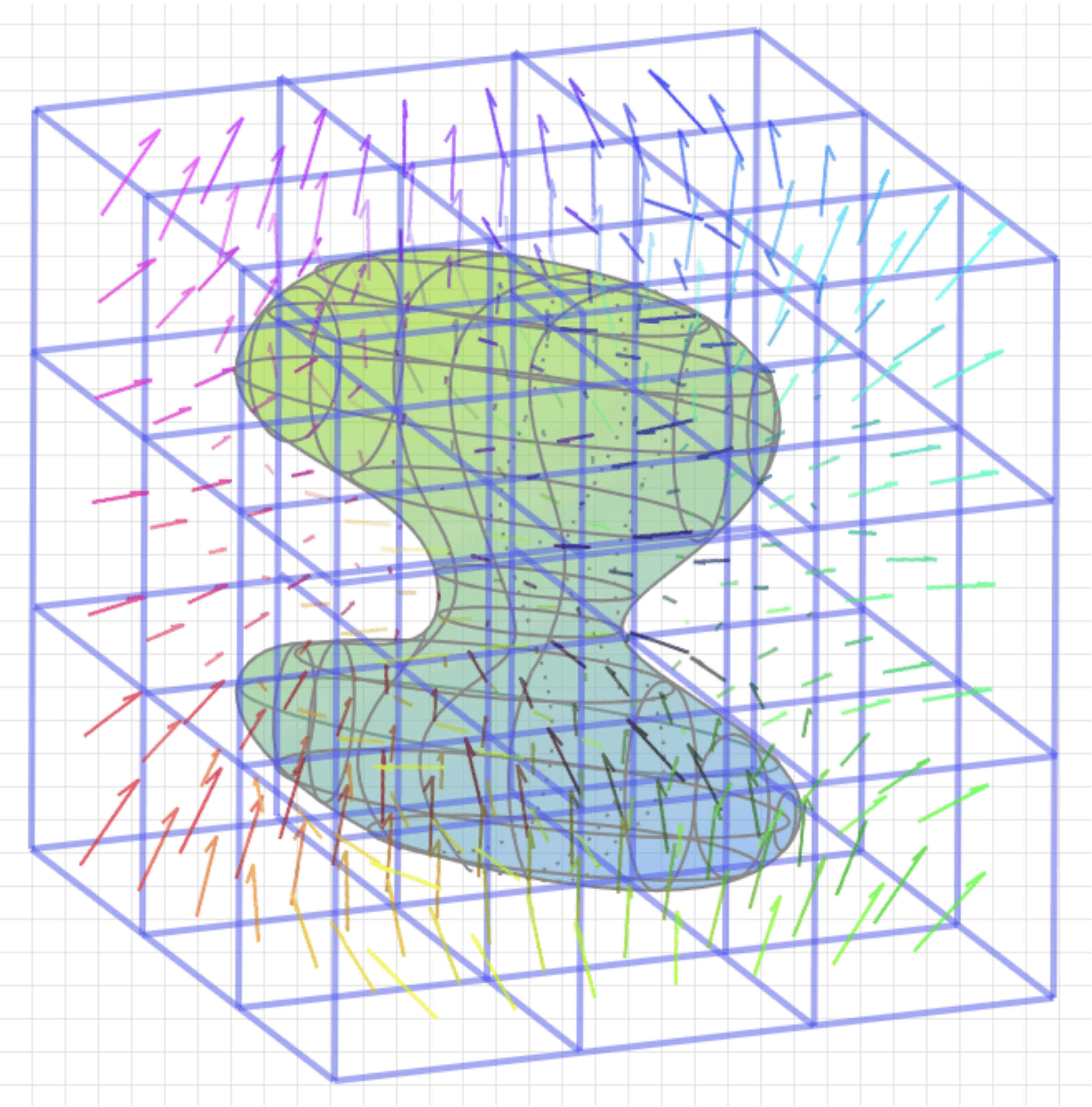
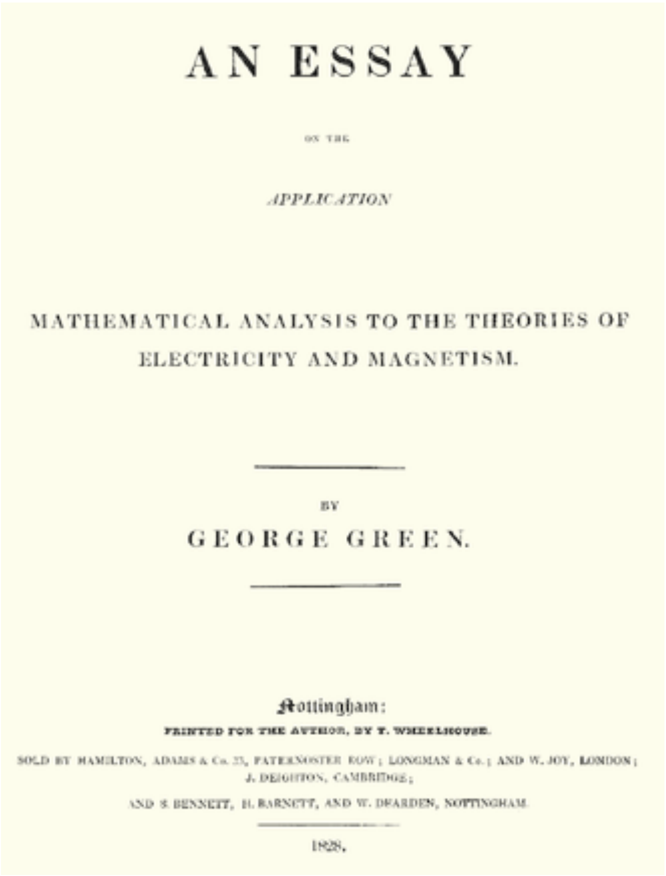
Electrodynamics

Preamble: Green's Theorem



Green's Theorem


AN ESSAY

ON THE

APPLICATION

MATHEMATICAL ANALYSIS TO THE THEORIES OF ELECTRICITY AND MAGNETISM.

BY

GEORGE GREEN.

Nottingham:

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1828.

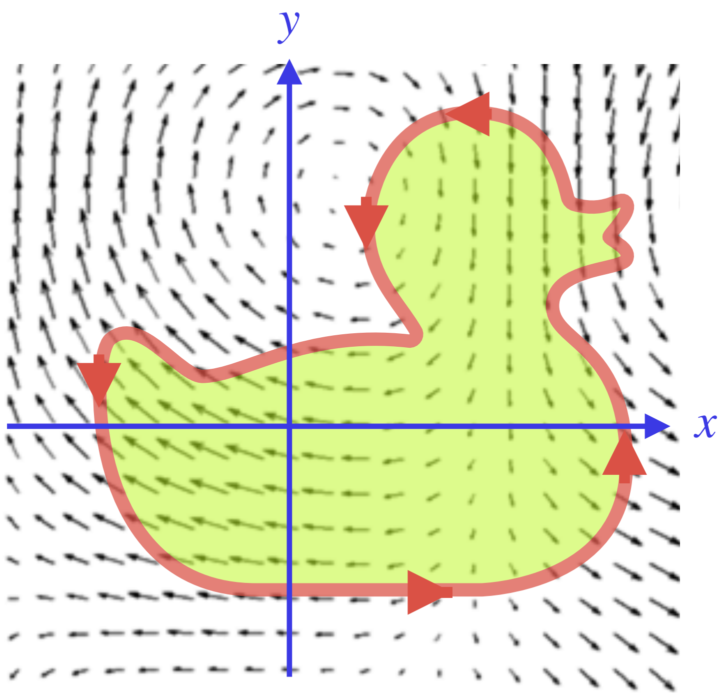
- Green's theorem is a statement in 2D, relating a line integral over a closed loop to a surface integral. Given two scalar functions f and g of the 2D position $\vec{x}$, we have:

$$\int_S d^2x \left(\frac{\partial f}{\partial x} - \frac{\partial g}{\partial y} \right) = \oint_{\vec{C}(S)} (g\, dx + f\, dy) \, .$$

- This theorem is in fact a special case of the ***Kelvin-Stokes Theorem***:

$$\int_S d\vec{S} \cdot \vec{\nabla} \times \vec{F} = \oint_{\vec{C}(S)} d\vec{l} \cdot \vec{F} \, .$$

Proof: take the $x - y$ plane, and identify $f \to F_y \, , g \to F_x$.



Stokes Theorem

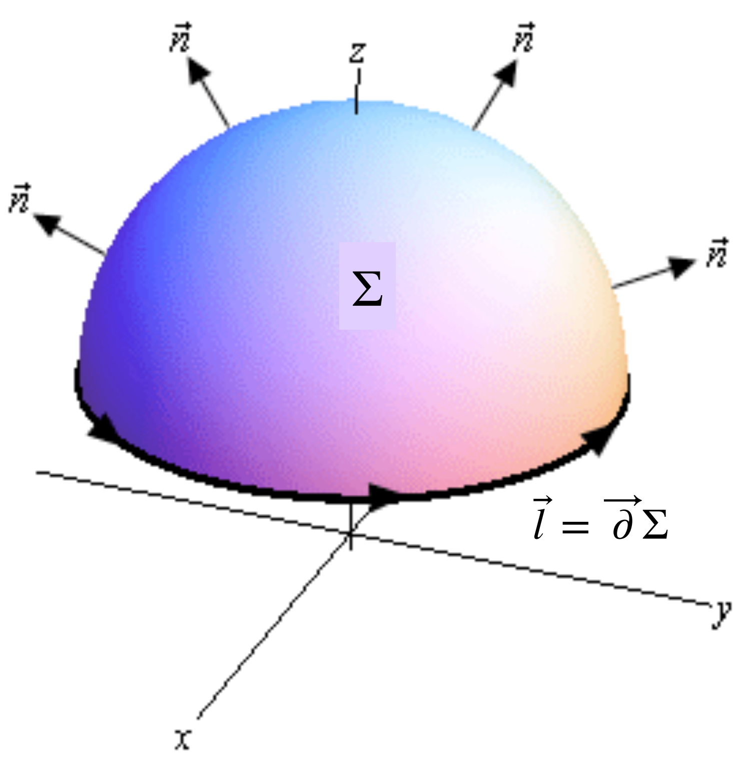
- The Kelvin-Stokes Theorem is a particular (3D, Euclidean) case:

$$\int_S d\vec{S} \cdot \vec{\nabla} \times \vec{F} = \oint_{\vec{C}(S)} d\vec{l} \cdot \vec{F}$$

- The ***Generalized Stokes Theorem***, also known as the ***Stokes-Cartan Theorem***, states that, for a "bulk" Σ and its "boundary" $\partial\Sigma$, the integral of:

$$\int_\Sigma d\omega = \oint_{\partial\Sigma} \omega \, ,$$

where $d\omega$ is the ***exterior derivative*** of the ***1-form*** ω .



Green's and Gauss's Theorem

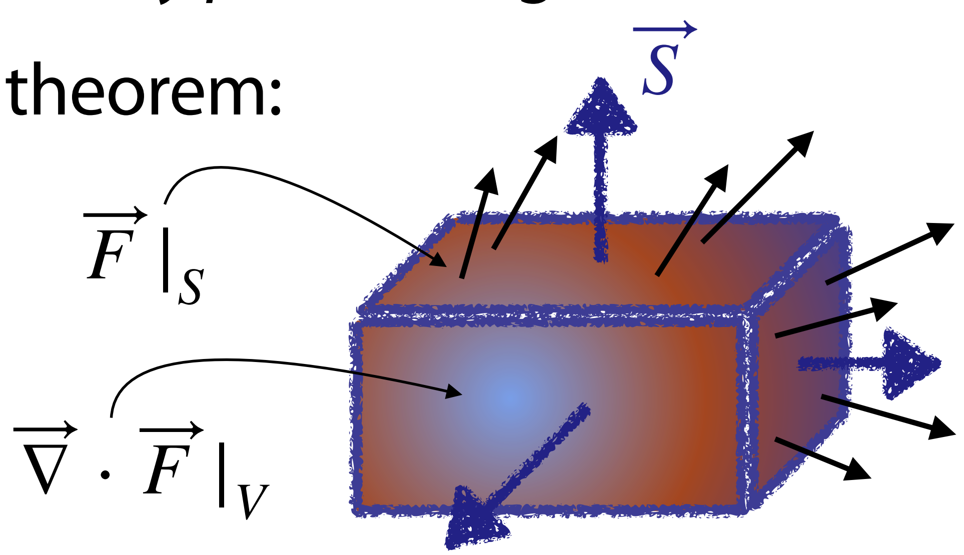
- The theorem is summed up by the following equality:

$$\int d^3x \left(f\,\nabla^2 g - g\,\nabla^2 f\right) = \oint d\vec{S} \cdot \left(f\,\vec{\nabla} g - g\,\vec{\nabla} f\right),$$

where f and g are *scalar* functions of the position $\vec{x}$

- The demonstration is straigthforward: just *integrate by parts*, using the fact that $\vec{\nabla} \cdot \vec{\nabla} f = \nabla^2 f$, and the Gauss (or divergence) theorem:

$$\int d^3x\,\vec{\nabla} \cdot \vec{F} = \oint d\vec{S} \cdot \vec{F}\,.$$



Green's Theorem and Surveying*

- Let's consider Green's theorem on a "pancacke" on the plane $z = 0$, with area A and height h .
- Evidently, the volume of the pancake is $V = A\,h$.
- Now, let's choose the functions f and g such that we have $f\,\nabla^2 g - g\,\nabla^2 f = 1$. Then, clearly, by Green's theorem we have that:

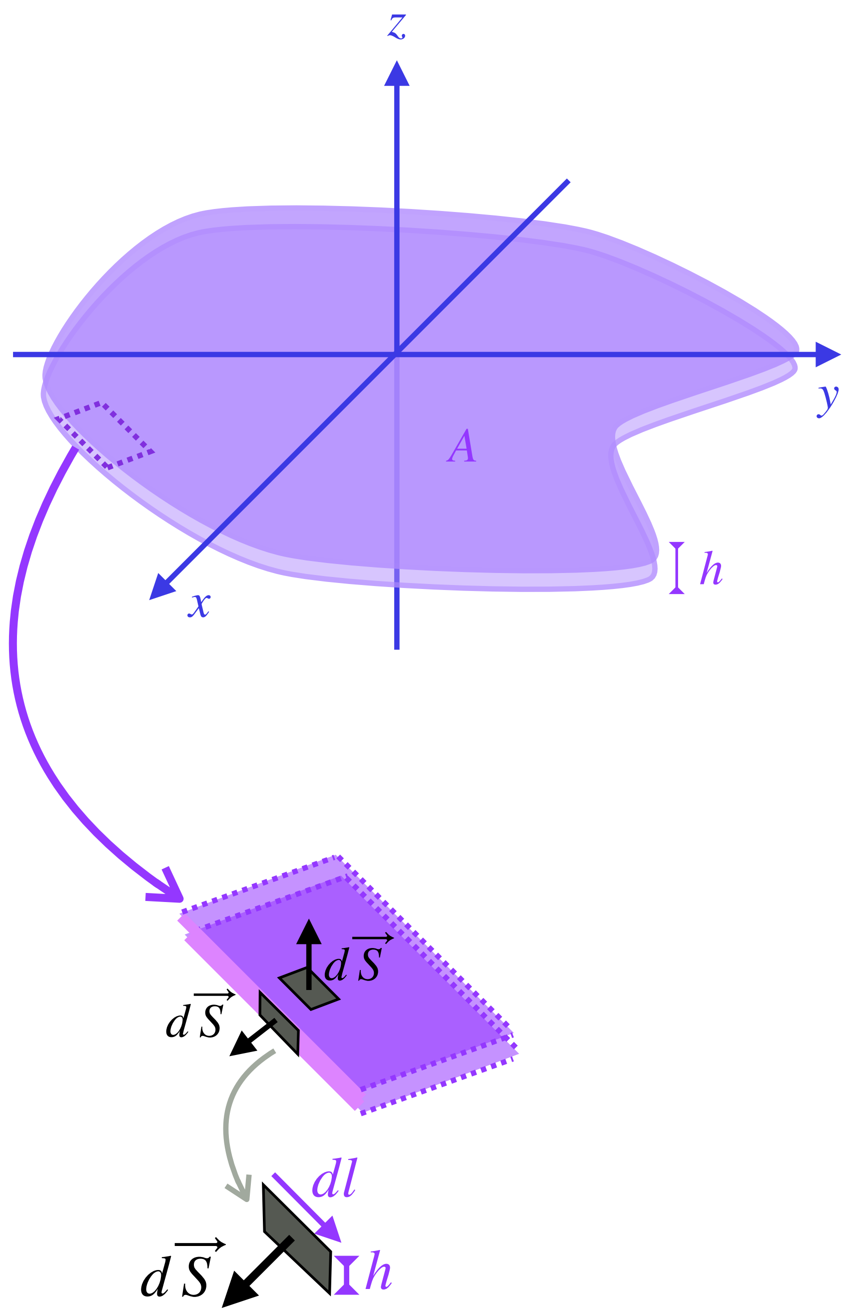
$$\int d^3x\,\left(f\,\nabla^2 g - g\,\nabla^2 f\right) = V = \oint d\vec{S} \cdot \left(f\,\vec{\nabla} g - g\,\vec{\nabla} f\right)$$

- Now let's suppose we can take

$f\,\vec{\nabla} g - g\,\vec{\nabla} f \perp \hat{z}$, such that

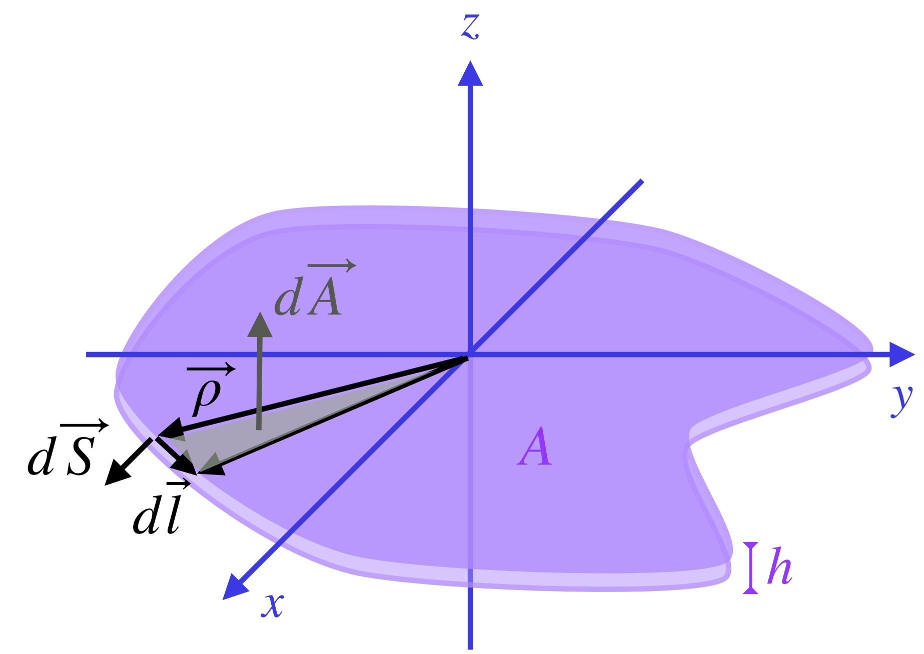
$d\vec{S} \cdot \left(f\,\vec{\nabla} g - g\,\vec{\nabla} f\right) \neq 0$ ***only*** on the ***side*** borders , where

$dS = h\,dl$



* Agrimensura

Green's Theorem and Surveying



- In particular, we can choose f and g such that:

$$f\,\nabla^2 g - g\,\nabla^2 f = 1 \quad \text{and} \quad f\,\vec{\nabla} g - g\,\vec{\nabla} f = \frac{1}{2}\vec{\rho}$$

- This means that:

$$A\,h = \oint d\vec{S}\cdot\left(f\,\vec{\nabla} g - g\,\vec{\nabla} f\right) = \int_{edge} \frac{d\vec{S}\cdot\vec{\rho}}{2}$$

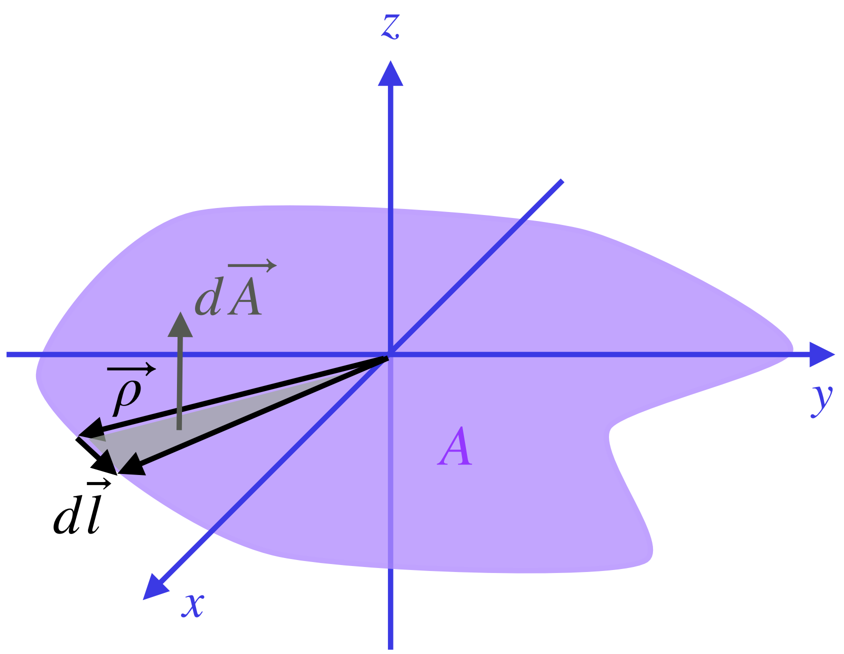
- Since $d\vec{S} = -\,h\,\hat{z}\times d\vec{l}$, we can write:

$$A\,h = -\,h\int_{edge}\frac{(\hat{z}\times d\vec{l})\cdot\vec{\rho}}{2} = -\,h\int_{edge}\frac{\hat{z}\cdot(d\vec{l}\times\vec{\rho}\,)}{2}$$

- By inspection, the area of the triangle in the figure is $\frac{1}{2}d\vec{l}\times\vec{\rho} = -\,dA\,\hat{z}$! Therefore,

$$A = \hat{z}\cdot\oint\frac{\vec{\rho}\times d\vec{l}}{2}\ ,$$ which is in fact evident!

Green's Theorem and Surveying



- Examples of functions which fulfill the requirement that

$$f\,\nabla^2 g - g\,\nabla^2 f = 1 \quad \text{and} \quad f\,\vec{\nabla} g - g\,\vec{\nabla} f = \frac{1}{2}\vec{\rho}\ :$$

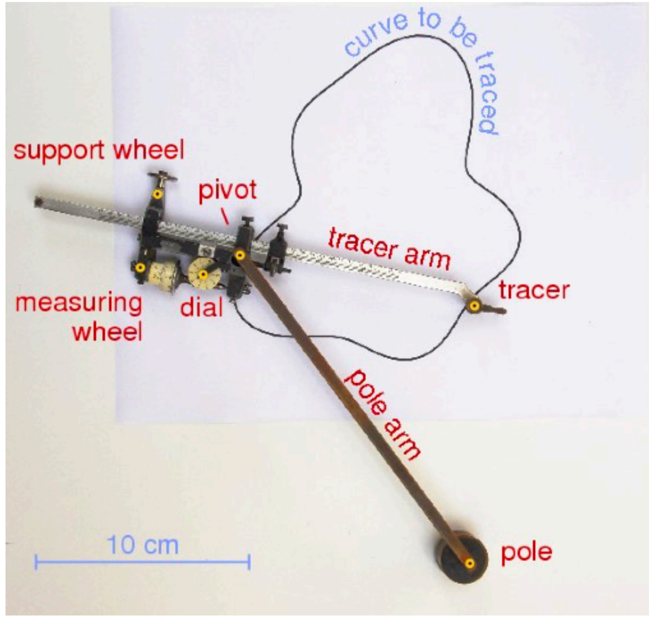
- Choose, for instance:

$$f = 1 \quad \text{and} \quad g = \frac{x^2+y^2}{4} = \frac{\rho^2}{4}$$

- But any choice of these two functions would work, as long as the conditions above are satisfied . Notice that, e.g., one can re-scale $f \to \alpha f \ ,\ g \to g/\alpha$.

- The location of the origin is also arbitrary — it could be even ***out*** of the area! (Check!)

- These results allow us to build simple machines* that can measure the area inside a closed curve just by following the path along that curve!



* http://www.ams.org/publicoutreach/feature-column/fcarc-surveying-two

Miscellaneous theorems of vector calculus

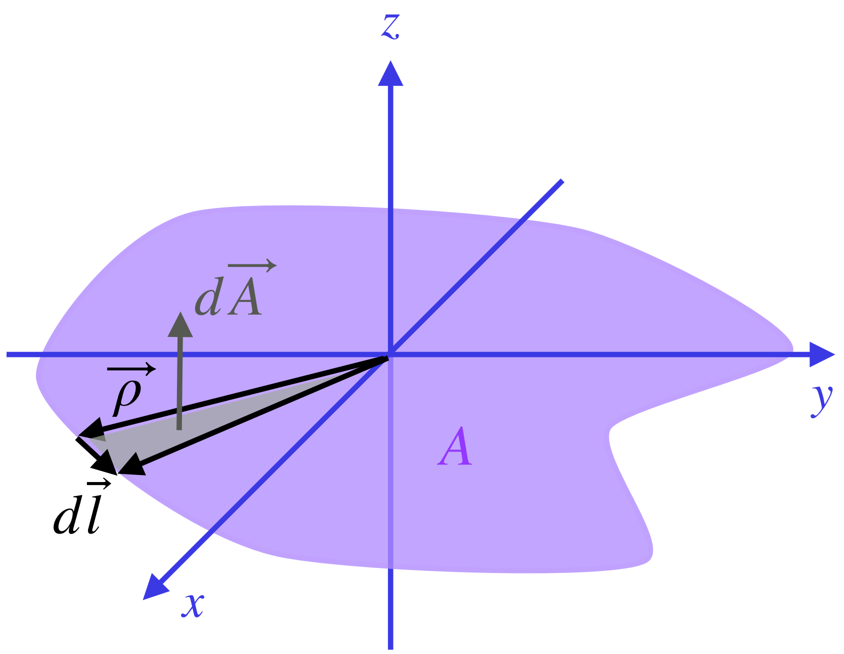
- Here are some other examples of identities ("theorems") of vector calculus:

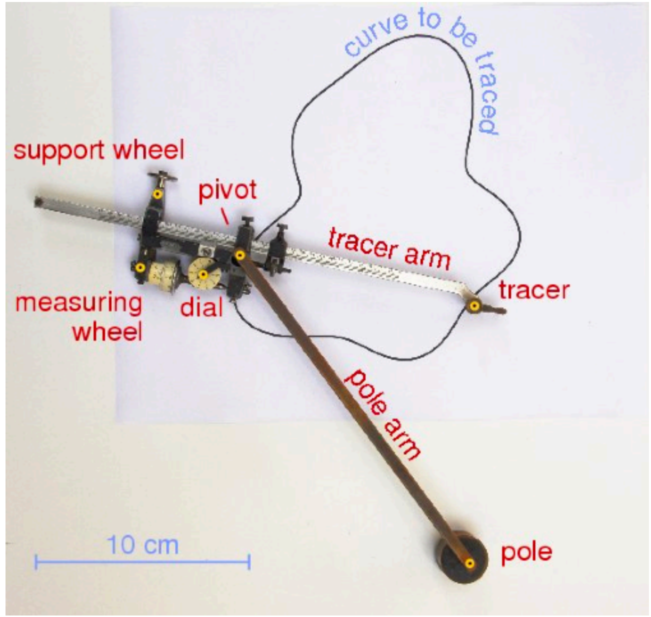
$$\int_S d\vec{S} \times \vec{\nabla} f = \oint_{C(S)} d\vec{l}\, f$$

$$\int_V dV\, \vec{\nabla} f = \oint_{S(V)} d\vec{S}\, f$$

$$\int_V dV\, \vec{\nabla} \times \vec{F} = \oint_{S(V)} d\vec{S} \times \vec{F}$$

$$\int_V dV \left(\vec{\nabla} \cdot \vec{F} + \vec{F} \cdot \vec{\nabla} \right) \vec{G} = \oint_{S(V)} \left(d\vec{S} \cdot \vec{F} \right) \vec{G}$$



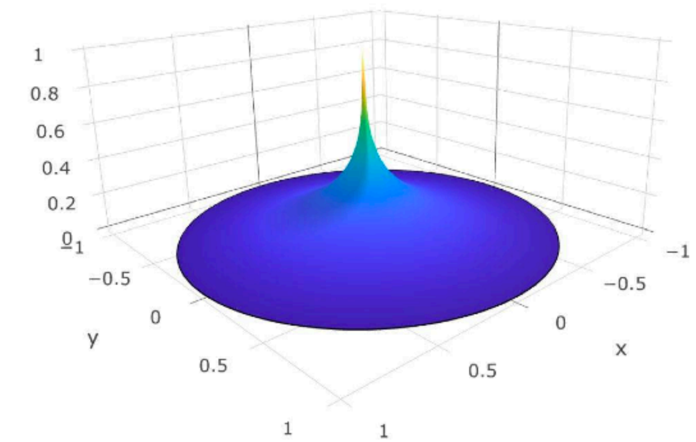


* http://www.ams.org/publicoutreach/feature-column/fcarc-surveying-two

Gauss's Law and Green's function for the Poisson Equation

- Let's go back to Gauss's Law (the Divergence Theorem):

$$\int d^3x\, \vec{\nabla} \cdot \vec{F} = \oint d\vec{S} \cdot \vec{F} \ .$$



- Consider the ***Poisson equation***:

$\nabla^2 \phi = s(\vec{x})$, where $s(\vec{x})$ is the ***source*** of the scalar field ϕ .

- We can solve this equation ***exactly*** if we find the **Green's function**:

$\nabla^2_x G(\vec{x}, \vec{x}') = \delta(\vec{x} - \vec{x}')$, where $\delta(\vec{x} - \vec{x}')$ is the ***Dirac delta function*** (a ***distribution***, actually!)

$$\Rightarrow \quad \phi(\vec{x}) = \int d^3x'\, G(\vec{x}, \vec{x}')\, s(\vec{x}')$$

- Verify:

$$\nabla^2_x \phi(\vec{x}) = \nabla^2_x \int d^3x'\, G(\vec{x}, \vec{x}')\, s(\vec{x}') = \int d^3x'\, \delta(\vec{x} - \vec{x}')\, s(\vec{x}') = s(\vec{x})$$

Gauss's Law and Green's function for the Poisson Equation

- So, the question now is: what is the Green's function for the Poisson equation,

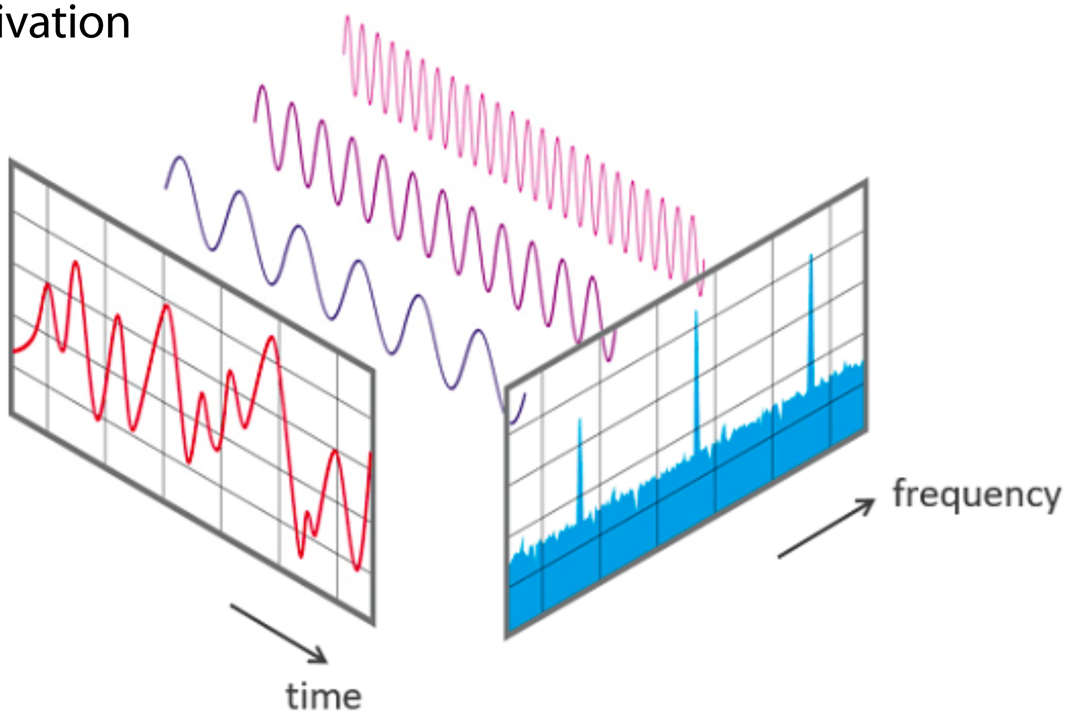
$$(*) \quad \nabla^2_x G(\vec{x}, \vec{x}\,') = \delta(\vec{x} - \vec{x}\,')$$

- There are many ways of going about solving for G. My favorite way, though totally overkill for this problem, is to use the ***Fourier transform***. I will use this derivation here for "pedagogical" effect.

- Given a function $f(\vec{x})$, its Fourier transform is defined as:

$$\tilde{f}(\vec{k}) = \int d^3x\, e^{i\vec{k}\cdot\vec{x}} f(\vec{x}) \quad , \quad \text{and the inverse ,}$$

$$f(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{-i\vec{k}\cdot\vec{x}} \tilde{f}(\vec{k})$$



- The relationships above are guaranteed by the fact that:

$$\int d^3x\, e^{\pm i\vec{k}\cdot\vec{x}} = (2\pi)^3\, \delta(\vec{k}) \quad , \quad \text{and conversely} \int \frac{d^3k}{(2\pi)^3}\, e^{\pm i\vec{k}\cdot\vec{x}} = \delta(\vec{x})$$

Gauss's Law and Green's function for the Poisson Equation

- OK, so let's try to solve for the Green's function of the Poisson equation using a Fourier transform. The equation is:

$$\nabla_x^2 G(\vec{x}, \vec{x}') = \delta(\vec{x} - \vec{x}')$$

- By looking at the Right-Hand-Side (RHS) of that equation we can "take the hint" that the Green's function should be a function only of the distance $\vec{R} = \vec{x} - \vec{x}'$, so $G(\vec{x}, \vec{x}') = G(\vec{x} - \vec{x}') = G(\vec{R})$.
- Let's then write:

$$\tilde{G}\,(\vec{k}) = \int d^3R\, e^{i\vec{k}\cdot\vec{R}}\, G(\vec{R}) \quad , \quad \text{and conversely}\, , \quad G\,(\vec{R}) = \int \frac{d^3k}{(2\pi)^3}\, e^{-i\vec{k}\cdot\vec{R}}\, \tilde{G}(\vec{k})\, .$$

- Substituting this last equality into the Poisson equation we have:

$$\nabla_x^2 G\,(\vec{R}) \;=\; \nabla_x^2 \int \frac{d^3k}{(2\pi)^3}\, e^{-i\vec{k}\cdot\vec{R}}\, \tilde{G}(\vec{k}) \;=\; \nabla_x^2 \int \frac{d^3k}{(2\pi)^3}\, e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}\, \tilde{G}(\vec{k}) \;=\; \int \frac{d^3k}{(2\pi)^3}\, (-k^2) e^{-i\vec{k}\cdot\vec{R}}\, \tilde{G}(\vec{k})$$

- Now, since the RHS of that equation (the *source term*) is $\delta(\vec{R})$ we can use the Fourier expression for it, arriving at:

$$\int \frac{d^3k}{(2\pi)^3}\, (-k^2) e^{-i\vec{k}\cdot\vec{R}}\, \tilde{G}(\vec{k}) \;=\; \delta(\vec{R}) \;=\; \int \frac{d^3k}{(2\pi)^3}\, e^{-i\vec{k}\cdot\vec{R}}$$

Gauss's Law and Green's function for the Poisson Equation

- Therefore, the integrands must be the same *, which means that:

$$\int \frac{d^3k}{(2\pi)^3}(-k^2)e^{-i\vec{k}\cdot\vec{R}}\,\tilde{G}(\vec{k}) = \int \frac{d^3k}{(2\pi)^3}\,e^{-i\vec{k}\cdot\vec{R}} \;\Rightarrow\; \tilde{G}(\vec{k}) = -\frac{1}{k^2}$$

- Now all we need to do is to integrate this back to "real" ("configuration") space:

$$G(\vec{R}) = -\int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2}e^{-i\vec{k}\cdot\vec{R}}$$

- Let's use a spherical coordinate system for $\vec{k}$, and choose the direction of the k_z axis to lie parallel to the position vector $\vec{R}$, so that we can write:

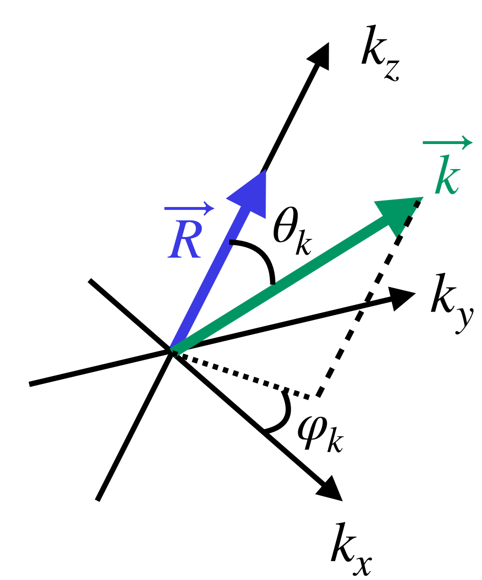
$$G(\vec{R}) = \frac{-1}{(2\pi)^3}\int_0^\infty dk\,k^2\int_0^{2\pi}d\varphi_k\int_0^\pi d\theta_k\,\sin\theta_k \times \frac{1}{k^2}e^{-ikR\cos\theta_k}$$

- Now using the short notation $\mu_k = \cos\theta_k$, and using the axial symmetry over φ_k we get:

$$G(\vec{R}) = \frac{-1}{(2\pi)^3}\int_0^\infty dk \times 2\pi \times \int_{-1}^1 d\mu_k\,e^{-ikR\mu_k}$$

$$= \frac{-1}{(2\pi)^2}\int_0^\infty dk \times 2\frac{\sin(kR)}{kR}$$

, where we used Euler's formula, $e^{iw} = \cos w + i\sin w$.



* Up to homogeneous terms

Gauss's Law and Green's function for the Poisson Equation

- We are almost done. This last result is in terms of a famous integral:

$$G(\vec{R}) = \frac{-1}{2\pi^2}\frac{1}{R}\int_0^\infty dw \times \frac{\sin w}{w} \quad , \quad \text{where we wrote } w = kR\,.$$

- This last integral is known as the **Dirichlet integral** (!), and it results in:

$$\int_0^\infty dw \times \frac{\sin w}{w} = \frac{\pi}{2}\,.$$

- We will prove this integral in a minute. For now let's write here our main result:

$$\boxed{G(\vec{R}) = \frac{-1}{4\pi}\frac{1}{R}}$$

- This is, therefore, **the Green's function of the Poisson equation**, and it is widely used in Electrostatics, Gravity, diffusion, etc.
- Notice that one of the more "popular" ways to write this Green's function is by noting that its gradient is:

$$\vec{\nabla} G(\vec{R}) = \frac{1}{4\pi}\frac{\vec{R}}{R^3} \quad \Longrightarrow \quad \vec{\nabla}_R^2 G(\vec{R}) = \vec{\nabla}_R \cdot \left[\vec{\nabla}_R G(\vec{R})\right] = \vec{\nabla}_R \cdot \left[\frac{1}{4\pi}\frac{\vec{R}}{R^3}\right] = \delta(\vec{R})$$

Which leads to the basic result of Gauss's law: $\int_V d^3R\, \vec{\nabla}_R \cdot \left[\frac{1}{4\pi}\frac{\vec{R}}{R^3}\right] = \oint_{S(V)} d^2\vec{S} \cdot \left[\frac{1}{4\pi}\frac{\vec{R}}{R^3}\right] = 1$ (***if*** V contains the origin!)

Gauss's Law and Green's function for the Poisson Equation

- Now let's get back to the Dirichlet integral:

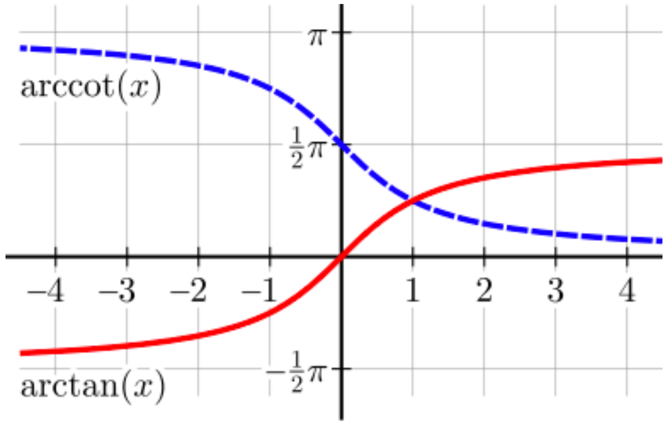
$$\int_0^\infty dw \times \frac{\sin w}{w} = \frac{\pi}{2} \; .$$

- Again, as in so many of these results, there are many proofs — one of them by Feynman himself! Let's write the function:

$$f(a) = \int_0^\infty dw \, e^{-wa} \, \frac{\sin w}{w}$$

- In order to find Dirichlet's integral we have to compute $f(0)$. But if we differentiate $f(a)$ with respect to a we get:

$$\frac{df}{da} = \int_0^\infty dw \, (-w) \, e^{-wa} \, \frac{\sin w}{w} = - \int_0^\infty dw \, e^{-wa} \, \sin w$$



- Now, this integral is trivial to compute using Euler's formula, with the result that:

$$\frac{df}{da} = - \frac{1}{1+a^2} \quad \Longrightarrow \quad f(a) = \int da \, \frac{df}{da} = - \int da \, \frac{1}{1+a^2} = - \arctan a + C$$

- The integration constant can be found by noting that, from the definition of $f(a)$, for $a \to \infty$ we have $f \to 0$. Since $\lim_{a\to\infty} \arctan a = \pi/2$ we have:

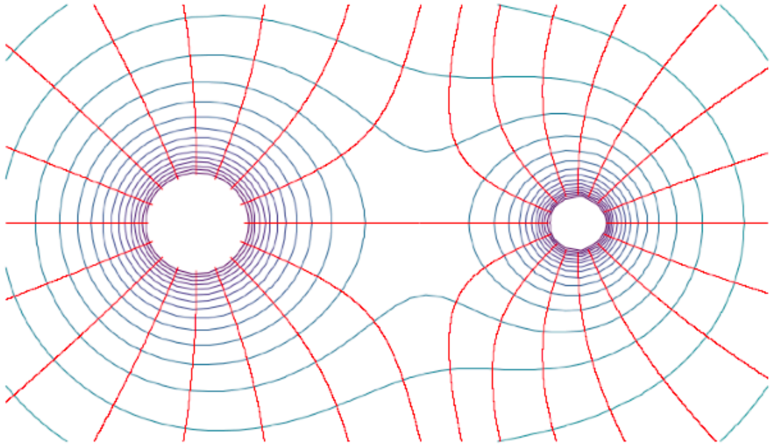
$$f(a) = - \arctan a + \frac{\pi}{2} \quad \Longrightarrow \quad f(0) = \frac{\pi}{2} \; .$$

Gauss's Law , Green's function and all that

- Therefore, one of the main conclusions we draw from today's lecture is that the Poisson equation:

$$\nabla^2 \phi = s(\vec{x})$$

can be solved as:



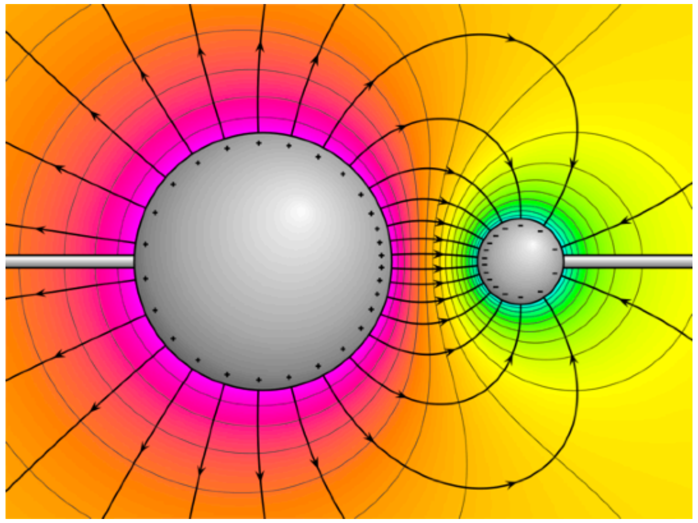
$$\phi(\vec{x}) = \int d^3x' \, G(\vec{x}, \vec{x}') \, s(\vec{x}') \quad , \text{ where}$$

$$G = \frac{-1}{4\pi} \frac{1}{|\vec{x} - \vec{x}'|} \quad , \text{ which solves the equation } \quad \nabla_x^2 G(\vec{x}, \vec{x}') = \delta(\vec{x} - \vec{x}')$$

- Equivalently, we can also state that the ***first integral*** of the identities above can be written in terms of:

$$\vec{\nabla}_x^2 G(\vec{x} - \vec{x}') = \vec{\nabla}_x \cdot \left[\vec{\nabla}_x G(\vec{x} - \vec{x}') \right] = \vec{\nabla}_x \cdot \left[\frac{1}{4\pi} \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} \right] = \delta(\vec{R})$$

Gauss's Law & Electrostatics



- In other words, the basic equation of Electrostatics, Gauss's Law:

$$\vec{\nabla} \cdot \vec{E} = -\nabla^2 \phi = \frac{1}{\epsilon_0} \rho(\vec{x}) \quad , \quad \text{where } \rho \text{ is the charge density and } \vec{E} = -\vec{\nabla}\phi \quad ,$$

has the solution:

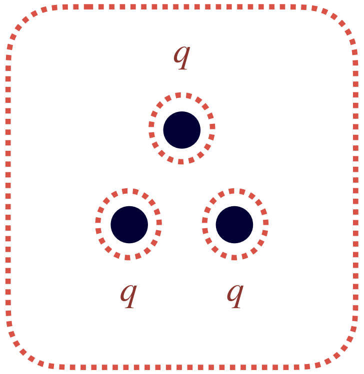
$$\phi(\vec{x}) = \frac{1}{4\pi\,\epsilon_0} \int d^3x' \, \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} \quad ,$$

where the charge density can be a ***continuous function*** (a ***distribution***), or it can be a sum of point charges:

$$\rho(\vec{x}) = \sum_i q_i \, \delta(\vec{x} - \vec{x}_i)$$

- Conversely, one can integrate these equations to find the integral form of Gauss's Law:

$$\int_V d^3x \, \vec{\nabla} \cdot \vec{E} = \oint_{S(V)} d\vec{S} \cdot \vec{E} = \int_V d^3x \, \frac{\rho}{\epsilon_0} = Q_V$$



$Q = 3q$

Next lecture:

- The Helmholtz Theorem
- The electric potential, and the energy of the electric field
- Boundary conditions

Reading material:

- Jackson, Ch. 1
- See also Zangwill, Ch. 1 (Helmholtz Th.)