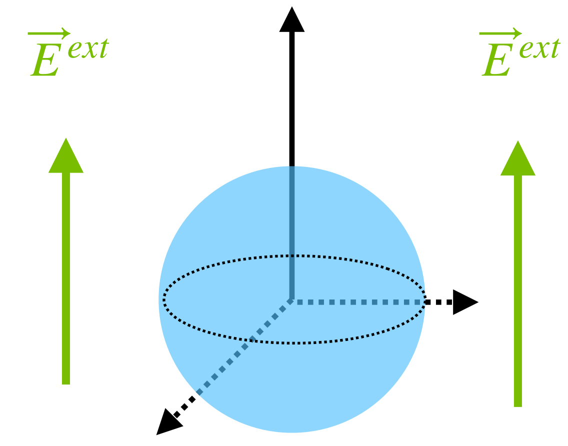
$$\phi(r = R_0) \;=\; \text{const} \;\to\; 0$$

- Because of the axial symmetry, we should try our general solution:

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell \, r^\ell + B_\ell \, r^{-1-\ell}\right) P_\ell(\cos\theta)$$

- Since we want the field and the potential outside the sphere, we would be tempted to simply throw away the basis functions r^ℓ , and try for a solution only with $r^{-\ell-1}$. However, that would be a **foolish** application of the **boundary conditions**! Why?…

Spherical coordinates + azimuthal symmetry



- The reason why we cannot simply disregard the first types of solutions that go as r^ℓ is the fact that, because of the external field, the boundary condition is such that:

$$\phi(r \to \infty) \to \phi^{ext} = -E_0\, z + \text{const}\,.$$

- The full expression is:

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell\, r^\ell + B_\ell\, r^{-1-\ell}\right) P_\ell(\cos\theta)$$

- If we look at the $\ell = 1$ term (called the **dipole**), we find that it has the structure:

$$r^1\, P_1(\cos\theta) = r\cos\theta = z$$

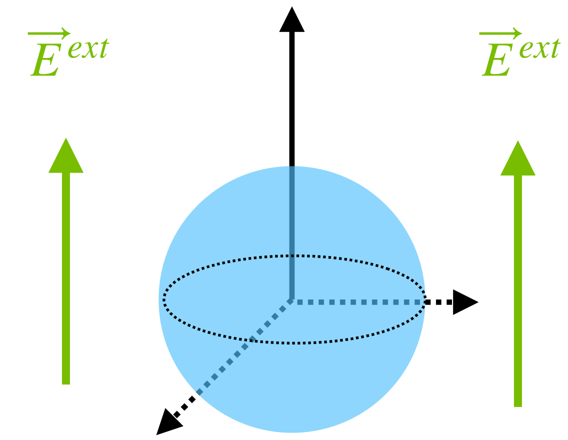
  hence, $A_1 = -E_0$ satisfies the boundary conditions at spatial infinity.

- Moreover, the term B_0 corresponds to a term of the potential that goes as r^{-1}, which will only appear if there is a non-zero total charge in the sphere! Since the sphere is neutral, $B_0 = 0$.

- Hence, we were already able to set $A_1 \to -E_0$, $B_0 \to 0$, and the term A_0 is irrelevant since it contributes a constant to the potential, and doesn't affect the field.

- It is easy to see that all $\ell \geq 2$ terms vanish for this problem.

Spherical coordinates + azimuthal symmetry



- We are left, finally, with adjusting the condition that:

$$\phi(r = R_0, \theta) = \left(A_1 R_0^1 + B_1 R_0^{-2}\right) P_1(\cos\theta) = \left(-E_0 R_0^1 + B_1 R_0^{-2}\right) \times \cos\theta = 0$$

which leads to:

$$B_1 = E_0 R_0^3$$

- Hence, the potential is that of a **dipole**:

$$\phi(r, \theta) = \left(-E_0 r + E_0 \frac{R_0^3}{r^2}\right) \cos\theta$$

- A simple calculation leads to the electric field:

$$\vec{E} = E_0 \left(1 + 2\frac{R_0^3}{r^3}\right) \cos\theta \, \hat{r} - E_0 \left(1 - \frac{R_0^3}{r^3}\right) \sin\theta \, \hat{\theta}$$

- It is obvious that, at $r = R_0$, the field is $\perp$ to the sphere. Exercise for you: compute the induced charge surface density on the sphere.

Multipolar expansion

- There are some important lessons from the expansion in Legendre polynomials and their associated radial functions. For the most common "out" solutions (and ignoring funny boundary conditions at infinity), we have the following structure for the radial dependence:

$$\text{Monopole } (\ell = 0) \quad : \quad \phi \sim \frac{1}{r} \quad , \quad E \sim \frac{1}{r^2}$$

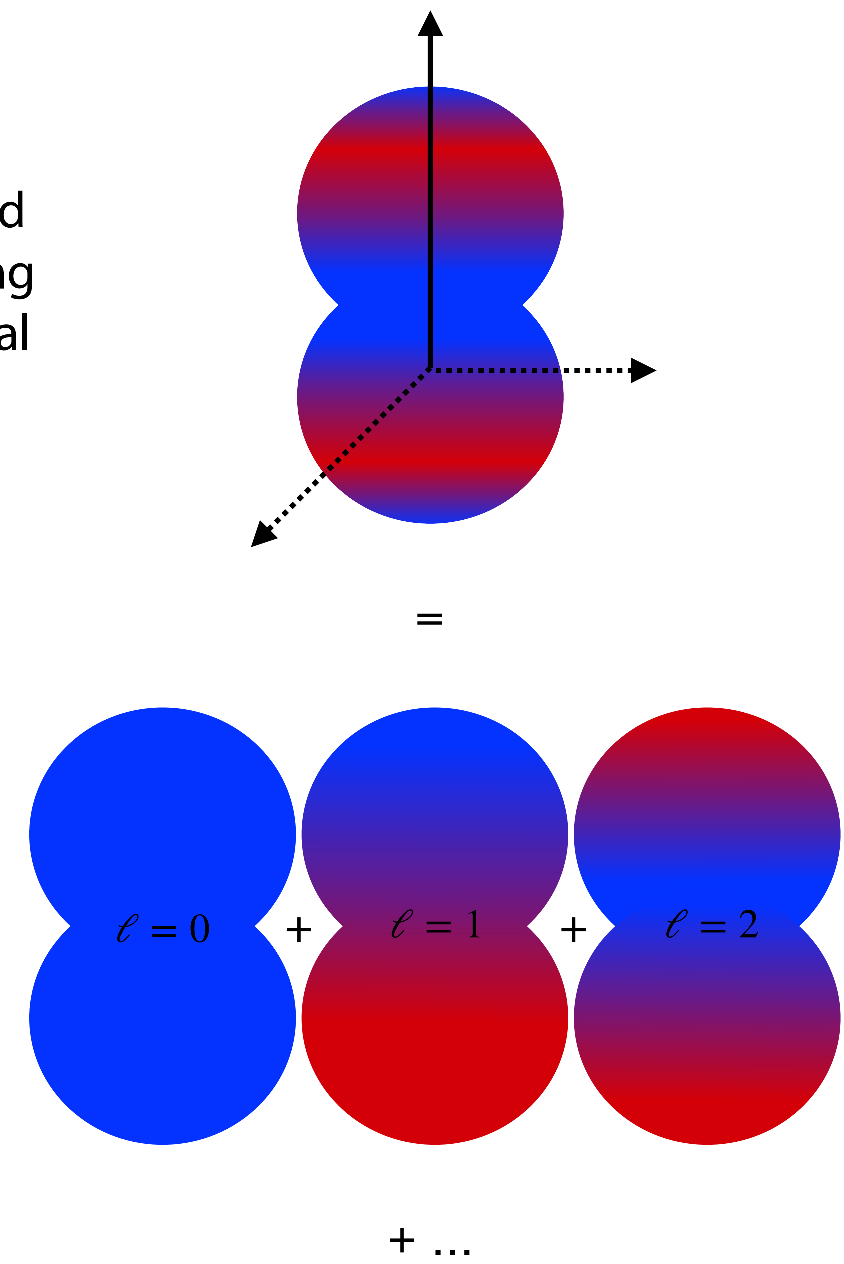
$$\text{Dipole } (\ell = 1) \quad : \quad \phi \sim \frac{1}{r^2} \quad , \quad E \sim \frac{1}{r^3}$$

$$\text{Quadrupole } (\ell = 2) \quad : \quad \phi \sim \frac{1}{r^3} \quad , \quad E \sim \frac{1}{r^4}$$

etc.

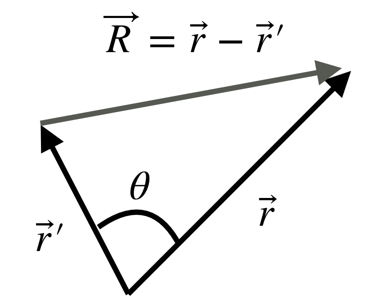
- The feature that remains at large distance is the monopole; as we come closer, the dipole appears; then the quadrupole; and so on.

- So, the fine details, inhomogeneities and anisotropies of the charge distribution (or b.c.) appear to us in terms of these **multipoles**.



Multipolar expansion

- As you know, this multipolar expansion can also be derived in terms of an expansion for the potential of a point charge (q). For a charge at the point $\vec{r}\,'$, the potential is:



$$\phi(\vec{r}\,) = \frac{q}{4\pi\epsilon_0}\frac{1}{|\,\vec{r} - \vec{r}\,'|} = \frac{q}{4\pi\epsilon_0}\frac{1}{R}$$

- We may expand the function $1/R$ in terms of r or in terms of r', depending on the problem: if we are far away from the source, then we use r; if the source is far away and we are close to the origin, we choose r'.

- Let's assume for now that the source is close to the origin, and we are far away, so the expansion becomes:

$$\frac{1}{R} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}} = \frac{1}{r}\frac{1}{\sqrt{1 + \frac{r'^2}{r^2} - 2\frac{r'}{r}\cos\theta}}$$

- The Taylor series around (r'/r) is precisely the series on Legendre polynomials:

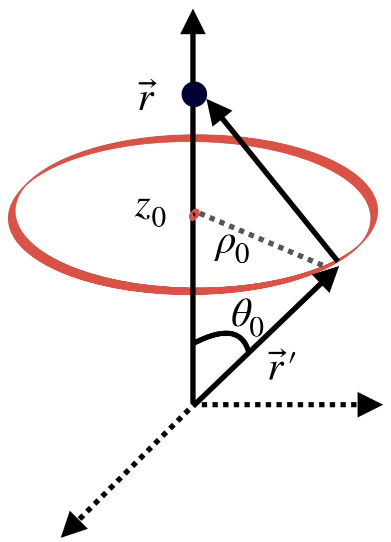
$$\frac{1}{R} = \frac{1}{r}\sum_\ell \left(\frac{r'}{r}\right)^\ell P_\ell(\cos\theta) \quad \text{if } r > r', \quad \text{and} \quad \frac{1}{R} = \frac{1}{r'}\sum_\ell \left(\frac{r}{r'}\right)^\ell P_\ell(\cos\theta) \quad \text{if } r' > r\,.$$

- The only difference is that our original multipolar series applies for a general charge distribution and/or a general boundary condition, while the series above is that for a **point source** at the position $\vec{r}\,'$.

Multipolar expansion

- Sometimes it is useful to employ the direct solution for point charges and construct solutions to problems with axial symmetry.
- Let's consider here the case of a **thin ring** of radius ρ_0, with a charge q that is homogeneously distributed along the ring, and let's say that the ring is placed at a height z_0 above the origin, as shown in the figure.
- We can start by checking what is the potential along the z axis. Then, due to the axial symmetry, each part of the ring is equidistant to the point $\vec{r} = z\hat{z}$, and the potential is therefore given by:

$$\phi = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta_0}} \quad , \quad \text{where} \quad \cos\theta_0 = \frac{z_0}{\sqrt{z_0^2 + \rho_0^2}}$$



- The Taylor series around (r'/r) is precisely **the same** as the series on Legendre polynomials:

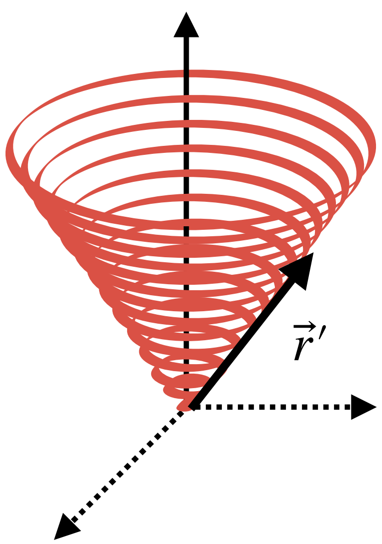
$$\phi(r, \cos\theta = 1) = \frac{q}{4\pi\epsilon_0} \frac{1}{r_>} \sum_\ell \left(\frac{r_<}{r_>}\right)^\ell P_\ell(\cos\theta_0)$$

- But this expansion already determines all the coefficients of the solution in terms of the Legendre polynomials for any angle — the coefficients are exactly the $P_\ell(\cos\theta_0)$! So, we get immediately that, for this ring, at **any** point:

$$\phi = \frac{q}{4\pi\epsilon_0} \frac{1}{r_>} \sum_\ell \left(\frac{r_<}{r_>}\right)^\ell P_\ell(\cos\theta_0) P_\ell(\cos\theta)$$

- We will now use this solution to construct other solutions, to more interesting problems.

Multipolar expansion: the tip effect



- Let's say that instead of a ring we chave a **cone** with constant surface charge density, such that the cone's tip is at the origin. This means that we construct that cone as a series of rings whose charges are proportional to their circumferences:

$$dq_{ring}(\vec{r}\,') = \sigma_0 \, dA'_{ring} = \sigma_0 \, 2\pi\rho' \, dr' = \sigma_0 \, 2\pi \sin\theta_0 \, r' dr'$$

- The potential for each ring is:

$$d\phi_{ring} = \frac{dq_{ring}}{4\pi\epsilon_0} \frac{1}{r_>} \sum_\ell \left(\frac{r_<}{r_>} \right)^\ell P_\ell(\cos\theta_0) P_\ell(\cos\theta)$$

$$\Rightarrow \quad \phi = \int d\phi_{ring} = \frac{\sigma_0}{2\epsilon_0} \sum_\ell P_\ell(\cos\theta_0) P_\ell(\cos\theta) \int dr' r' \frac{r_<^\ell}{r_>^{\ell+1}}$$

- It is useful to separate this integral explicitly in two parts: (a) when $r' < r$, so $r_> \to r \ , \ r_< \to r'$; and (b) when $r' > r$, so $r_> \to r' \ , \ r_< \to r$.

- Let's also assume that we are outside the cone, in a region not too far from the origin, so we can cut off the integral at some large distance R_0.

Multipolar expansion: the tip effect

- Therefore, we obtain, using the cut-off distance scale R_0:

$$\phi = \frac{\sigma_0}{2\epsilon_0} \sum_\ell P_\ell(\cos\theta_0) P_\ell(\cos\theta) \times \left[\int_0^r dr' r' \frac{r'^\ell}{r^{\ell+1}} + \int_r^{R_0} dr' r' \frac{r^\ell}{r'^{\ell+1}} \right]$$

- You can easily compute the integrals inside the square brackets, and show that:

$$\ell = 0 \quad \to \quad R_0 - \frac{1}{2} r$$

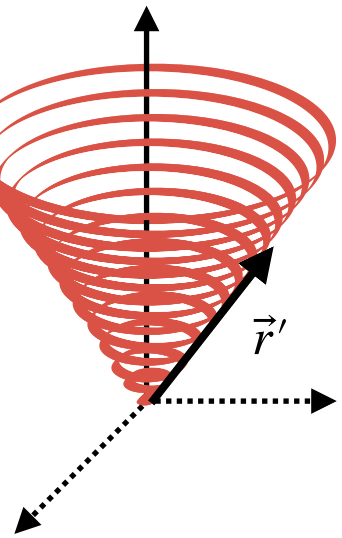
$$\ell = 1 \quad \to \frac{1}{3} r + r \log \frac{R_0}{r}$$

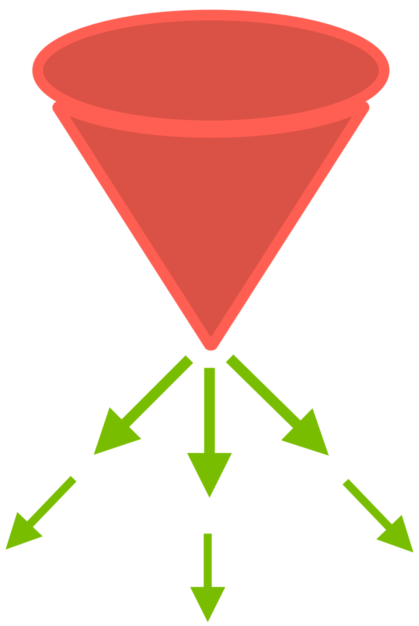
$$\ell \geq 2 \quad \to \quad \frac{r}{\ell+2} + \frac{r}{\ell-1} \left(1 - \frac{r^{\ell-1}}{R_0^{\ell-1}} \right) \simeq \frac{2\ell+1}{(\ell+2)(\ell-1)} r$$

- You can (again) substitute these expressions back into the series solution above, but what is interesting here is that we have a "funny term" that showed up: the dipole term $r \log R_0/r$. For **small distances**, this term dominates (by far!), and so we can write that, near the tip of that cone the electric field becomes:

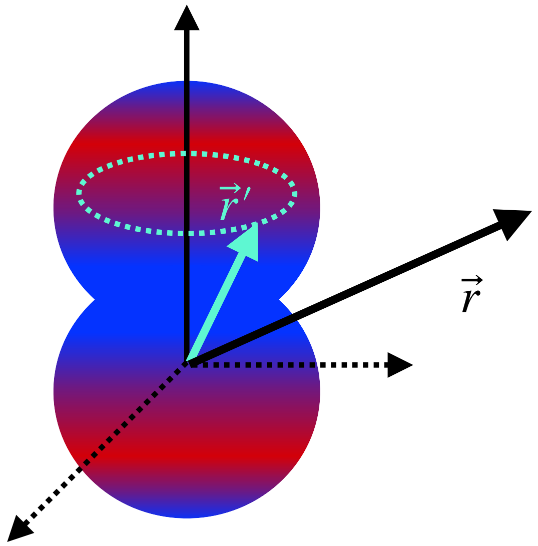
$$\vec{E} \simeq - \frac{\sigma_0}{2\epsilon_0} \sin\theta_0 \cos\theta_0 \log(R_0/r) \, \cos\theta \, \hat{r} + \dots$$

- This is the "**tip effect**": if you place a charge on a **pointy** conductor, the field at the tip is **very strong**! See Jackson's book (Ch. 3.4) for a more complete discussion.





Physical multipoles



- The previous discussion shows that we can talk about the multipoles of the charge distributions. Let's assume that we have some (axially symmetric, for now) charge density as shown in the figure.
- Let's also assume that we are interested in the region far outside of the charge density, so it is safe to assume that $r_> \to r$ and $r_< \to r'$. For each volume element of the charge density of a ring we have:

$$d\phi = \frac{\rho(\vec{r}')dV'}{4\pi\epsilon_0}\frac{1}{r}\sum_\ell \left(\frac{r'}{r}\right)^\ell P_\ell(\cos\theta')P_\ell(\cos\theta) \quad , \quad \text{and therefore:}$$

$$\phi = \int \frac{\rho(\vec{r}')dV'}{4\pi\epsilon_0}\frac{1}{r}\sum_\ell \left(\frac{r'}{r}\right)^\ell P_\ell(\cos\theta')P_\ell(\cos\theta)$$

$$= \sum_\ell \frac{1}{r^{\ell+1}}P_\ell(\cos\theta)\ \int_0^\infty r'^2 dr' \int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\varphi' \frac{\rho(r',\theta')}{4\pi\epsilon_0} r'^\ell P_\ell(\cos\theta')$$

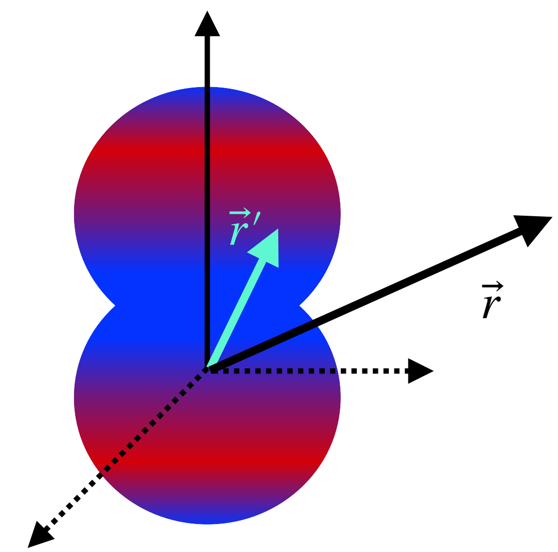
- These integrals define the multipoles of the charge distribution:

$$\rho_\ell \ = \int d^3r'\,\rho(r',\theta')\,r'^\ell P_\ell(\cos\theta') \qquad \text{(Notice that the multipole } \ell \text{ has dimensions } q\,L^{\ell-1}\text{)}$$

- In terms of these coefficients we have that:

$$\phi = \frac{1}{4\pi\epsilon_0}\sum_\ell \frac{\rho_\ell}{r^{\ell+1}}P_\ell(\cos\theta)$$

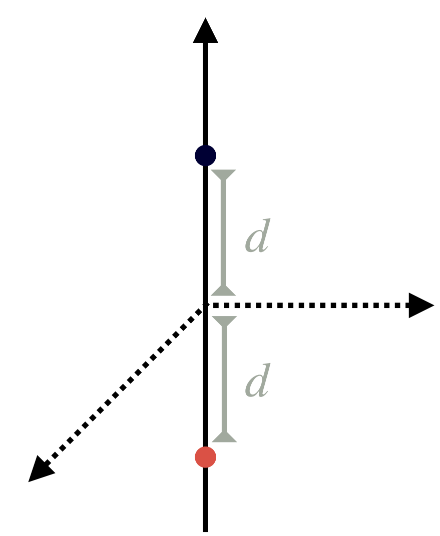
Physical multipoles



- The lowest multipole is the monopole, and for a charge distribution it expresses the total charge:

$$\rho_0 \;=\; \int d^3r'\,\rho(\vec{r}\,')$$

- The next multipole is the dipole, which can be written as:



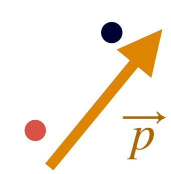
$$\rho_1 \;=\; \int d^3r'\,\rho(r',\theta')\,r'^1 P_1(\cos\theta') \;=\; \int d^3r'\,\rho(r',\theta') \times z'$$

So, for a pair of positive/negative charges at a distance $2d$ from each other we have:

$$\rho_1 \;=\; \int d^3r'\,\left[q\delta(\vec{r}\,' - d\hat{z}) - q\delta(\vec{r}\,' + d\hat{z})\right] \times z' = qd - q(-d) = 2qd$$

- In general we can have dipoles oriented in any direction, so we write:

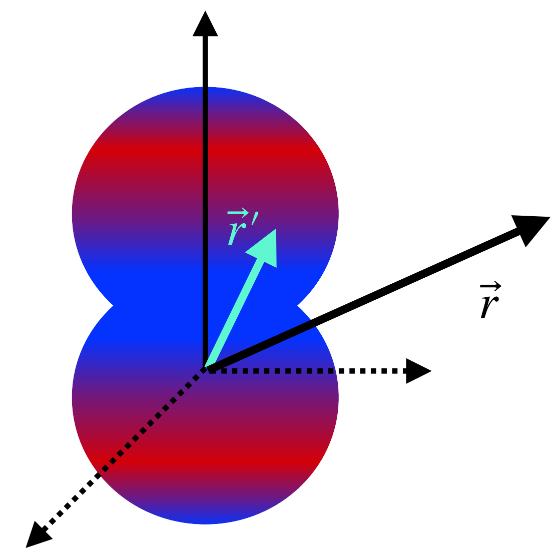
$$\rho_1 = \hat{r}\cdot\overrightarrow{p} = \hat{r}\cdot\int d^3r'\,\vec{r}\,'\rho(\vec{r}\,')$$



Physical multipoles

- For the next multipole ($\ell = 2$, the quadrupole), it starts to get a bit more complicated. We have:

$$\rho_2 = \int d^3r' \, \rho(r',\theta') \, r'^2 P_2(\cos\theta') = \int d^3r' \, \rho(r',\theta') \times \frac{1}{2}\left[3z'^2 - r'^2\right]$$

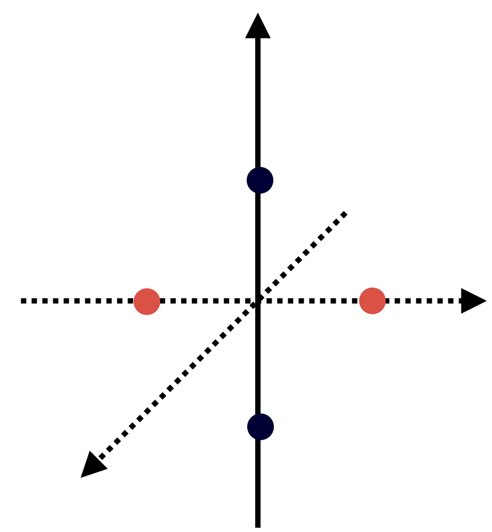


- But notice that we can write:

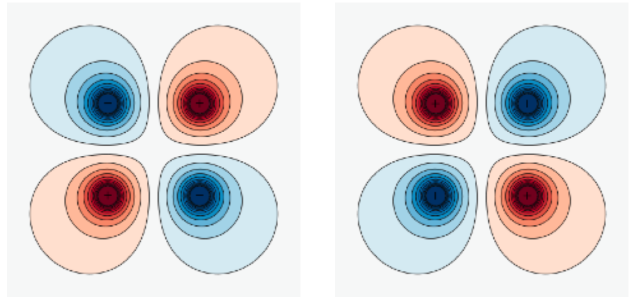
$$3z'^2 - r'^2 \to 3(\hat{r}\cdot\vec{r}\,')^2 - r'^2\,\hat{r}\cdot\hat{r} = \sum_{i,j=1}^{3}\left[3r'_i r'_j - r'^2\delta_{ij}\right]\hat{r}_i\hat{r}_j$$

- Therefore, a generic quadrupole can be expressed as:

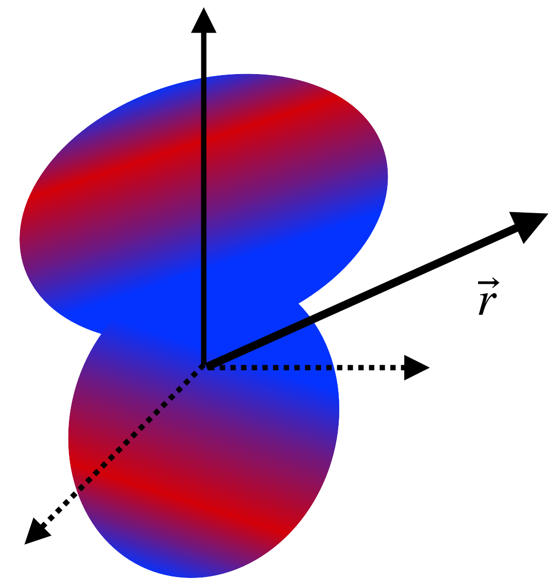
$$\rho_2 = \sum_{i,j=1}^{3}\hat{r}_i\hat{r}_j\,Q_{ij} \quad , \quad \text{where} \quad Q_{ij} = \frac{1}{2}\int d^3r'\left[3r'_i r'_j - r'^2\delta_{ij}\right]\rho(\vec{r}\,')$$



- The quadrupole is not a vector, however we can give it a vector-like interpretation. We can associate with it the **direction** perpendicular to a plane where we can place the charge distribution that generates the dipole.
- Notice also that the quadrupole has a certain **symmetry under rotations**: is invariant under rotations of π (180°) along that axis perpendicular to the plane of the quadrupole charges — and it changes sign under rotations of $\pi/2$.
- In fact, the multipole ℓ is invariant under rotations of $2\pi/\ell$. But we will come back to that!



Spherical Harmonics



- So far we have only considered problems with axial symmetry. But now we must abandon that assumption and start treating a completely general problem.
- Let's go back to the Laplace equation in spherical coordinates:

$$\nabla^2 \phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \phi}{\partial \varphi^2} \ = \ 0$$

- However, instead of assuming that nothing depends on the azimuthal angle, we will take that dependence into account. We will try (again) solutions in the form:

$$\phi = \frac{R(r)}{r} P(\theta)\,\Psi(\varphi) \quad ,$$

- Isolating the functions we obtain the equations:

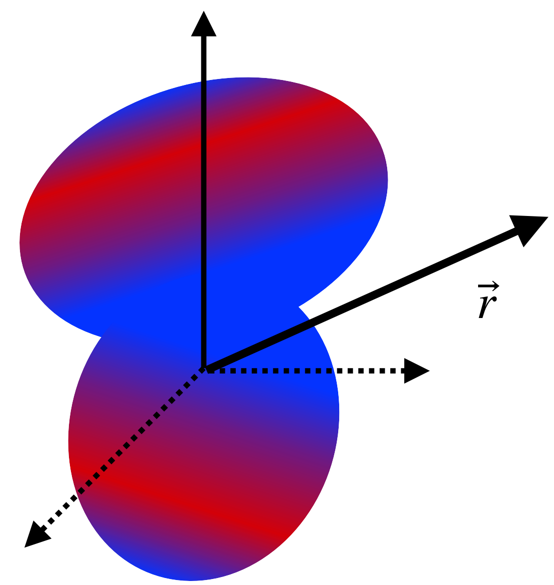
$$r^2 \sin^2\theta \left[\frac{1}{R}\frac{d^2 R}{dr^2} + \frac{1}{P\, r^2\, \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right)\right] + \frac{1}{\Psi}\frac{d^2\Psi}{d\varphi^2} = 0$$

- Notice that the structure is now a bit different: the term with Ψ can itself be a constant (not necessarily zero), so the solutions to the functions R and P cannot be simply the ones we had before.

Spherical Harmonics

- Playing the "separation of variables" game, we call

$$\frac{1}{\Psi}\frac{d^2\Psi}{d\varphi^2} = -m^2 \quad , \quad \text{which leads directly to the solutions } \Psi \to e^{im\varphi} \quad .$$



- So, if we set $m = 0$ we come back to the case of axial symmetry. So, using that correspondence we get, for the functions R and P:

$$\frac{d^2R}{dr^2} - \frac{\ell(\ell+1)}{r^2}R = 0 \quad ,$$

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dP}{d\mu}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-\mu^2}\right]P = 0 \quad , \quad \text{where } \mu = \cos\theta \, .$$

- The radial functions are the same as before, (power laws), but now the functions $P(\theta)$ are a kind of *generalization* of the Legendre polynomials — the ***associated Legendre polynomials***.

- In order to obtain finite solutions in $\mu \in [-1,1]$ we must have $\ell \geq 0$, and $m = -\ell, -\ell+1, \ldots, \ell-1, \ell$. So, we have:

$$P(\theta) \to P_\ell^{(m)}(\mu) = (-1)^m(1-\mu^2)^{m/2}\frac{d^m}{d\mu^m}P_\ell(\mu) \quad .$$

The $(-1)^m$ factor is for "historical"

Spherical Harmonics

- The associated Legendre functions also obey orthogonality relations:

$$\int_{-1}^{1} d\mu \, P_\ell^{(m)}(\mu) \, P_{\ell'}^{(m)}(\mu) = \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!} \delta_{\ell\ell'} \quad .$$

- Notice that the functions of the azimuthal angle φ are *also* orthogonal, in the sense:

$$\int_0^{2\pi} d\varphi e^{im\varphi} e^{-im'\varphi} = 2\pi \, \delta_{mm'} \quad .$$

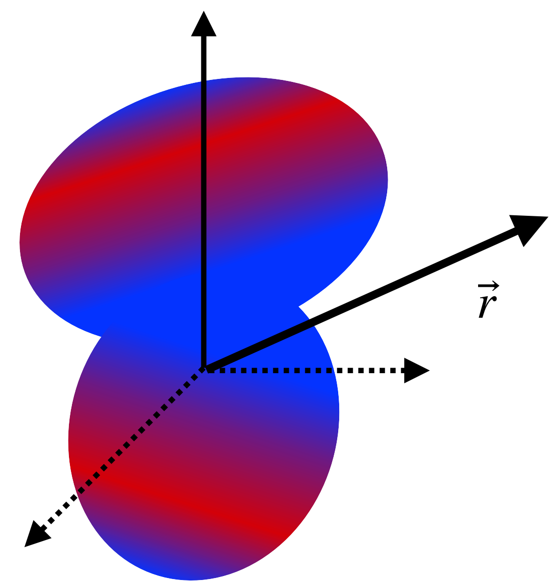
- This means that we could ***combine*** the two functions of the angular variables and obtain:

$$\int_{-1}^{1} d\mu \int_0^{2\pi} d\varphi \left[e^{im\varphi} \, P_\ell^{(m)}(\mu) \right] \left[e^{-im'\varphi} P_{\ell'}^{(m')}(\mu) \right] = \frac{4\pi}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!} \, \delta_{\ell\ell'} \, \delta_{mm'}$$

- But the integrals above are simply the integral over **solid angle**,

$\int d\mu \int d\varphi = \int \sin\theta d\theta \int d\varphi = \int d^2\Omega$, so we can write:

$$\int \frac{d^2\Omega}{4\pi} \left[e^{im\varphi} \, P_\ell^{(m)}(\mu) \right] \left[e^{-im'\varphi} P_{\ell'}^{(m')}(\mu) \right] = \frac{1}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!} \, \delta_{\ell\ell'} \, \delta_{mm'}$$



Spherical Harmonics

- If we now normalize these angular functions appropriately,

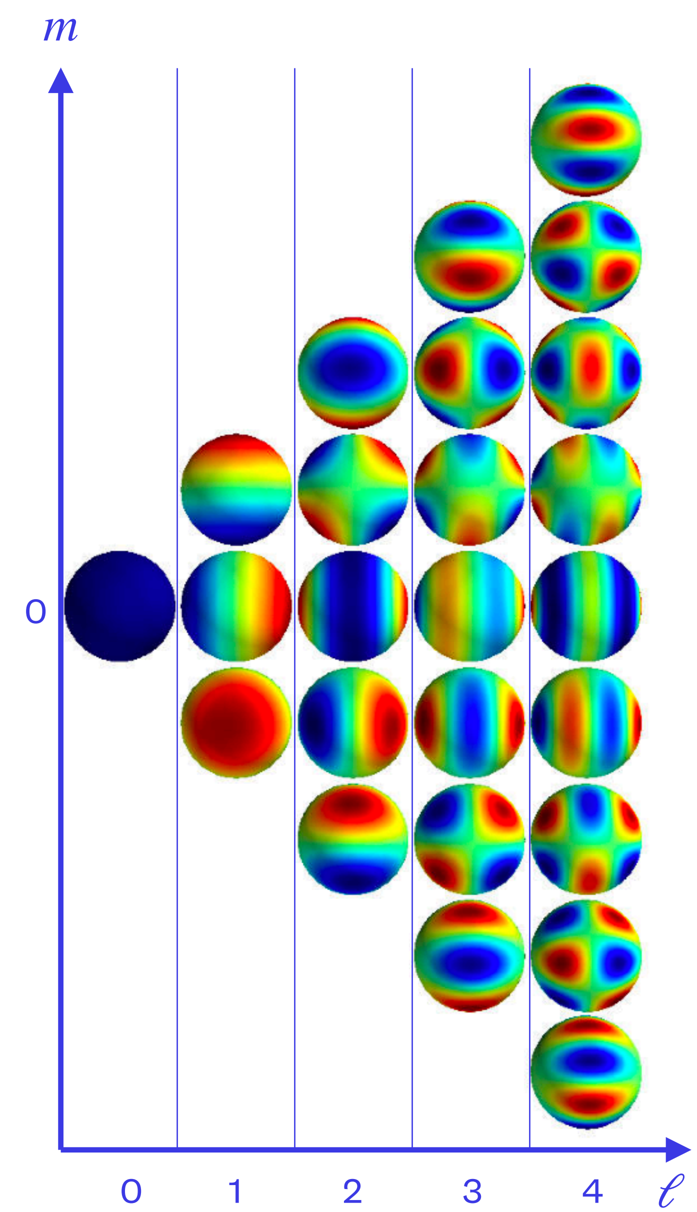
$$e^{im\varphi}\, P_\ell^{(m)}(\mu) \;\to\; (-1)^m \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\; e^{im\varphi}\, P_\ell^{(m)}(\mu) \;=\; Y_\ell^{(m)}(\theta,\varphi)$$

  we obtain functions that are normalized to unit under integration over the solid angle:

$$\int d^2\Omega\, Y_\ell^{(m)}(\theta,\varphi)\, Y_{\ell'}^{(m')*}(\theta,\varphi) \;=\; \delta_{\ell\ell'}\,\delta_{mm'} \quad .$$

- These are the **spherical harmonic functions**, and incredibly useful and versatile set of functions that are absolutely fundamental to treat a huge variety of problems, from classical mechanics, quantum mechanics, wave mechanics, medical physics, astronomy, cosmology… and, of course, electromagnetism!

- The list of properties, symmetries and relations of these angular basis functions is almost endless. I will list some of them here, and some I will leave for later.



Spherical Harmonics

- First of all, the dependence on the azimuthal angle is encoded in the index m , and by definition:

$$Y_\ell^{(m)*}(\theta,\varphi) = (-1)^m Y_\ell^{(-m)}(\theta,\varphi) \quad .$$

- In addition to the orthogonality, the spherical harmonics also obey a ***completeness relation***, which is orthogonality in the target space of the basis functions:

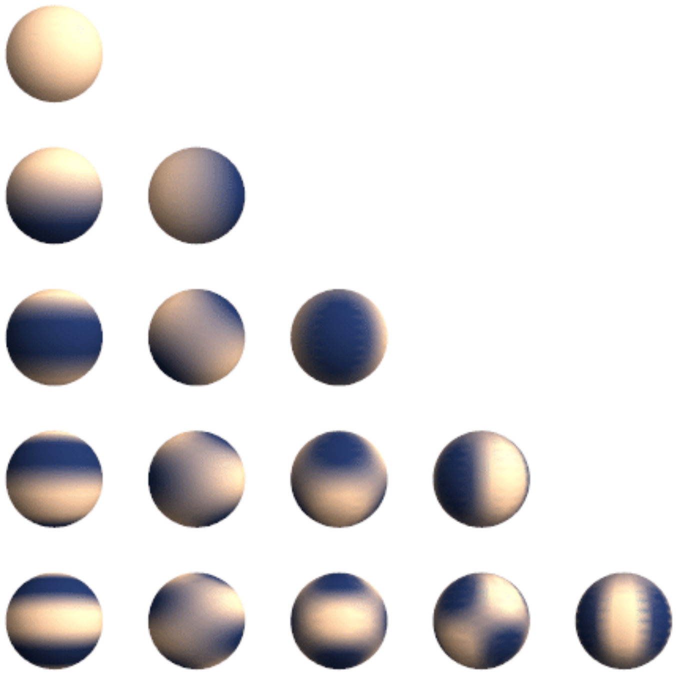
$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_\ell^{(m)*}(\theta,\varphi)Y_\ell^{(m)}(\theta',\varphi') \;=\; \delta(\varphi-\varphi')\,\delta(\cos\theta-\cos\theta')$$

- A soft version of the completeness relation involves only the azimuthal angle:

$$\sum_{m=-\ell}^{\ell} Y_\ell^{(m)*}(\theta,\varphi)Y_\ell^{(m)}(\theta',\varphi') \;=\; \frac{2\ell+1}{4\pi}P_\ell(\cos\gamma) \quad ,$$

where $\cos\gamma = \hat{n}\cdot\hat{n}'$, with $\hat{n}$ and $\hat{n}'$ denoting the *unit vectors* to the angular positions (θ,φ) and (θ',φ'). In particular, if $\hat{n}=\hat{n}'$ we get:

$$\sum_{m=-\ell}^{\ell} \left| Y_\ell^{(m)}(\theta,\varphi)\right|^2 \;=\; \frac{2\ell+1}{4\pi}$$



See more visualizations of the spherical harmonics in, e.g., http://www-udc.ig.utexas.edu/external/becker/teaching-sh.html

Spherical Harmonics and angular momentum

- It is worth recalling the relation between the spherical harmonics and angular momentum. Let's start with the equations that the spherical harmonics obey:

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dY_\ell^{(m)}}{d\mu}\right]+\left[\ell(\ell+1)-\frac{m^2}{1-\mu^2}\right]Y_\ell^{(m)}=0$$

- Now remember that, in Quantum Mechanics, angular momentum is given by the operator:

$$\vec{L}=-i\hbar\,\vec{r}\times\vec{\nabla}=L_x\hat{x}+L_y\hat{y}+L_z\hat{z}$$

and, in particular,

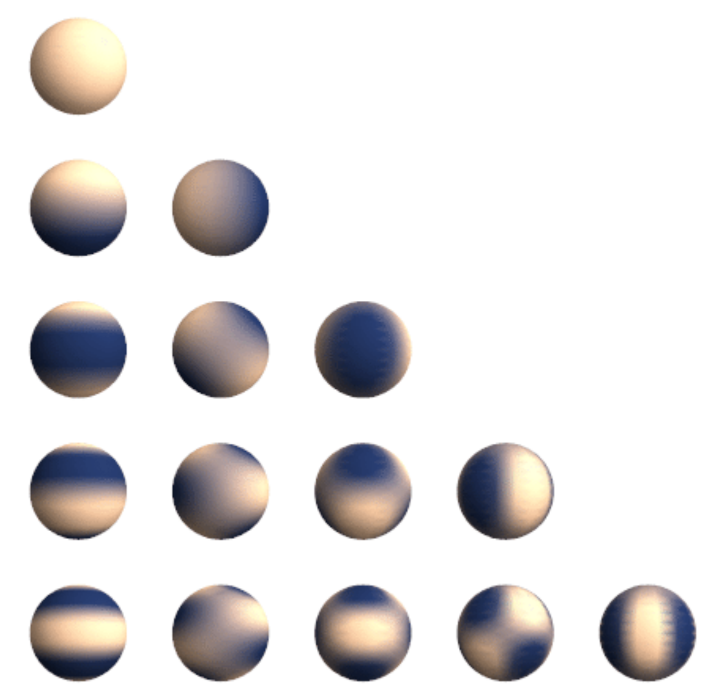
$$L_z=-i\hbar\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)=-i\hbar\frac{\partial}{\partial\varphi}$$

- The total angular momentum is:

$$-\frac{1}{\hbar^2}L^2=\nabla_\Omega^2=\frac{d}{d\mu}\left[(1-\mu^2)\frac{d}{d\mu}\right]+\frac{1}{1-\mu^2}\frac{d^2}{d\varphi^2}$$

- It is clear then that the spherical harmonics are **eigenfunctions** of the angular momentum operator:

$$L^2\,Y_\ell^{(m)}=\hbar^2\,\ell(\ell+1)Y_\ell^{(m)}\quad,\quad\text{and}\quad L_z\,Y_\ell^{(m)}=\hbar\,m\,Y_\ell^{(m)}$$



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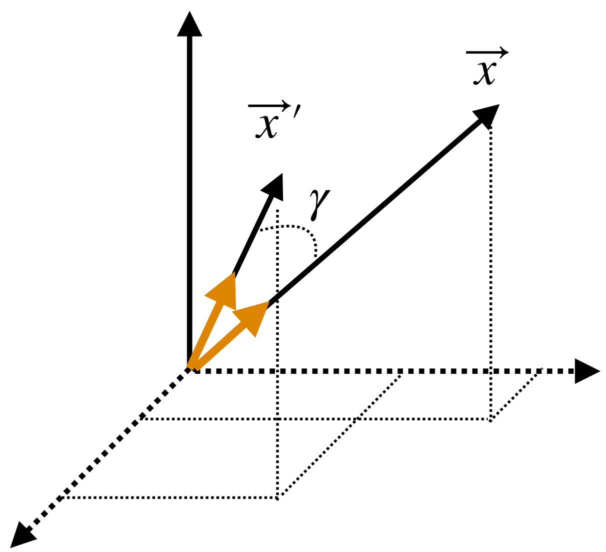
Spherical Harmonics: applications



- A familiar example of an application of spherical harmonics is another way that we can expand the function $1/|\vec{x} - \vec{x}\,'|$:

$$\frac{1}{|\vec{x} - \vec{x}\,'|} = \sum_{\ell} \frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(\hat{n} \cdot \hat{n}') \quad .$$

- Using the relation that we obtained above, i.e.,

$$\sum_{m=-\ell}^{\ell} Y_\ell^{(m)*}(\hat{n})\, Y_\ell^{(m)}(\hat{n}') = \frac{2\ell + 1}{4\pi} P_\ell(\hat{n} \cdot \hat{n}') \quad ,$$

which leads to:

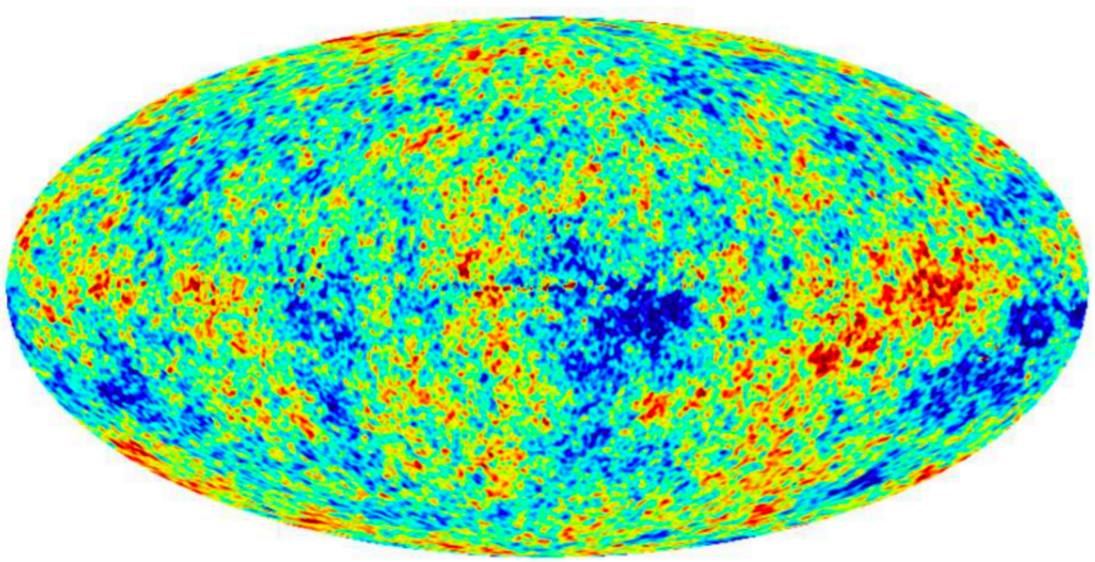
$$\frac{1}{|\vec{x} - \vec{x}\,'|} = \sum_{\ell,m} \frac{4\pi}{2\ell + 1} \frac{r_<^{\ell}}{r_>^{\ell+1}} Y_\ell^{(m)}(\hat{n})\, Y_\ell^{(m)*}(\hat{n}') \quad .$$

Spherical Harmonics: applications

- The fact that the spherical harmonics are a complete set of orthogonal, normalized functions, means that we can expand **any angular function** in 3D into a set of coefficients:

$$f(\theta,\varphi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_{\ell m}\, Y_\ell^{(m)}(\theta,\varphi) \quad \text{, where the coefficients are given by:} \quad f_{\ell m} = \int d^2\Omega\, f(\theta,\varphi)\, Y_\ell^{(m)*}(\theta,\varphi)$$

- A nice example is the way in which cosmologists like myself describe the *cosmic microwave background radiation*, which are is light coming from far away in the Universe, that arrive from all directions of the sky.



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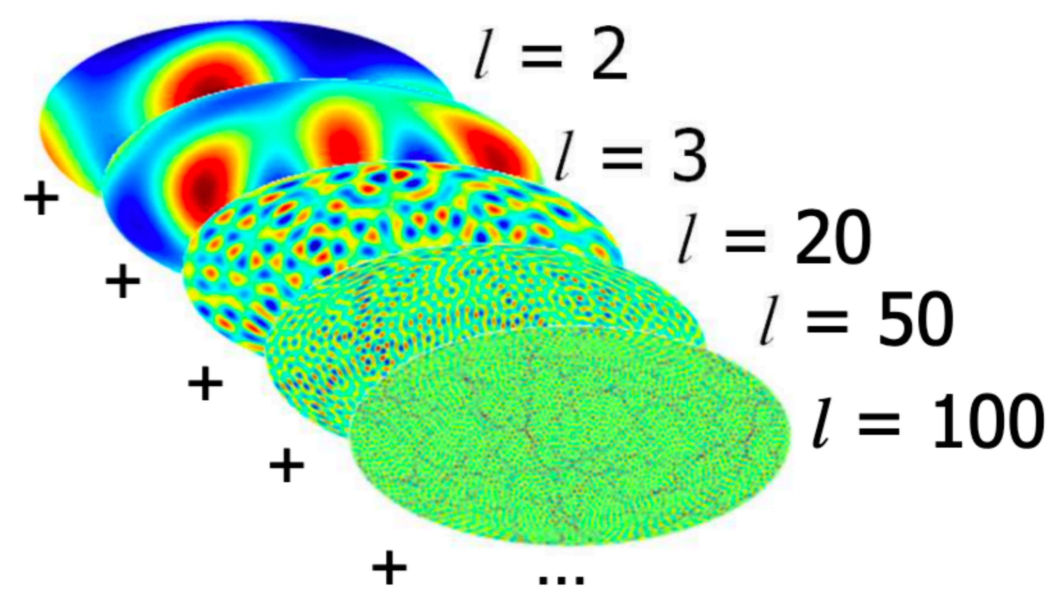

Next class:

- Laplace equation in cylindrical and spherical coordinates
- Green's functions
- More boundary condition problems
- Jackson, Ch. 3; Zangwill, Ch. 4