Sistemas Lineares

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases} \quad , \quad y(t) = (y_1(t), \dots, y_n(t)).$$

$$\begin{cases} y_1'(t) = f_1(t, y_1(t), \dots, y_n(t)) & y_1(t_0) = y_{10} \\ \vdots \quad\quad \vdots & \vdots \\ y_n'(t) = f_n(t, y_1(t), \dots, y_n(t)) & y_n(t_0) = y_{n0} \end{cases}$$

Geral.

LINEAR $\quad f_j(t, y_1(t), \dots, y_n(t)) = \sum_{k=1}^{n} a_{jk}(t)\, y_k(t)$ $\quad$ LINEAR EM $y_1, \dots, y_n$.

$$\begin{cases} y_1'(t) = a_{11}(t)\, y_1(t) + \dots + a_{1n}(t)\, y_n(t) \\ \vdots \\ y_n'(t) = a_{n1}(t)\, y_1(t) + \dots + a_{nn}(t)\, y_n(t) \end{cases}$$

VAMOS DEFINIR $A: \mathbb{R} \to \underbrace{\mathbb{R}^{n\times n}}_{\text{MATRIZES } n\times n}$ POR $(A(t))_{ij} = a_{ij}(t)$

$$A(t) = \begin{pmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \dots & a_{nn}(t) \end{pmatrix}$$

VAMOS DEFINIR $Y: \mathbb{R} \to \underbrace{\mathbb{R}^{n}}_{\simeq \text{ MATRIZ } n\times 1}$ POR $Y(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$.

EM TERMOS MATRICIAIS.

$$\frac{d}{dt}\begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \dots & a_{nn}(t) \end{pmatrix} \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$$

$$\boxed{\frac{d}{dt} Y(t) = A(t)\, Y(t)}$$

CONDIÇÕES INICIAIS $\quad Y(t_0) = \begin{pmatrix} y_{10} \\ \vdots \\ y_{n0} \end{pmatrix} := Y_0 \quad \exists!$ $\boxed{\begin{array}{l} \dfrac{dY}{dt}(t) = A(t)Y(t) \\ Y(t_0) = Y_0 \end{array}}$

VAMOS COMEÇAR CONSIDERANDO $A(t) = A$, $\forall t \in \mathbb{R}$. $\quad (a_{ij}(t) = a_{ij})$

<u>COEFICIENTES CONSTANTES</u>

$$\boxed{\begin{array}{l} \dfrac{dY}{dt}(t) = AY(t) \\ Y(0) = Y_0 \qquad \text{(PODERIA TAMBEM TER } t_0 \neq 0\text{).} \end{array}}$$

MOTIVAÇÃO:

1) APARECE EM MODELOS FÍSICOS.

EXEMPLOS (AULA PASSADA):

i) $\left.\begin{array}{l} m x''(t) = -k x(t) \\ x(0) = x_0 \\ x'(0) = v_0 \end{array}\right\} \Leftrightarrow \underbrace{\frac{d}{dt}\begin{pmatrix} q(t) \\ p(t) \end{pmatrix}}_{\frac{dY(t)}{dt}} = \underbrace{\begin{pmatrix} 0 & \frac{1}{m} \\ -k & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} q(t) \\ p(t) \end{pmatrix}}_{Y(t)} \quad \begin{pmatrix} q(0) \\ p(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ mv_0 \end{pmatrix}$

ii) $\left.\begin{array}{l} m x''(t) = -kx(t) - bx'(t) \\ x(0) = x_0 \\ x'(0) = v_0 \end{array}\right\} \Leftrightarrow \underbrace{\frac{d}{dt}\begin{pmatrix} q(t) \\ p(t) \end{pmatrix}}_{\frac{dY}{dt}} = \underbrace{\begin{pmatrix} 0 & \frac{1}{m} \\ -k & -\frac{b}{m} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} q(t) \\ p(t) \end{pmatrix}}_{Y}.$

iii) $y^{(m)}(t) + a_1 y^{(m-1)}(t) + \dots + a_m y(t) = 0$,

$y(0) = \bar{y}_1$

$\vdots$

$y^{(m-1)}(0) = \bar{y}_m$

TRUQUE $y_1 = y,\ y_2 = y',\ \dots,\ y_m = y^{(m-1)}$

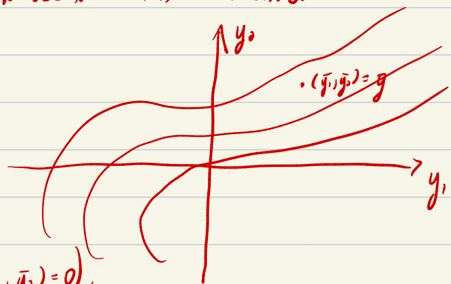
$$\left.\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_{m-1}' &= y_m \\ y_m' &= y^{(m)} = -a_1 y^{(m-1)} - \dots - a_m y \end{aligned}\right\} \Leftrightarrow \frac{d}{dt}\underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m-1} \\ y_m \end{pmatrix}}_{Y(t)} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ -a_m & -a_{m-1} & -a_{m-2} & \dots & -a_1 \end{pmatrix}}_{A} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m-1} \\ y_m \end{pmatrix}$$

$$Y(0) = \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_m \end{pmatrix}$$

2) LINEARIZAÇÃO É IMPORTANTE TAMBÉM EM PROBLEMAS NÃO LINEARES.

$$y'(t) = f(y(t)) \qquad \begin{cases} y_1'(t) = f_1(y_1(t), y_2(t)) \\ y_2'(t) = f_2(y_1(t), y_2(t)). \end{cases}$$

$$y = (y_1(t), y_2(t))$$



VAMOS SUPOR QUE $f(\bar{y}_1, \bar{y}_2) = 0 \ (f_1(\bar{y}_1, \bar{y}_2) = f_2(\bar{y}_1, \bar{y}_2) = 0)$.

COMO SE COMPORTAM AS SOLUÇÕES PRÓXIMAS AO PONTO $(\bar{y}_1, \bar{y}_2)$.

IDEIA: $f(y(t)) = f(y(t) - \bar{y} + \bar{y}) \underset{\text{TAYLOR}}{\approx} f(\bar{y}) + Df(\bar{y})(y(t) - \bar{y}) = Df(\bar{y})(y(t) - \bar{y})$.

LOGO $y'(t) \approx Df(\bar{y})(y(t) - \bar{y}) \Leftrightarrow \frac{d}{dt}\underbrace{(y(t) - \bar{y})}_{Y(t)} \approx \underbrace{Df(\bar{y})}_{A}\underbrace{(y(t) - \bar{y})}_{Y(t)}$

OBSERVAÇÃO: TAYLOR

$$(f_1(y_1,y_2), f_2(y_1,y_2)) \approx (f_1(\bar y_1,\bar y_2), f_2(\bar y_1,\bar y_2)) + \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(\bar y_1,\bar y_2) & \frac{\partial f_1}{\partial y_2}(\bar y_1,\bar y_2) \\ \frac{\partial f_2}{\partial y_1}(\bar y_1,\bar y_2) & \frac{\partial f_2}{\partial y_2}(\bar y_1,\bar y_2) \end{pmatrix} \begin{pmatrix} y_1-\bar y_1 \\ y_2-\bar y_2 \end{pmatrix}$$

$$f(y) \approx f(\bar y) + \underbrace{Df(\bar y)}_{\text{MATRIZ JACOBIANA}} (y-\bar y)$$

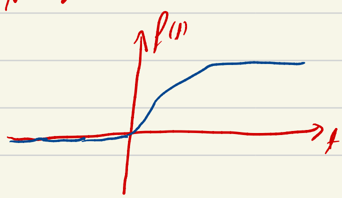
COMO RESOLVER SISTEMAS LINEARES?

RECORDAÇÃO: EXPANSÃO EM SÉRIES.

SE $f: \mathbb{R} \to \mathbb{R}$ DE CLASSE C^∞, ENTÃO $f(t) = \sum_{j=0}^{N} \frac{1}{j!} f^{(j)}(0)\, t^j + R_N(t)$.

ALGUMAS FUNÇÕES C^∞, TEMOS $R_N(t) \to 0$, $N \to \infty$.

CURIOSIDADE: $f: \mathbb{R} \to \mathbb{R}$ $\quad f(t) = \begin{cases} e^{-\frac{1}{t}}, & t > 0 \\ 0, & t \leq 0 \end{cases}$



$$f(t) = \sum_{j=0}^{N} \frac{1}{j!} \underbrace{f^{(j)}(0)}_{0}\, t^j + R_N(t) = R_N(t), \qquad \underset{t>0}{R_N(t) \not\to 0} \quad N \to \infty.$$

NO ENTANTO, FUNÇÕES COMO SENO, COSSENO, EXPONENCIAL, $R_N(t) \to 0$.

$$e^x = \sum_{j=0}^{\infty} \frac{1}{j!} x^j, \quad \cos(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} x^{2j}, \quad \sin(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} x^{2j+1}$$

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \qquad 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \qquad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

O problema linear + simples é em $n=1$.

$$\begin{cases} \frac{dy}{dt}(t) = a\,y(t) \\ y(0) = y_0 \end{cases}$$

$$\left(\int_0^t \frac{y'}{y} = \int_0^t a \Rightarrow \ln\left(\frac{y(t)}{y_0}\right) = at \qquad y(t) = e^{at} y_0 \right)$$

Aula passada $y(t) = e^{at} y_0$

$\updownarrow$ = (usamos Taylor).

$$\boxed{y(t) = \sum_{j=0}^{\infty} \frac{1}{j!} (at)^j y_0}$$

$$y(t) = e^{at} y_0 = y_0 e^{at}$$

$$Y(t) = e^{tA} Y_0 \neq \cancel{Y_0 e^{tA}}$$

Para dimensões maiores.

$$\begin{cases} \frac{dY}{dt}(t) = A\,Y(t) \\ Y(0) = Y_0 \end{cases}, \quad Y \in \mathbb{R}^n, \quad A \text{ é matriz } n \times n.$$

Por analogia, podemos adivinhar a solução

$$Y(t) = \sum_{j=0}^{\infty} \frac{1}{j!} t^j A^j Y_0$$

Isto nos motiva a definir exponencial de matriz

Dado uma matriz X n por n, definimos a matriz e^X ($n \times n$ também) por

$$e^X = \sum_{j=0}^{\infty} \frac{1}{j!} X^j, \quad X^0 := I.$$

Logo $\frac{dY}{dt}(t) = AY(t)$, $Y(0) = Y_0$ tem solução $Y(t) = e^{tA} Y_0$.

QUESTÕES: 1) O QUE QUER DIZER SOMATÓRIA DE MATRIZ?
2) SEMPRE CONVERGE?
3) DE FATO, $e^{tA}Y_0$ É SOLUÇÃO?

ANTES DE MAIS NADA, PARA FIXAR IDEIAS, VAMOS VER EXEMPLOS

EXEMPLOS DE EXPONENCIAL DE MATRIZES

1) $X = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. QUEM É e^X? $\left(e^X := \exp(X) \right)$

$$e^X = \sum_{j=0}^{\infty} \frac{1}{j!} X^j \quad , \quad X^0 := I.$$

$$X^2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}$$

$$X^3 = X^2 X = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^3 & 0 \\ 0 & \lambda_2^3 \end{pmatrix}$$

$\vdots$

$$X^j = \begin{pmatrix} \lambda_1^j & 0 \\ 0 & \lambda_2^j \end{pmatrix}$$

$$e^X = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \dots = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} + \begin{pmatrix} \frac{\lambda_1^2}{2!} & 0 \\ 0 & \frac{\lambda_2^2}{2!} \end{pmatrix} + \begin{pmatrix} \frac{\lambda_1^3}{3!} & 0 \\ 0 & \frac{\lambda_2^3}{3!} \end{pmatrix} + \dots = \begin{pmatrix} \sum_{j=0}^{\infty} \frac{1}{j!}\lambda_1^j & 0 \\ 0 & \sum_{j=0}^{\infty} \frac{1}{j!}\lambda_2^j \end{pmatrix} = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}$$

$$\exp \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix} /\!/$$

APLICAÇÃO $\begin{cases} y_1'(t) = \alpha y_1(t) & y_1(0) = \bar{y}_1 \\ y_2'(t) = \beta y_2(t) & y_2(0) = \bar{y}_2 \end{cases}$

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}}_{Y(t)} \qquad Y(t) = e^{tA} Y_0 = \begin{pmatrix} e^{t\alpha} & 0 \\ 0 & e^{t\beta} \end{pmatrix} \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}$$

A SOLUÇÃO É $\quad y_1(t) = e^{t\alpha} \bar{y}_1$

$\quad y_2(t) = e^{t\beta} \bar{y}_2$

2) $X = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \qquad \exp(X) = \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix}$

3) $X = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$

$X^0 := I$

$X = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$

$X^2 = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$X^3 = 0$

$\vdots$

$X^n = 0, \ \forall n \geq 2.$

$$e^X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

APLICAÇÃO: $\boxed{\begin{array}{l} y_1'(t) = \alpha y_2(t) \\ y_2'(t) = 0 \qquad (*) \\ y_1(0) = \bar{y}_1 \\ y_2(0) = \bar{y}_2 \end{array}}$

$$\underbrace{\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix}}_{Y'} = \underbrace{\begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}}_{Y}$$

$$X = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} \Rightarrow e^X = \begin{pmatrix} 1 & \lambda \\ 0 & \lambda \end{pmatrix}$$

$$\lambda = \alpha t$$

$$A = \begin{pmatrix} 0 & \alpha t \\ 0 & 0 \end{pmatrix} \to e^{tA} = \begin{pmatrix} 1 & \alpha t \\ 0 & 1 \end{pmatrix}$$

A SOLUÇÃO É $Y(t) = e^{tA} Y_0$

$$tA = \begin{pmatrix} 0 & t\alpha \\ 0 & 0 \end{pmatrix} \Rightarrow e^{tA} = \begin{pmatrix} 1 & t\alpha \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 1 & t\alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} = \begin{pmatrix} \bar{y}_1 + \alpha t \bar{y}_2 \\ \bar{y}_2 \end{pmatrix}$$

$$\boxed{\begin{aligned} y_1(t) &= \bar{y}_1 + \alpha t \bar{y}_2 \\ y_2(t) &= \bar{y}_2 \end{aligned}}$$

É SOLUÇÃO DE $(*)$

4) $X = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$.

$$X^2 = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} = \begin{pmatrix} -\lambda^2 & 0 \\ 0 & -\lambda^2 \end{pmatrix} = -\lambda^2 I$$

$$X^3 = X^2 X = -\lambda^2 I X = -\lambda^2 X = -\lambda^2 \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\lambda^3 \\ \lambda^3 & 0 \end{pmatrix} = -\lambda^2 X$$

$$X^4 = X^3 X = -\lambda^2 X X = -\lambda^2 X^2 = (-\lambda^2)(-\lambda^2 I) = \lambda^4 I.$$

$$X^5 = X^4 X = \lambda^4 I X = \lambda^4 X.$$

$$\boxed{\begin{aligned} X^{2j} &= (-1)^j \lambda^{2j} I \\ X^{2j+1} &= (-1)^j \lambda^{2j} X \end{aligned}}$$

$$e^X = \sum_{j=0}^{\infty} \frac{1}{j!} X^j = \underset{\text{PARES}}{\sum_{j=0}^{\infty} \frac{1}{(2j)!} X^{2j}} + \underset{\text{ÍMPARES}}{\sum_{j=0}^{\infty} \frac{1}{(2j+1)!} X^{2j+1}} = \underbrace{\left(\sum_{j=0}^{\infty} \frac{1}{(2j)!} (-1)^j \lambda^{2j}\right)}_{\cos(\lambda)} I + \underbrace{\left(\sum_{j=0}^{\infty} \frac{1}{(2j+1)!} (-1)^j \lambda^{2j}\right)}_{\frac{\sin(\lambda)}{\lambda}} X$$

$$\cos(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} x^{2j} \qquad \sin(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} x^{2j+1}$$

$$\cos(\lambda)\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{\operatorname{sen}(\lambda)}{\lambda}\begin{pmatrix}0&\lambda\\-\lambda&0\end{pmatrix}=\begin{pmatrix}\cos(\lambda)&\operatorname{sen}(\lambda)\\-\operatorname{sen}(\lambda)&\cos(\lambda)\end{pmatrix}$$

APLICAÇÃO: $y_1'(t)=\frac{1}{m}y_2(t)$ (SISTEMA MASSA-MOLA)

$y_2'(t)=-k\,y_1(t)$

$\tilde{y}_1=\sqrt{mk}\,y_1$ $\qquad\tilde{y}_1'(t)=\sqrt{mk}\,y_1'(t)=\sqrt{mk}\,\frac{1}{m}y_2(t)=\sqrt{\frac{k}{m}}\,\tilde{y}_2(t)$

$\tilde{y}_2=y_2$ $\qquad\tilde{y}_2'(t)=-k\,y_1(t)=-k\frac{1}{\sqrt{mk}}\tilde{y}_1(t)=-\sqrt{\frac{k}{m}}\,\tilde{y}_1(t)$

$\omega:=\sqrt{\frac{k}{m}}$ $\qquad\begin{pmatrix}\tilde{y}_1'(t)\\ \tilde{y}_2'(t)\end{pmatrix}=\underbrace{\begin{pmatrix}0&\omega\\-\omega&0\end{pmatrix}}_{A}\begin{pmatrix}\tilde{y}_1(t)\\ \tilde{y}_2(t)\end{pmatrix}$

$$\tilde{Y}(t)=e^{tA}\,\tilde{Y}(0)$$

$$\tilde{Y}(t)=\begin{pmatrix}\cos(\omega t)&\operatorname{sen}(\omega t)\\-\operatorname{sen}(\omega t)&\cos(\omega t)\end{pmatrix}\begin{pmatrix}\tilde{y}_1(0)\\ \tilde{y}_2(0)\end{pmatrix}$$

$$\boxed{\begin{aligned}\tilde{y}_1(t)&=\cos(\omega t)\,\tilde{y}_1(0)+\operatorname{sen}(\omega t)\,\tilde{y}_2(0)\\ \tilde{y}_2(t)&=-\operatorname{sen}(\omega t)\,\tilde{y}_1(0)+\cos(\omega t)\,\tilde{y}_2(0)\end{aligned}}$$

NAS COORDENADAS y_1, y_2

$$\boxed{\begin{aligned}y_1(t)&=\cos(t\omega)\,y_1(0)+\frac{1}{\sqrt{mk}}\operatorname{sen}(t\omega)\,y_2(0)\\ y_2(t)&=-\sqrt{mk}\operatorname{sen}(t\omega)\,y_1(0)+\cos(t\omega)\,y_2(0)\end{aligned}}$$