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THE GYRATOR, A NEW ELECTRIC NETWORK ELEMENT

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Summary

Besides the capacitor, the resistor, the inductor, and the ideal transformer a fifth, linear, constant, passive network element is conceivable which violates the reciprocity relation and which is defined by (10). We have denoted it by the name of "ideal gyrator". By its introduction the system of network elements is completed and network synthesis is much simplified. The gyrator can be realized by means of a medium consisting of particles carrying both permanent electric and permanent magnetic dipoles or by means of a gyromagnetic effect of a ferromagnetic medium.

Résumé

A côté de la capacité, de la résistance, de l'inductance et du transformateur idéal on peut concevoir un cinquième élément de circuit, linéaire, constant et passif, qui transgresse la relation de réciprocité et qui est défini par (10). Nous l'avons appelé „gyrateur idéal". En l'introduisant, le système d'éléments de circuit est complété et la synthèse de circuits s'en trouve très simplifiée. Le gyrateur peut être réalisé à l'aide d'un milieu composé de particles portant à la fois des dipôles électriques permanents et des dipôles magnétiques permanents ou au moyen d'un effet gyromagnétique d'un milieu ferromagnétique.

1. Introduction

The physical investigation of electric phenomena led to the creation of several devices which afterwards were used in engineering for the construction of electric networks.

From electrostatics resulted the *capacitor*, whose properties are described by

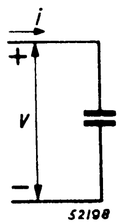

Fig. 1. Capacitor.

$$i = C\frac{\mathrm{d}v}{\mathrm{d}t}, \tag{1}$$

where i is the current and v the voltage. The coefficient C is called the *capacitance* and is always positive.

From the investigation of currents in conductors resulted the *resistor*, described by

$$v = Ri\,. \tag{2}$$

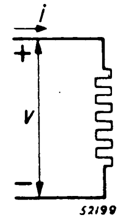

Fig. 2. Resistor.

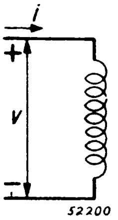

Fig. 3. Coil.

The coefficient R is called the *resistance* and is always positive.

From electromagnetism resulted the *inductor* or *coil*, described by

$$v = L \frac{di}{dt}. \tag{3}$$

The coefficient L is called the *self-inductance* and is always positive.

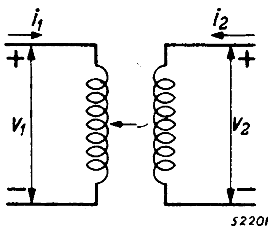

Fig. 4. Two coupled coils.

Two *coupled coils* are described by a set of two equations

$$\left.\begin{aligned} v_1 &= L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}; \\ v_2 &= M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}. \end{aligned}\right\} \tag{4}$$

The coefficient M is called the *mutual inductance* and is restricted by $M^2 \leqq L_1 L_2$.

Further development of physics has not resulted in the creation of other similar devices and coefficients. Engineering has taken these devices into use as network elements, and we shall begin by recalling broadly what has been done with them.

1·1. The network elements

As to the *network elements* themselves, many types of capacitor, resistor and inductor are constructed for various purposes, with different values, for different currents and voltages, fixed and variable. It was soon discovered that elements whose properties are given by (1), (2), and (3) must be considered as *ideal elements*, which in practice can only be approximated; every capacitor, for instance, has some losses and some inductance, and similar remarks hold for resistors and inductors.

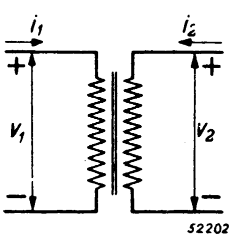

Fig. 5.
Ideal transformer.

From the system of two coupled coils a more radical development started. By making the coupling coefficient M^2/L_1L_2 as nearly as possible equal to 1, the transformer was created, whose properties in the ideal state are described by

$$\left.\begin{aligned} i_1 &= -u i_2, \\ v_2 &= u v_1. \end{aligned}\right\} \tag{5}$$

The coefficient u is called the *transformation ratio*.

For general considerations on networks we can better regard this *ideal transformer* as being the fourth network element rather than the general system of two coupled coils.

As to energy relations: an ideal capacitor and an ideal inductor can only store energy; an ideal resistor can only dissipate energy; an ideal transformer, however, can neither store nor dissipate energy but can only transfer energy, for, by (5), the absorbed power is always zero:

$$i_1 v_1 + i_2 v_2 = 0 \,. \tag{6}$$

1·2. The networks

By connecting the four types of network element together, *networks* can be constructed which give rise to several problems. In the first place, when given voltage or current sources are connected to a given network one may ask for the voltages and currents of the various branches of the network. These problems are denoted by the name of network *analysis* and may be considered generally solved.

For engineering purposes networks are often not given, but it may be asked to construct networks with specified properties. The problems arising therefrom are denoted by the name of network *synthesis*, and we shall go into them in some detail.

To solve these problems the networks are classified in several ways. In the first place the networks are classified by the number of terminals, which we shall suppose always to be combined in pairs such that the current entering the network by one terminal of a pair is always equal to the current leaving the network by the other terminal of the same pair. Thus we distinguish between two-poles, four-poles, six-poles, ..., $2n$-poles when a network has 1, 2, 3, ..., n terminal pairs, respectively. In the second place the networks are classified as resistanceless networks and as networks with resistance. In the third place the networks are classified, according to the order of the differential equation to which they give rise, as networks of zeroth order, first order, second order, and so on.

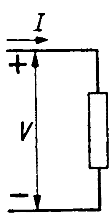

Fig. 6. Two-pole.

A two-pole is characterized by a relation between the voltage and the current of the terminals, which may be written in complex form, for instance, as

$$V = ZI \,. \tag{7}$$

The two-pole parameter Z is the impedance of the two-pole and is a function of frequency.

A four-pole is characterized by two relations between the voltages and the currents of the terminals, which may be written in complex form, for instance, as

$$\left.\begin{aligned} V_1 &= Z_{11}\, I_1 + Z_{12}\, I_2 \,, \\ V_2 &= Z_{21}\, I_1 + Z_{22}\, I_2 \,. \end{aligned}\right\} \tag{8}$$

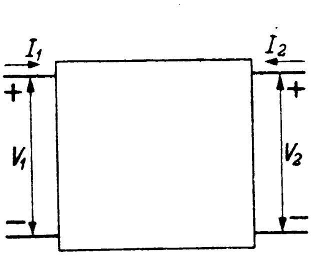

Fig. 7. Four-pole.

The four-pole parameters Z_{11}, Z_{12}, Z_{21}, Z_{22} are functions of frequency.

A $2n$-pole is thus characterized by n similar relations between the voltages and the currents of the terminals. The corresponding parameters of two $2n$-poles with different networks may be equal functions of frequency. Such $2n$-poles are said to be *equivalent.*

The problem of network synthesis may be stated as the problem of finding the necessary and sufficient conditions for a system of functions of frequency in order that these may represent the parameters of a $2n$-pole composed of the above-mentioned four types of network element, and further as the problem of indicating for each such set of parameters at least one way to construct a corresponding network. However, when it is required to construct a $2n$-pole with specified properties, as a rule its parameters cannot be considered as given but must first be found. Now, in general, the more complicated we are prepared to make the network, the better we shall be able to satisfy given requirements. As the classification of networks by their order is essentially a classification by their complexity, the problem of network synthesis may also be stated as the problem of finding the parameters of the most general $2n$-poles of a certain order that are realizable with the help of the four types of network element, and of indicating for each such set of parameters at least one way to construct a corresponding network [1]).

The synthesis problem is solved in both senses for resistanceless two-poles, for resistanceless four-poles, and for two-poles with resistance. For four-poles with resistance it is only solved in the first sense.

The synthesis of resistanceless two-poles was accomplished by Foster [2]). The result, for the zeroth to the fourth order, is given by fig. 8. For every order there are two types of two-pole. Fig. 8 contains all those two-pole networks of these orders composed of L's and C's for which the sum of the numbers of L's and C's is equal to the order of the two-pole. For the third order both types of two-pole can be realized by two such equivalent networks, for the fourth order by four.

The synthesis of resistanceless four-poles of a certain order was accomplished by the author [1]). Apart from order zero there are four types of four-pole of odd order and five types of even order.

The synthesis of two-poles with resistance was accomplished by Bruné [3]). For the 0th, 1st, 2nd, 3rd, and 4th orders there are, respectively, 1, 2, 5, 12, and 29 types of two-pole.

The synthesis of four-poles with resistance was studied by Gewertz [4]), who succeeded in finding necessary and sufficient conditions for the four-pole parameters. He did not, however, tackle the problem of finding all four-pole parameters of a certain order; so we might try to solve this problem and to construct the simplest corresponding networks. This problem is not an academic one, as four-poles are most extensively used in engineering. However, we have seen above that the synthesis both of resistanceless four-poles and of two-poles with resistance is more complicated than that of resistanceless two-poles, and we must therefore expect the synthesis of four-poles with resistance to be still more complicated.

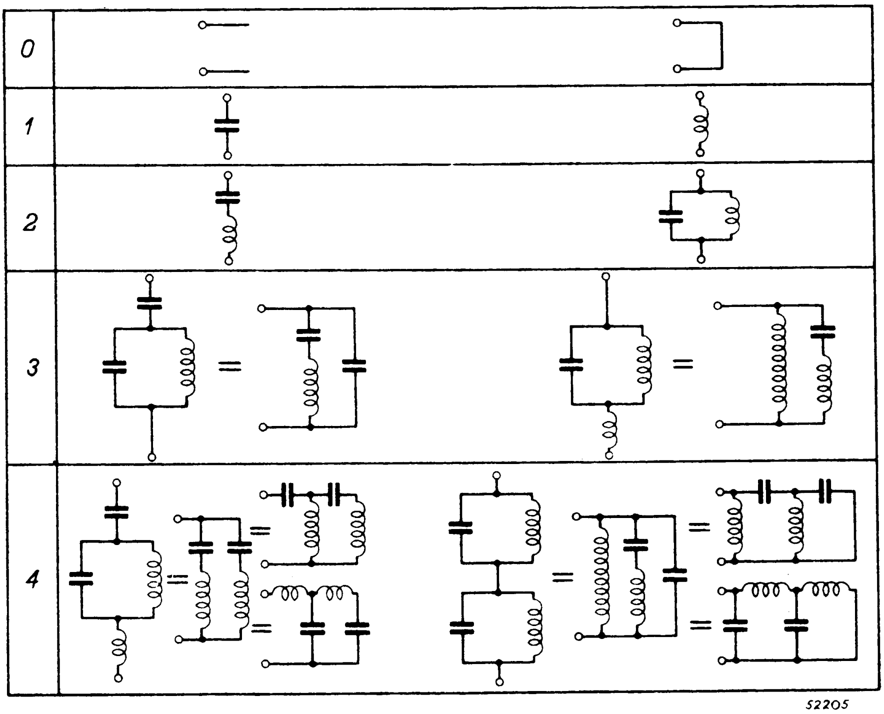

Fig. 8. Synthesis of resistanceless two-poles from the zeroth to the fourth order.

2. Statement of the problem — the ideal gyrator

For these reasons, before starting to try solving this problem, let us stop for a moment and look backwards. What are we doing as a matter of fact? Our four network elements have their origin in physics, and engineering has taken them for granted. It is worth while asking whether the system of four network elements is complete. The physicist has no reason to raise this question. He studies the phenomena of nature as they present themselves to him. For the technician, who wants to create useful systems, this question is of the utmost importance, and so we ask: "*Besides the four known network elements are other, similar elements conceivable?*"

At first this question seems rather vague: what is similar? — what is conceivable? However, the question appears to be a very definite one if we pay attention to the methods and results of network synthesis.

These are based upon the finding, first, of a number of general properties of $2n$-poles composed of the four known network elements, and then trying to construct any $2n$-pole possessing these properties by means of these elements. These properties are:

(a) the relation between the voltages and the currents of the terminals is formed by a system of ordinary *linear* differential equations, with
(b) *constant* coefficients;
(c) the $2n$-pole is *passive*, i.e., it can deliver no energy;
(d) the *reciprocity* relation.

This last property is expressed by the equality of those coefficients of the four-pole equations that lie symmetrically with respect to the principal diagonal if these equations express both voltages in both currents, or vice versa — the voltages and currents being taken positive according to fig. 7. Thus in (8) $Z_{21} = Z_{12}$, and also (4) shows an example of this form of the reciprocity relation. If, however, the current of one pair of terminals and the voltage of the other pair are expressed in the voltage of the first pair and the current of the second pair, the reciprocity relation is expressed by the opposite equality of the corresponding coefficients; for from (8) it follows that

$$\left.\begin{aligned} I_1 &= (1/Z_{11})\ V_1 \qquad\qquad - (Z_{12}/Z_{11})\ I_2\,, \\ V_2 &= (Z_{21}/Z_{11})\ V_1 + (Z_{22} - Z_{21}Z_{12}/Z_{11})\ I_2\,. \end{aligned}\right\} \qquad (9)$$

Equations (5) exhibit an example of this form of the reciprocity relation.

The above-mentioned investigations on network synthesis show that any two-pole and four-pole possessing these four properties can be realized by a network composed of the four elements. It seems very unlikely that this will be otherwise for $2n$-poles with $n > 2$. Therefore, if we restrict ourselves to $2n$-poles possessing the four mentioned properties the answer to our principal question is in the negative: no other similar network elements are conceivable. If we want to extend the possibilities we must drop one or more of the four properties.

If we drop the first property, the linearity, the principle of superposition will no longer hold and the systems will become much more complicated. Consequently, we want to keep this property.

If we drop the second property, the coefficients may become functions of the time, for instance periodic functions. Frequency conversion may then arise and this also complicates the system considerably, so that we want to keep the second property too.

If we drop the third property, the system must contain some source of energy. Amplifying valves, for instance, whose properties when dealing with small alternating voltages and currents are described by linear equations with constant coefficients, need D.C. sources of energy and thus constitute elements more complicated than those passive elements considered so far. Consequently, we want to keep also the third property.

As to the fourth property, the reciprocity, however, compared with the former three properties this is of much less importance. A $2n$-pole possessing the first three properties but lacking the fourth may very well be termed similar to the $2n$-poles composed of the four normal network elements. Therefore it seems worth while to investigate what it will lead to if we maintain the first three properties but drop the fourth.

We shall need some new type of network element to realize these $2n$-poles, in particular an element violating the reciprocity relation. This requirement has no significance for a two-pole element, such as C, R and L, so we must look for a new four-pole element. The simplest types of four-pole are the resistanceless ones of order zero. These are types for which $i_1v_1 + i_2v_2 = 0$, as this expresses that energy can neither be dissipated nor stored in the four-pole. The ideal transformer, whose equations are given by (5):

$$\left.\begin{aligned} i_1 &= -u\, i_2\,, \\ v_2 &= u\, v_1\,, \end{aligned}\right\} \tag{5}$$

is an example of such a four-pole satisfying the reciprocity relation. Another such four-pole, but violating the reciprocity relation, is described by

$$\left.\begin{aligned} v_1 &= -s\, i_2\,; \\ v_2 &= s\, i_1\,. \end{aligned}\right\} \tag{10}$$

In fact, from (10) it follows that $i_1v_1 + i_2v_2 = 0$ whereas the coefficients in (10) are not equal, as the reciprocity relation requires, but oppositely equal.

For reasons given below we shall denote such a four-pole by the name of *ideal gyrator*. We shall consider the ideal gyrator as a fifth network element.

3. Properties of the ideal gyrator

The ideal gyrator has the property that it "gyrates" a current into a voltage, and vice versa. The coefficient s, which has the dimension of a resistance, we shall call the *gyration resistance*, whilst $1/s$ we shall call the *gyration conductance*. In circuit diagrams we shall represent the ideal gyrator by the symbol of fig. 9.

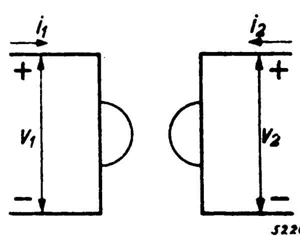

Fig. 9. Proposed symbol for the ideal gyrator.

The following properties of the ideal gyrator are easily derived from (10).

If we leave the secondary terminals open, $i_2 = 0$, the primary terminals are short-circuited, $v_1 = 0$, and vice versa. If we connect the secondary terminals by an inductance L, between the primary terminals we find a capacitance $C = L/s^2$. Conversely, if we connect the secondary terminals by a capacitance C, between the primary terminals we find an inductance $L = s^2C$. Generally, if we connect the secondary terminals by an impedance Z we find between the primary terminals an impedance s^2/Z. An impedance Z in series with or in parallel to the secondary terminals is equivalent to an impedance s^2/Z in parallel to respectively in series with the primary terminals, and vice versa (fig. 10).

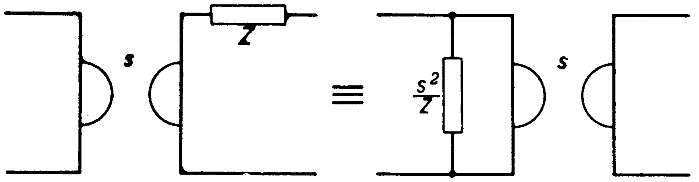

Fig. 10. An impedance in series with one pair of terminals of an ideal gyrator is equivalent to another impedance in parallel to the other pair of terminals.

Two ideal gyrators in cascade constitute an ideal transformer; an ideal gyrator and an ideal transformer in cascade constitute another ideal gyrator.

4. Networks with ideal gyrators

If ideal gyrators were available we could investigate anew all network problems arising in engineering, and since the extension of the system of four network elements to five is a relatively large one, we may expect considerably improved solutions to be possible for most network problems. As an example we may mention the system of two equal, critically coupled, tuned circuits such as are commonly used in the intermediate-frequency stages of superheterodyne radio receivers. If the circuits are coupled in an appropriate way by a gyrator and a resistance, the amplification per stage, compared with that obtained with inductively or capacitively coupled circuits under similar conditions, can be made larger by a factor $1 + \sqrt{2}$. Further details will be given in a subsequent paper.

Before investigating how an ideal gyrator might be realized or approximated, we shall first devote attention to the theory of networks that may contain ideal gyrators. As the reciprocity relation is of only subordinate importance in the methods of network analysis, this part of network theory is not much influenced by the introduction of the gyrator. Network synthesis, however, is influenced by it to a great extent and proves to be much simplified.

The synthesis of resistanceless two-poles does not change by the introduction of the gyrator. We may add that, by connecting them to an ideal gyrator, the two types of two-pole of a certain order are transformed one into the other, as mentioned above for the two-poles of the first order, the L and the C.

The synthesis of resistanceless four-poles is much simplified by the introduction of the gyrator. For every order there are two types, which can be transformed one into the other by connecting an ideal gyrator to any one of their terminal pairs.

As to the synthesis of two-poles with resistance, the addition of the gyrator does not create new possibilities, for the properties of a two-pole do not depend upon the reciprocity relation. However, the number of networks necessary for the realization of the most general two-pole of a certain order can be reduced to one by the use of ideal gyrators. We can construct this general type of two-pole of a certain order by taking any one of the two types of resistanceless four-pole of the same order and connecting any one of its two pairs of terminals by a resistance, whereby the four-pole changes into a two-pole.

The same simplicity is expected to hold for $2n$-poles. For every order of a resistanceless $2n$-pole there are probably two types, which can be transformed one into the other by connecting a gyrator to any one of their terminal pairs. For every order of a $2(n-1)$-pole with resistance there is probably one type, which can be constructed by taking any one of the two types of resistanceless $2n$-pole of the same order and connecting any one of its n pairs of terminals by a resistance.

These results of network synthesis will be fully dealt with in subsequent papers. They show how much network synthesis is simplified by the introduction of the ideal gyrator, and demonstrate that it is only by adding the ideal gyrator to the four known network elements that a complete set of elements arises.

5. Related problems in mechanics and electromechanics

Before turning to the problem of realizing the gyrator we shall give a short survey of related problems in mechanics and electromechanics.

Systems whose properties are described by a set of ordinary linear differential equations with constant coefficients were first studied in mechanics under the theory of small vibrations. A full account of these studies is given by Thomson and Tait in their "Treatise on natural philosophy". In these equations special terms may occur, called by those authors "gyroscopic" or "gyrostatic" terms, "because they occur when fly-wheels each given in a state of rapid rotation form part of the system by being mounted on frictionless bearings connected throug framework with other parts of the system; and because they occur when the motion considered is motion of the given system relatively to a rigid body revolving with a constrainedly constant angular velocity round a fixed axis" [5]).

As an example let us consider a system described by

$$\left.\begin{aligned} a_1 \frac{\mathrm{d}^2\psi_1}{\mathrm{d}t^2} + c_1\psi_1 - b\,\frac{\mathrm{d}\psi_2}{\mathrm{d}t} &= \Psi_1\,, \\ a_2 \frac{\mathrm{d}^2\psi_2}{\mathrm{d}t^2} + c_2\psi_2 + b\,\frac{\mathrm{d}\psi_1}{\mathrm{d}t} &= \Psi_2\,. \end{aligned}\right\} \qquad (11)$$

In these equations ψ_1 and ψ_2 denote two general coordinates of the system, Ψ_1 and Ψ_2 the corresponding general forces; a_1 and a_2 are the general masses, c_1 and c_2 the general stiffnesses. The terms $-b\,\mathrm{d}\psi_2/\mathrm{d}t$ and $b\,\mathrm{d}\psi_1/\mathrm{d}t$, whose coefficients are oppositely equal, are called gyrostatic terms.

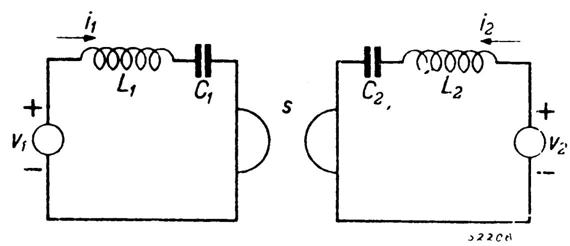

Fig. 11. Two circuits coupled by an ideal gyrator.

Let us compare with such a mechanical system the electrical system of fig. 11 in which two tuned circuits, L_1-C_1 and L_2-C_2, incorporating e.m.f.'s v_1 and v_2, respectively, are coupled by an ideal gyrator with gyration resistance s. The equations of this system are

$$\left.\begin{aligned} L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} + \frac{1}{C_1}\int i_1 \mathrm{d}t - s\,i_2 &= v_1\,, \\ L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t} + \frac{1}{C_2}\int i_2 \mathrm{d}t + s\,i_1 &= v_2\,. \end{aligned}\right\} \qquad (12)$$

If we put $i_1 = \mathrm{d}Q_1/\mathrm{d}t$ and $i_2 = \mathrm{d}Q_2/\mathrm{d}t$, we come to

$$\left.\begin{aligned} L_1 \frac{\mathrm{d}^2Q_1}{\mathrm{d}t^2} + \frac{1}{C_1}\,Q_1 - s\,\frac{\mathrm{d}Q_2}{\mathrm{d}t} &= v_1\,, \\ L_2 \frac{\mathrm{d}^2Q_2}{\mathrm{d}t^2} + \frac{1}{C_2}\,Q_2 + s\,\frac{\mathrm{d}Q_1}{\mathrm{d}t} &= v_2\,, \end{aligned}\right\} \qquad (13)$$

which is of quite the same form as (11). The gyrostatic terms of (11) correspond to the terms of (13) arising from the gyrator, and it is because of this correspondence that we have chosen the name of gyrator for the new network element. Recently Bloch [6]) devised an ideal gyroscopic coupler which is the exact mechanical equivalent of our gyrator.

Another field where gyrostatic terms arise is in the theory of electromechanical transducers. Poincaré, in his theory of the telephone receiver [7] [8]), deduced equations that in complex notation are of the form

$$\left.\begin{aligned} V &= Z_e\, I - A\, W\,, \\ F &= A\, I + Z_m\, W\,. \end{aligned}\right\} \tag{14}$$

V, F, I, W are, respectively, the (complex) voltage, force, current, velocity; Z_e is the (complex) electrical and Z_m the (complex) mechanical impedance; $-AW$ and AI are the coupling terms, in which A is real.

We can derive from (14) the equations of a corresponding mechanical system by replacing the voltage by a force and the current by a velocity. The coupling terms then become a pair of gyrostatic terms, and so in equation (14) they are also often called gyrostatic terms.

Equations of the form (14) occur in the theory of magnetic or moving-coil transducers. In the theory of electrostatic or piezo-electric transducers the coefficients of the coupling terms have equal signs [8]). Thus, if a magnetic and an electrostatic transducer are connected in cascade, so that by the former electrical oscillations are transduced into mechanical oscillations and by the latter the mechanical oscillations are again transduced into electrical oscillations, the resultant four-pole will violate the reciprocity relation of ordinary networks. This was explicitly stated by Jefferson [9]) and by Mc. Millan [10]).

So we could try to approximate the ideal gyrator by some special electromechanical apparatus. However, if we want to develop a gyrator that can also be used in high-frequency networks we must remember that high-frequency oscillations in electromechanical systems are only possible by the use of resonance (such as in piezo-quartz oscillators), and so we could at most arrive at a gyrator for a very narrow frequency range, contrary to the ideal gyrator, which has an unlimited frequency range. Our object must therefore be to approximate the ideal gyrator by other than electromechanical means.

6. The origin of the reciprocity relation

As the gyrator violates the reciprocity relation of ordinary networks we begin by recalling the origin of this relation, in the expectation that this will show us how to dispense with it *).

Let us first consider two insulated conductors 1 and 2 with potentials v_1 and v_2 and charges Q_1 and Q_2. The charges are linear functions of the potentials of the form

$$\left.\begin{aligned} Q_1 &= C_{11}\, v_1 + C_{12}\, v_2\,, \\ Q_2 &= C_{21}\, v_1 + C_{22}\, v_2\,. \end{aligned}\right\} \tag{15}$$

*) As the ideal gyrator has no dissipation we restrict our investigation to systems without dissipation. The reciprocity relation of systems with dissipation has been studied by Onsager [11]).

To change the charges by amounts $\mathrm{d}Q_1$ and $\mathrm{d}Q_2$ there must be supplied an energy

$$v_1\,\mathrm{d}Q_1 + v_2\,\mathrm{d}Q_2 = (C_{11}\,v_1 + C_{21}\,v_2)\,\mathrm{d}v_1 + (C_{12}\,v_1 + C_{22}\,v_2)\,\mathrm{d}v_2\,. \tag{16}$$

As this must be a total differential $\mathrm{d}U$,

$$\left.\begin{aligned} C_{11}\,v_1 + C_{21}\,v_2 &= \partial U/\partial v_1\,,\\ C_{12}\,v_1 + C_{22}\,v_2 &= \partial U/\partial v_2\,,\end{aligned}\right\} \tag{17}$$

from which we arrive at

$$C_{21} = \frac{\partial^2 U}{\partial v_1 \partial v_2} = \frac{\partial^2 U}{\partial v_2 \partial v_1} = C_{12}\,, \tag{18}$$

the reciprocity relation of electrostatics.

We may generalize this result (Ehrenfest [12]).

There are many more pairs of quantities in physics, such as Q and v, whose product is an energy; e.g., magnetic flux Φ and current i, mechanical displacement s and force K, turning angle φ and moment M, entropy S and temperature T. If a pair of such quantities characterizing the state of a system are slightly changed, the energy supplied to the system is equal to one of these quantities multiplied by the increment of the other quantity:

$$v\mathrm{d}Q,\ i\mathrm{d}\Phi,\ K\mathrm{d}s,\ M\mathrm{d}\varphi,\ T\mathrm{d}S.$$

The quantities are thereby divided into two classes: v, i, K, M, T belong to one class of quantities and Q, Φ, s, φ, S belong to the other class.

Suppose now, we have a system whose state is characterized by two pairs of such quantities, denoted by x_1, y_1 and x_2, y_2, and let a linear relationship exist between them such as

$$\left.\begin{aligned} y_1 &= a_{11}\,x_1 + a_{12}\,x_2\,,\\ y_2 &= a_{21}\,x_1 + a_{22}\,x_2\,.\end{aligned}\right\} \tag{19}$$

Then, if x_1 and x_2 belong to one class of quantities and therefore y_1 and y_2 to the other class, by a reasoning similar to that in the electrostatic case we may deduce $a_{12} = a_{21}$. On the other hand, if x_1 and x_2 belong to different classes of quantities we arrive at $a_{12} = -a_{21}$.

7. The violation of the reciprocity relation

The network equations, such as (8), are usually written with voltages and currents as variables. Now, these do not form a pair of quantities such as those mentioned above, for their product is not an energy but a power. We can change from the former to the latter pairs of quantities by differentiation with respect to the time. From the electrostatic equations (15) we thus come to

$$\left.\begin{aligned} i_1 &= C_{11}\frac{dv_1}{dt} + C_{12}\frac{dv_2}{dt}, \\ i_2 &= C_{21}\frac{dv_1}{dt} + C_{22}\frac{dv_2}{dt}. \end{aligned}\right\} \tag{20}$$

By comparing this with what has been written in section 2 we see that $C_{12} = C_{21}$ corresponds to the reciprocity relation of networks.

If we start from the equations of two coupled coils

$$\left.\begin{aligned} \Phi_1 &= L_{11}\, i_1 + L_{12}\, i_2, \\ \Phi_2 &= L_{21}\, i_1 + L_{22}\, i_2, \end{aligned}\right\} \tag{21}$$

where the i's are the currents and the Φ's are the fluxes through the coils, we arrive at $L_{12} = L_{21}$. By differentiation we come to

$$\left.\begin{aligned} v_1 &= L_{11}\frac{di_1}{dt} + L_{12}\frac{di_2}{dt}, \\ v_2 &= L_{21}\frac{di_1}{dt} + L_{22}\frac{di_2}{dt}. \end{aligned}\right\} \tag{22}$$

Here, too, the relation $L_{12} = L_{21}$ corresponds to the reciprocity relation of networks.

To come to a violation of the reciprocity relation of networks we must start from a system characterized by a pair of quantities Q_1, v_1 and a pair of quantities Φ_2, i_2. The equations will then take the form

$$\left.\begin{aligned} Q_1 &= C\, v_1 + A\, i_2, \\ \Phi_2 &= A\, v_1 + L\, i_2, \end{aligned}\right\} \tag{23}$$

where the two coefficients A will be equal. By differentiation we now arrive at

$$\left.\begin{aligned} i_1 &= C\,\frac{dv_1}{dt} + A\,\frac{di_2}{dt}, \\ v_2 &= A\,\frac{dv_1}{dt} + L\,\frac{di_2}{dt}, \end{aligned}\right\} \tag{24}$$

which, according to section 2, violates the reciprocity relation of networks if $A \neq 0$.

The equations (24) bear much resemblance to the equations (4). The energy of the system is equal to

$$U = \int i_1 v_1\, dt + \int i_2 v_2\, dt = \tfrac{1}{2}\, C\, v_1^2 + A\, v_1 i_2 + \tfrac{1}{2}\, L\, i_2^2, \tag{25}$$

if we take this energy as zero when $v_1 = 0$ and $i_2 = 0$. As this energy can never become negative in a passive system, C and L must be positive and A is restricted by $A^2 \leqq CL$.

In the same way as we can approximate the ideal transformer from the system described by (4), we can approximate the ideal gyrator from a system described by (24) by making the "coupling coefficient" A^2/CL as nearly as possible equal to 1.

8. The realization of the gyrator

Our object, therefore, is to devise a system described by (23) with a coupling coefficient as nearly as possible equal to 1. To investigate how this can be done we first show how such a system could be constructed if a medium were available characterized by relations between the field vectors of the type

$$\left.\begin{aligned} \mathbf{D} &= \varepsilon\,\mathbf{E} + \gamma\,\mathbf{H}\,, \\ \mathbf{B} &= \gamma\,\mathbf{E} + \mu\,\mathbf{H}\,, \end{aligned}\right\} \qquad (26)$$

and with $\gamma^2/\varepsilon\mu$ nearly equal to 1.

Let us consider the system of fig. 12. This consists of two flat parallel electrodes the space between which is filled with a medium described by (26). Besides, there is a yoke of magnetic material, having a very large permeability, on which a coil is wound. The electrodes constitute the terminals 1 of the system and the coil terminals constitute the terminals 2.

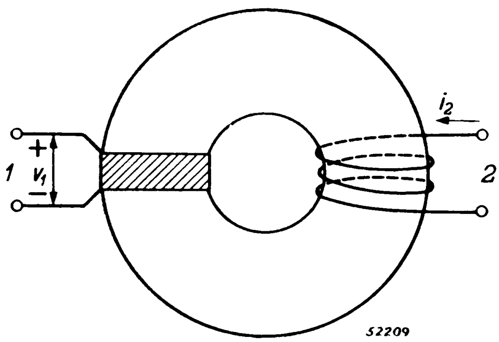

Fig. 12. A construction of the gyrator.

Let the area of the electrodes and of the cross-section of the yoke be S, then the charge Q_1 on the electrodes will be *)

$$Q_1 = S\,D = \varepsilon S\,E + \gamma S\,H \qquad (27)$$

and the flux Φ_2 through the coil will be

$$\Phi_2 = nS\,B = \gamma nS\,E + \mu nS\,H\,, \qquad (28)$$

where n is the number of turns of the coil.

The field vectors in the medium will all be perpendicular to the electrodes.

Let the distance between the electrodes be l, then the voltage v_1 between the electrodes will be

$$v_1 = lE \qquad (29)$$

and the current i_2 through the coil will be

$$i_2 = H\,l/n\,. \qquad (30)$$

*) We use the rationalized Giorgi system of units.

Putting (29) and (30) in (27) and (28) we get

$$\left.\begin{aligned} Q_1 &= \frac{\varepsilon S}{l} \cdot v_1 + \frac{\gamma n S}{l} \cdot i_2 , \\ \Phi_2 &= \frac{\gamma n S}{l} \cdot v_1 + \frac{\mu n^2 S}{l} \cdot i_2 . \end{aligned}\right\} \tag{31}$$

This corresponds to (23) and shows that $A^2/CL = \gamma^2/\varepsilon\mu$. Therefore, if $\gamma^2/\varepsilon\mu$ is nearly equal to 1, the same holds for A^2/CL.

8·1. The medium

To investigate media as described by (26) we first introduce the electric polarization **P** and the magnetic polarization **J** by putting

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \quad \text{and} \quad \mathbf{B} = \mu_0 \mathbf{H} + \mathbf{J}$$

and further putting $\varepsilon - \varepsilon_0 = \varkappa$ and $\mu - \mu_0 = \chi$.
By substituting this in (26) we get

$$\left.\begin{aligned} \mathbf{P} &= \varkappa \mathbf{E} + \gamma \mathbf{H} , \\ \mathbf{J} &= \gamma \mathbf{E} + \chi \mathbf{H} . \end{aligned}\right\} \tag{32}$$

Now, what does this represent? The coefficients $\varkappa$ and χ are ordinary electric and magnetic susceptibilities, the coefficient γ is something new. The term $\gamma\mathbf{E}$ expresses that when the medium is exposed to an electric field it will become magnetically polarized; the term $\gamma\mathbf{H}$ expresses that when the medium is exposed to a magnetic field it will become electrically polarized.

How can we get a medium with such properties? When a material medium is placed in a field, polarization of the medium may result from two different causes. Firstly, the elements of the medium, e.g. the molecules or the atoms, can acquire a dipole moment by the action of the field, i.e., they are polarizable; and, secondly, the elements of the medium can be permanently polarized and are oriented by the action of the field.

It is difficult to imagine how magnetic and electric polarizations due to the first cause can be coupled to each other in such a way that a medium with a coefficient γ results. When we consider polarization due to the second cause, however, we see immediately that our purpose can be attained if the elements of the medium bear both permanent electric and permanent magnetic dipoles, and if in all elements these dipoles are parallel or are anti-parallel. If we place such a medium in an electric field the electric dipoles will be oriented and thus the magnetic dipoles will be oriented at the same time, and the same will take place if the medium is placed in a magnetic field. The magnetic and the electric polarizations will then be proportional to each other. This means that **P**/**J** will be independent of **E** and **H**, from which we deduce by (23) that $\varkappa/\gamma = \gamma/\chi$ or $\gamma^2/\varkappa\chi = 1$. If, besides bearing permanent dipoles, the elements of the medium have some polarizability, $\varkappa$ and χ will in general become larger, so that $\gamma^2/\varkappa\chi \leqq 1$.

As mentioned above, $\gamma^2/\varepsilon\mu$ should be nearly equal to 1. Now

$$\frac{\gamma^2}{\varepsilon\mu} = \frac{\gamma^2}{(\varepsilon_0 + \varkappa)(\mu_0 + \chi)} \leqq \frac{\gamma^2}{\varkappa\chi} \leqq 1.$$

So we see that to reach this end we must make $\varkappa \gg \varepsilon_0$ and $\chi \gg \mu_0$, whilst these large susceptibilities must exclusively result from the orientation of permanent dipoles.

Media with large permanent, orientable, electric dipoles exist in abundance, e.g. water, which maintains its great electric susceptibility up to very high frequencies.

Media with permanent, orientable, magnetic dipoles are the paramagnetic media, whose magnetic susceptibility, however, is small compared with μ_0 and thus does not comply with the requirements just mentioned. In ferromagnetic media the susceptibility is due to the orientation of the electron spins, but it is hard to imagine how these could be directly coupled to a permanent electric dipole. We can, however, imagine a medium consisting of small ferromagnetic particles with permanent moments, small permanent magnets, suspended in some appropriate liquid.

We made some preliminary experiments in this direction by grinding some magnet steel and sifting out the smallest particles, which under a microscope were found to have dimensions of about one micron. From this a stable suspension was made which was put into a test-tube carrying a coil, and the self-inductance of the coil was measured. Then the suspension was magnetized by pouring it between the poles of a permanent magnet, after which it was again put into the test-tube. The self-inductance was increased by a factor 1·3. However, it is rather difficult to achieve a suspension that is stable during a long time, and one calls for very small magnetized particles.

9. Another realization of the gyrator

As experiments in this direction seem to be rather difficult it was considered worth while first to investigate whether there are conceivable fundamentally different possibilities of realizing the gyrator. Since the ideal transformer shows some analogy to the ideal gyrator we may include this in these investigations.

From the point of view of network theory the ideal transformer and the ideal gyrator are network elements defined by the equations (5) and (10), irrespective of any method of realizing them physically. To investigate how this could be done we first turn our attention to the terminals. Any terminal pair consists of two terminals situated closely together, so that even at very high frequencies we can speak of the voltage between the terminals. From such a terminal pair there start two wires of well-conducting material connecting the terminals with the element proper. Within the element these wires may be separated or connected. If the wires are separated we can deform the ends of them into two parallel electrodes. If we supply a voltage to the terminals, an electric charge Q will flow to the electrodes, and thus a current $i = \mathrm{d}Q/\mathrm{d}t$ will arise. If the wires are connected we can deform them into a coil. If we supply a current to the terminals, a magnetic flux Φ will flow through the coil, and thus a voltage $v = \mathrm{d}\Phi/\mathrm{d}t$ will arise. Therefore, we shall refer to the first type of terminal pair as the electric type and to the second as the magnetic type. According to the types of terminal pair of a four-pole element we shall refer to it as a double-electric, a double-magnetic, or an electromagnetic type. We could try to realize the transformer and the gyrator by means of each of these three types.

9·1. The general medium

Between the electrodes of an electric pair of terminals and within the coil of a magnetic pair of terminals we may now introduce some material medium. The type of four-pole that then results will depend to a large extent upon the properties of the medium. Therefore we ask what are the most general, linear relations by which the electric and magnetic properties of a medium can be phenomenologically described. These relations will consist of relations between the electric and magnetic polarizations **P** and **J** and the electric and magnetic field strengths **E** and **H** through which these polarizations may arise.

An example of relations of this kind is presented by the equations (32), but these do not constitute the most general form these relations can take. In the first place the medium may be anisotropic. For such a medium the equations must take a form wherein the three components of **P** and the three components of **J** along the directions of some rectangular system of coordinates are expressed in the three components of **E** and the three components of **H** along the same directions. In the second place we can suppose all components to vary sinusoidally with time with the same angular frequency ω. Then the components can be represented by complex quantities whereby the amplitude and the phase of each component can be expressed. We thus arrive at a system of six linear, homogeneous equations between twelve complex quantities and with complex coefficients.

As the ideal transformer and the ideal gyrator have no dissipation, we shall confine our conderations to media without dissipation. For such a medium some properties of the above-mentioned coefficients are deduced in the appendix. The equations for a medium without dissipation may thus be written in full as:

$$\begin{aligned}
P_x &= \varkappa_{xx}E_x+(\varkappa_{xy}+j\lambda_{xy})E_y+(\varkappa_{zx}+j\lambda_{zx})E_z+(\gamma_{xx}+j\delta_{xx})H_x+(\gamma_{xy}+j\delta_{xy})H_y+(\gamma_{xz}+j\delta_{xz})H_z,\\
P_y &= (\varkappa_{xy}-j\lambda_{xy})E_x+\varkappa_{yy}E_y+(\varkappa_{yz}+j\lambda_{yz})E_z+(\gamma_{yx}+j\delta_{yx})H_x+(\gamma_{yy}+j\delta_{yy})H_y+(\gamma_{yz}+j\delta_{yz})H_z,\\
P_z &= (\varkappa_{zx}-j\lambda_{zx})E_x+(\varkappa_{yz}-j\lambda_{yz})E_y+\varkappa_{zz}E_z+(\gamma_{zx}+j\delta_{zx})H_x+(\gamma_{zy}+j\delta_{zy})H_y+(\gamma_{zz}+j\delta_{zz})H_z,\\
J_x &= (\gamma_{xx}-j\delta_{xx})E_x+(\gamma_{yx}-j\delta_{yx})E_y+(\gamma_{zx}-j\delta_{zx})E_z+\chi_{xx}H_x+(\chi_{xy}+j\zeta_{xy})H_y+(\chi_{zx}+j\zeta_{zx})H_z,\\
J_y &= (\gamma_{xy}-j\delta_{xy})E_x+(\gamma_{yy}-j\delta_{yy})E_y+(\gamma_{zy}-j\delta_{zy})E_z+(\chi_{xy}-j\zeta_{xy})H_x+\chi_{yy}H_y+(\chi_{yz}+j\zeta_{yz})H_z,\\
J_z &= (\gamma_{xz}-j\delta_{xz})E_x+(\gamma_{yz}-j\delta_{yz})E_y+(\gamma_{zz}-j\delta_{zz})E_z+(\chi_{zx}-j\zeta_{zx})H_x+(\chi_{yz}-j\zeta_{yz})H_y+\chi_{zz}H_z.
\end{aligned} \tag{33}$$

The coefficients in the principal diagonal are real; the coefficients lying symmetrically with respect to this diagonal are conjugate complex.

The coefficients consist of six groups of quantities, denoted by $\varkappa$, λ, γ, δ, χ, ζ, representing six different properties of the medium. If the terminal pairs of a four-pole are coupled to each other by means of one of these properties of the medium constituting the four-pole, this four-pole will be of a certain type.

In the first place this four-pole will or will not satisfy the reciprocity relation. To investigate this we differentiate the equations (33) with respect to t. The left-hand sides will then become dP_x/dt, etc. and dJ_x/dt, etc., and the right-hand sides we may multiply by $j\omega$. Now dP/dt is a part of a current, dJ/dt is a part of a voltage, E is a part of a voltage, and H is a part of a current. So, bearing in mind what has been said in section 2 about the way the reciprocity relation is expressed by equality or opposite equality of certain four-pole coefficients, we see that those four-poles of which the terminal pairs are coupled to each other by means of the property of the medium represented by $\varkappa$, δ, or χ, respectively by λ, γ, or ζ, will, respectively will not, satisfy the reciprocity relation of networks.

Furthermore, P and E are related to electric pairs of terminals and J and H to magnetic pairs. Therefore, if a transformer or a gyrator could be realized by coupling two terminal pairs by means of one of the above-mentioned six properties of a medium, coupling by

$\varkappa$ could lead only to a double-electric transformer,
λ to a double-electric gyrator,
γ to an electromagnetic gyrator,
δ to an electromagnetic transformer,
χ to a double-magnetic transformer,
ζ to a double-magnetic gyrator.

Thus we see that both for the transformer and for the gyrator there are three fundamentally different ways in which we could try to realize them.

Let us examine the properties of the medium more closely. The $\varkappa$'s represent the electric susceptibility, which in the general, anisotropic case has the form of a symmetric tensor, characterized by six components. Likewise, the χ's represent the magnetic susceptibility. The γ's are the generalizations of the coefficient γ discussed in section 8·1, where we pointed out the experimental difficulties to be overcome in realizing a medium with γ-properties. The λ's and ζ's show two other ways of realizing the gyrator.

As previously mentioned, the λ's can only lead to a double-electric gyrator. Double-electric elements depend on the properties of the electric field. Now the electric field, because of curl $\mathbf{E} = 0$, has limited possibilities: a voltage step-up is impossible in a dielectric. As a consequence the only ideal transformer realizable by means of a dielectric is a transformer with a one-to-one ratio, which hardly deserves the name of transformer. As similar limitations may be expected when trying to realize an ideal gyrator by means of a dielectric with a λ-property, we shall not go into the questions what this λ-property represents and how a medium with such a property could be achieved.

Thus we are left with the problem of investigating the ζ-property of a medium and the realization of a gyrator by means of it. To investigate the ζ-property we suppose the medium to be characterized by χ's and ζ's only, the $\varkappa$'s, λ's, γ's, and δ's being zero. The equations (33) then reduce to three, expressing the components of $\mathbf{J}$ in the components of $\mathbf{H}$. Instead of studying these equations it is simpler to study the inverse equations by which the components of $\mathbf{H}$ are expressed in the components of $\mathbf{J}$. If in these equations we replace j by $(1/\omega)\mathrm{d}/\mathrm{d}t$ we arrive at equations of the form

$$\left.\begin{aligned} H_x &= \xi_{xx} J_x + (\xi_{xy} - \eta_z\,\mathrm{d}/\mathrm{d}t)\,J_y + (\xi_{zx} + \eta_y\,\mathrm{d}/\mathrm{d}t)\,J_z, \\ H_y &= (\xi_{xy} + \eta_z\,\mathrm{d}/\mathrm{d}t)\,J_x + \xi_{yy} J_y + (\xi_{yz} - \eta_x\,\mathrm{d}/\mathrm{d}t)\,J_z, \\ H_z &= (\xi_{zx} - \eta_y\,\mathrm{d}/\mathrm{d}t)\,J_x + (\xi_{yz} + \eta_x\,\mathrm{d}/\mathrm{d}t)\,J_y + \xi_{zz} J_z. \end{aligned}\right\} \quad (34)$$

The ξ's constitute an inverse susceptibility. It is the terms with η in which we are particularly interested. The equations show that these terms give a contribution to the field that may be written in vector notation as

$$\mathbf{H}' = \vec{\eta} \times \frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t}, \quad (35)$$

where $\times$ denotes the vector product and the components of $\vec{\eta}$ are η_x, η_y, η_z.

This shows that the η's describe some transverse effect in the medium: a rate of change of the magnetic polarization of the medium, $d\mathbf{J}/dt$, has to give rise to a component of the magnetic field, $\mathbf{H}'$, at right angles to it. Such an effect can be expected in a ferromagnetic medium when this is magnetized to saturation in a certain direction, e.g. by placing it in a sufficiently strong constant magnetic field. The spins of the electrons contributing to the ferromagnetism will then all be parallel to one another. When the magnetization changes in a direction perpendicular to the direction of saturation the spins will turn. As the electrons carry not only a magnetic moment but also an angular momentum, there is a tendency for the spins to deviate in a transverse direction, perpendicular to the direction of the saturation and the direction in which we want them to turn. This tendency is equivalent to the action of a magnetic-field component in that direction and can thus be balanced by a component in the opposite direction. This effect is related to the various known types of gyromagnetic effect. In a subsequent paper we shall show how this effect enables us to realize the gyrator.

10. Appendix

The mean dissipated energy in a medium per unit of time and per unit of volume is equal to the real part of

$$\left(\mathbf{E}^* \cdot \frac{d\mathbf{D}}{dt} + \mathbf{H}^* \cdot \frac{d\mathbf{B}}{dt}\right),$$

where the asterisk denotes the conjugate complex value and the dot the scalar product. For a medium without dissipation this energy must be zero, so we come to

$$\mathrm{Re}\,(\mathbf{E}^* \cdot j\omega\,\mathbf{D} + \mathbf{H}^* \cdot j\omega\mathbf{B}) = 0.$$

Substituting $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$ and $\mathbf{B} = \mu_0\mathbf{H} + \mathbf{J}$, we get

$$\mathrm{Im}\,(\mathbf{E}^* \cdot \mathbf{P} + \mathbf{H}^* \cdot \mathbf{J}) = 0. \tag{36}$$

Let us suppose first that $\mathbf{H} = 0$ and that $\mathbf{E}$ has only a component E_x. Then (36) becomes

$$\mathrm{Im}\,(E_x^*\, P_x) = 0. \tag{37}$$

In this case the equation for P_x reduces to an equation of the form

$$P_x = (\varkappa_{xx} + j\lambda_{xx})\, E_x\,. \tag{38}$$

Substituting this in (37) we arrive at $\lambda_{xx} = 0$. Thus the coefficient of E_x in the equation for P_x is real. By similar reasoning we may show that also the coefficient of E_y in the equation for P_y, etc., and the coefficient of H_x in the equation for J_x, etc., are real.

Let us now suppose that **H** = 0 and that **E** has only components E_x and E_y. Then (36) becomes

$$\mathrm{Im}\,(E_x^*\,P_x + E_y^*\,P_y) = 0. \qquad (39)$$

In this case the equations for P_x and P_y reduce to equations of the form

$$\left.\begin{aligned} P_x &= \varkappa_{xx}\,E_x + (\varkappa_{xy} + j\lambda_{xy})\,E_y\,, \\ P_y &= (\varkappa_{yx} + j\lambda_{yx})\,E_x + \varkappa_{yy}\,E_y\,, \end{aligned}\right\} \qquad (40)$$

the coefficient of E_x in the first equation and the coefficient of E_y in the second equation being real on account of what has just been proved. Substituting this in (39) we get

$$\mathrm{Im}\,\{(\varkappa_{xy} + j\lambda_{xy})\,E_x^*\,E_y + (\varkappa_{yx} + j\lambda_{yx})\,E_x\,E_y^*\} = 0.$$

As $\mathrm{Re}(E_xE_y^*) = \mathrm{Re}(E_x^*E_y)$ and $\mathrm{Im}(E_xE_y^*) = -\,\mathrm{Im}(E_x^*E_y)$, we get

$$(\varkappa_{xy} - \varkappa_{yx})\,\mathrm{Im}(E_x^*E_y) + j(\lambda_{xy} + \lambda_{yx})\,\mathrm{Re}(E_x^*E_y) = 0.$$

As this must be true for every value of $E_x^*E_y$, we finally get

$$\varkappa_{yx} = \varkappa_{xy} \text{ and } \lambda_{yx} = -\,\lambda_{xy}.$$

Thus the coefficient of E_y in the equation for P_x and the coefficient of E_x in the equation for P_y are conjugate complex. By similar reasoning we may show that many other analogous pairs of coefficients are conjugate complex.

Eindhoven, September 1947

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Errata

The author has provided the following list of corrections:

1. The last sentence of the next-to-last paragraph of Section 4 should read: "For every order of a $2n$-pole with resistance there is one type, which can be constructed by taking any one of the two types of resistanceless $4n$-poles of the same order and connecting any n of its $2n$ pairs of terminals by a resistance."

2. The last sentence of Section 9 • 1 mentions a subsequent paper. That paper has never been written.