$$\begin{cases} Y'(t) = AY(t) \\ Y(0) = Y_0 \end{cases} \underset{\substack{\downarrow \\ \text{AULA} \\ \text{PASSADA}}}{\Rightarrow} \boxed{Y(t) = \underbrace{\left(\sum_{j=0}^{\infty} \frac{1}{j!}(tA)^j\right) Y_0}_{e^{tA}Y_0}} \Rightarrow Y(t) = e^{tA}Y_0$$

$$\frac{d}{dt} e^{tA} = Ae^{tA} \Leftrightarrow \lim_{h \to 0} \left\| \frac{e^{(t+h)A} - e^{tA}}{h} - Ae^{tA} \right\| = 0 \Leftrightarrow \frac{d}{dt}(e^{tA})_{ij} = (Ae^{tA})_{ij} \quad \forall i,j$$

AULA PASSADA VIMOS EXISTÊNCIA DE SOLUÇÕES

Proposição (Unicidade). Se $Y: \mathbb{R} \to \mathbb{R}^n$ $(\simeq M_{n \times 1}(\mathbb{R}))$ for solução de

$Y'(t) = AY(t)$, $Y(0) = Y_0$, então $Y(t) = e^{tA}Y_0$.

(Já vimos que $t \mapsto e^{tA}Y_0$ é solução. Vamos mostrar que é única)

Demo: Vamos definir $g(t) = e^{-tA}Y(t)$

Logo $\frac{dg}{dt}(t) = \frac{d}{dt}\left(e^{-tA}Y(t)\right) \underset{\substack{\downarrow \\ \text{AULA PASSADA: } \frac{d}{dt}(AB) = A'B + AB'.}}{=} \frac{d}{dt}\left(e^{-tA}\right)Y(t) + e^{-tA}\frac{dY}{dt}(t)$

$$\underset{\substack{\downarrow \\ \frac{d}{dt}(e^{-tA}) = -Ae^{-tA} \\ \frac{d}{dt}Y(t) = AY(t)}}{=} -Ae^{-tA}Y(t) + e^{-tA}AY(t) \underset{\substack{\downarrow \\ AB = BA, \\ e^A B = Be^A \\ (B \leftrightarrow A \quad A \leftrightarrow -tA)}}{=} -Ae^{-tA}Y(t) + Ae^{-tA}Y(t) = 0$$

Como $g'(t) = 0$, $\forall t \in \mathbb{R}$, então $g(t) =$ constante.

Logo $g(t) = g(0)$, $\forall t \in \mathbb{R}$. Mas $g(0) = \underbrace{e^{-0A}}_{e^0}\underbrace{Y(0)}_{Y_0} = Y_0$.

$$e^{-tA}Y(t) = Y_0$$

$$Y(t) = (e^{-tA})^{-1}Y_0$$

$$e^0 = I$$

$$e^0 = \sum_{j=0}^{\infty} \frac{1}{j!} 0^j = I, \quad 0^0 := I, \quad 0^j = 0, \; j \geq 1.$$

MAS $(e^{-tA})^{-1} = e^{tA}$ (VEREMOS A SEGUIR).

LOGO $Y(t) = (e^{-tA})^{-1} Y_0 = e^{tA} Y_0$ $\boxed{B}$

PROPOSIÇÃO: e^A É INVERSÍVEL E $(e^A)^{-1} = e^{-A}$.

DEMO: $g(t) = e^{-tA} e^{tA}$.

$$g'(t) = \underbrace{(e^{-tA})'}_{= -Ae^{-tA}} e^{tA} + e^{-tA} \underbrace{(e^{tA})'}_{= Ae^{tA}} = -Ae^{-tA}e^{tA} + e^{-tA}Ae^{tA}$$

$$= -Ae^{-tA}e^{tA} + Ae^{-tA}e^{tA} = 0.$$

LOGO g É CONSTANTE. PORTANTO PARA TODO $t \in \mathbb{R}$, TEMOS

$$g(t) = g(0) = e^{-0A}e^{0A} = e^0 e^0 = I^2 = I$$

EM PARTICULAR, PARA $t = 1$, TEMOS $g(1) = e^{-A}e^{A} = I$.

$$\Rightarrow (e^A)^{-1} = e^{-A}$$ $\boxed{B}$

PROPOSIÇÃO: SE $AB = BA$, ENTÃO $e^{A+B} = e^A e^B$.

OBS: JÁ TÍNHAMOS VISTO QUE $AB = BA$, ENTÃO $e^A e^B = e^B e^A$.

DEMO: VAMOS DEFINIR $g(t) = e^{-t(A+B)} e^{tA} e^{tB}$.

$$g'(t) = \frac{d}{dt}\left(e^{-t(A+B)} e^{tA} e^{tB}\right) = \underbrace{\frac{d}{dt}\left(e^{-t(A+B)}\right)}_{-(A+B)e^{-t(A+B)}} e^{tA} e^{tB} + e^{-t(A+B)} \underbrace{\frac{d}{dt}(e^{tA})}_{Ae^{tA}} e^{tB} + e^{-t(A+B)} e^{tA} \underbrace{\frac{d}{dt}(e^{tB})}_{Be^{tB}}$$

$$g'(t) = \underbrace{-(A+B)}e^{-t(A+B)}e^{tA}e^{tB} + e^{-t(A+B)}\underbrace{A}e^{tA}e^{tB} + e^{-t(A+B)}e^{tA}\underbrace{B}e^{-tB}.$$

COMO $AB = BA$, ENTÃO AS MATRIZES ACIMA COMUTAM.

$$g'(t) = [-A - B + A + B]\, e^{-t(A+B)}e^{tA}e^{tB} = 0$$

LOGO $\forall t \in \mathbb{R}$, $g(t) = g(0) = e^{-0(A+B)}e^{0A}e^{0B} = I^3 = I.$

$$\Rightarrow \quad e^{-t(A+B)}e^{tA}e^{tB} = I \Rightarrow e^{tA}e^{tB} = e^{t(A+B)}.$$

$$(e^{-t(A+B)})^{-1} = e^{t(A+B)}$$

PARA $t = 1$, TEMOS $e^{A+B} = e^{A}e^{B}$ ☒

COROLÁRIO: $e^{tA}e^{sA} = e^{(t+s)A}$, $\forall t, s \in \mathbb{R}$.

DEMO: NOTE QUE $(tA)(sA) = ts A^2 = (sA)(tA)$

LOGO $e^{tA}e^{sA} = e^{tA+sA} = e^{(t+s)A}$ ☒

EXEMPLO: $\exists\, A, B$ T.Q. $e^{A+B} \neq e^{A}e^{B}$.

$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$$\left.\begin{aligned} AB &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ BA &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{aligned}\right\} \begin{array}{l} AB \neq BA. \\ \text{NÃO ESTAMOS NAS CONDIÇÕES DA PROPOSIÇÃO!} \end{array}$$

$$e^{B} = \exp\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^1 & 0 \\ 0 & e^0 \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

$$e^A = \sum_{j=0}^{\infty} \frac{1}{j!} A^j \underset{\substack{\text{PEGAMOS} \\ \text{1º E 2º TERMO}}}{=} A^0 + \frac{1}{1!} A + 0 = I + A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$e^A e^B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e & 1 \\ 0 & 1 \end{pmatrix}$$

$$e^{A+B} \; ? \quad C := A + B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$C^2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = C$$

$$C^3 = C \underbrace{C^2}_{= C} = C^2 = C$$

$$\vdots$$

$$C^j = C, \quad \forall j \geq 1.$$

$$\exp C = \sum_{j=0}^{\infty} \frac{1}{j!} C^j = I + \sum_{j=1}^{\infty} \frac{1}{j!} C^j \underset{C^j = C}{=} I + C \sum_{j=1}^{\infty} \frac{1}{j!} \underset{\substack{\text{SOMAMOS} \\ \text{E SUBTRAÍMOS 1}}}{=} I + C\Big(\underbrace{\sum_{j=0}^{\infty} \frac{1}{j!}}_{\sum_{j=0}^{\infty} \frac{1}{j!} x^j \big|_{x=1} = e^1 = e} - 1\Big)$$

$$= I + C(e - 1)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (e-1)\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} e & e \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} e & e-1 \\ 0 & 1 \end{pmatrix}$$

$$\left.\begin{aligned} \exp(A+B) &= \begin{pmatrix} e & e-1 \\ 0 & 1 \end{pmatrix} \\ \exp(A)\exp(B) &= \begin{pmatrix} e & 1 \\ 0 & 1 \end{pmatrix} \end{aligned}\right\} \text{ SÃO DIFERENTES!}$$

APLICAÇÕES:

EX: $\begin{cases} y_1'(t) = \lambda y_1(t) + y_2(t) & y_1(0) = \bar{y}_1 \\ y_2'(t) = \lambda y_2(t) & y_2(0) = \bar{y}_2 \end{cases}$

1ª FORMA: $y_2' = \lambda y_2 \Rightarrow \boxed{y_2(t) = e^{\lambda t}\bar{y}_2}$

$$y_1'(t) = \lambda y_1(t) + e^{\lambda t}\bar{y}_2$$

$$\begin{aligned} \left(e^{-\lambda t} y_1(t)\right)' &= -\lambda e^{-\lambda t} y_1(t) + e^{-\lambda t} y_1'(t) \\ &= \cancel{-\lambda e^{-\lambda t} y_1(t)} + \cancel{e^{-\lambda t}\lambda y_1(t)} + \cancel{e^{-\lambda t}}\,\cancel{e^{\lambda t}}\,\bar{y}_2 \end{aligned}$$

$$\left(e^{-\lambda t} y_1(t)\right)' = \bar{y}_2 \Rightarrow \int_0^t \left(e^{-\lambda s} y_1(s)\right)' ds = \int_0^t \bar{y}_2\, ds$$

$$\Rightarrow e^{-\lambda t} y_1(t) - \overset{=1}{e^{-\lambda 0}}\, \overset{=\bar{y}_1}{y_1(0)} = \bar{y}_2\, t$$

$$\Rightarrow \boxed{y_1(t) = \bar{y}_1 e^{\lambda t} + \bar{y}_2 e^{\lambda t} t}$$

2ª FORMA: ESCREVEMOS DE FORMA MATRICIAL

$$\underbrace{\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix}}_{Y'} = \begin{pmatrix} \lambda y_1(t) + y_2(t) \\ \lambda y_2(t) \end{pmatrix} = \underbrace{\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}}_{Y} \qquad Y_0 = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}$$

$$Y(t) = \exp(tA)\, Y_0$$

VAMOS CALCULAR

$$\exp\begin{pmatrix} \lambda t & t \\ 0 & \lambda t \end{pmatrix} = \exp\left(\underbrace{\begin{pmatrix} \lambda t & 0 \\ 0 & \lambda t \end{pmatrix}}_{C := \lambda t I} + \underbrace{\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}}_{D} \right)$$

$$CD = \lambda t I D = \lambda t D I = D(\lambda t I) = DC$$

$$= \exp\begin{pmatrix} \lambda t & 0 \\ 0 & \lambda t \end{pmatrix} \exp\underbrace{\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}}_{D} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$$

$$D^2 = 0 \qquad \exp(D) = I + tD = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$\boxed{Y(t) = \begin{pmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}\begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} = \begin{pmatrix} e^{\lambda t}\bar{y}_1 + t e^{\lambda t}\bar{y}_2 \\ e^{\lambda t}\bar{y}_2 \end{pmatrix}}$$

Ex:

$$y_1'(t) = a y_1(t) + b y_2(t) \qquad y_1(0) = \bar{y}_1$$
$$y_2'(t) = -b y_1(t) + a y_2(t) \qquad y_2(0) = \bar{y}_2$$

$$\underbrace{\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix}}_{Y'(t)} = \begin{pmatrix} a y_1 + b y_2 \\ -b y_1 + a y_2 \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ -b & a \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}}_{Y(t)} \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \underbrace{\begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}}_{Y_0}$$

$$Y(t) = e^{tA} Y_0 = \exp\begin{pmatrix} ta & tb \\ -tb & ta \end{pmatrix} \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}$$

$$= \exp\left(\underbrace{\begin{pmatrix} ta & 0 \\ 0 & ta \end{pmatrix}}_{:=C} + \underbrace{\begin{pmatrix} 0 & tb \\ -tb & 0 \end{pmatrix}}_{:=D} \right) \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}$$

$CD = DC$, POIS C É MULTIPLO DA IDENTIDADE.

$$= \exp\begin{pmatrix} ta & 0 \\ 0 & ta \end{pmatrix} \exp\begin{pmatrix} 0 & tb \\ -tb & 0 \end{pmatrix} \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}$$

$$= \begin{pmatrix} e^{ta} & 0 \\ 0 & e^{ta} \end{pmatrix} \begin{pmatrix} \cos(tb) & \sin(tb) \\ -\sin(tb) & \cos(tb) \end{pmatrix} \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} = \boxed{\begin{pmatrix} e^{ta}\cos(tb)\bar{y}_1 + e^{ta}\sin(tb)\bar{y}_2 \\ -e^{ta}\sin(tb)\bar{y}_1 + e^{ta}\cos(tb)\bar{y}_2 \end{pmatrix}}$$

COMO CALCULAR e^A EM GERAL?

EM GERAL, PODEMOS ACHAR UM MATRIZ P T.Q.

$P^{-1}AP = B$ E B É UMA MATRIZ "SIMPLES."

VAMOS ESTUDAR COMO ACHAR P EM DETERMINADAS SITUAÇÕES

Proposição: Se $A, P \in M_{n\times n}(\mathbb{C})$, P inversível e $P^{-1}AP = B$, então $e^A = Pe^BP^{-1}$.

Demo: $P^{-1}AP = B \Rightarrow \underbrace{PP^{-1}}_{I}AP = PB \Rightarrow AP = PB \Rightarrow A\underbrace{PP^{-1}}_{I} = PBP^{-1}$

$\Rightarrow A = PBP^{-1}$

$$\exp(A) = \sum_{j=0}^{\infty} \frac{1}{j!}A^j = \sum_{j=0}^{\infty}\frac{1}{j!}(PBP^{-1})^j = \sum_{j=0}^{\infty}\frac{1}{j!}PB^jP^{-1} = P\left(\sum_{j=0}^{\infty}\frac{1}{j!}B^j\right)P^{-1}$$

$$= Pe^BP^{-1}$$

$$(PBP^{-1})^j = PB\underbrace{P^{-1}P}_{I}B\underbrace{P^{-1}P}_{I}BP^{-1}\cdots\underbrace{P}_{I}BP^{-1} = PB^jP^{-1}$$ $\boxtimes$

Matrizes Diagonalizáveis

Vamos estudar matrizes A para as quais $\exists P$ t.q. $P^{-1}AP = D$, D diagonal.

Como achar P?

Recordação de Álgebra Linear.

Definição: Seja $A \in M_{n\times n}(\mathbb{C})$.

i) Dizemos que $X \in \mathbb{R}^n \simeq M_{n\times 1}(\mathbb{C})$ é um autovetor de A se $X \neq 0$, e $AX = \lambda X$, $\lambda \in \mathbb{C}$.

ii) Dizemos que $\lambda \in \mathbb{C}$ é um autovalor de A se $AX = \lambda X$, para algum $X \neq 0$.

$$AX = \lambda X$$

λ AUTOVALOR

X AUTOVETOR.

EXEMPLO: $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\quad X_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \lambda_1 = 0, \; \lambda_2 = 1.$

$$AX_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \overset{\text{AUTOVALOR}}{0} \overset{\text{AUTOVETOR}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

$$AX_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \underset{\text{AUTOVALOR}}{1} \underset{\text{AUTOVETOR.}}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$