03/12/2020

Reference: N.N. Lebedev, "Special functions and Applications . Dover . 1972

The Gamma Function:

Factorial Function: $\Pi(x) = \int_0^{\infty} t^x e^{-t}\, dt$

$x \in \mathbb{R} \mid x > -1$

$x \neq -1$ for it to be defined at $t=0$.

On integrating it by parts, we get:

$$\int_0^{\infty} t^{x+1} e^{-t}\, dt = \underbrace{-t^{x+1} e^{-t} \Big|_0^{\infty}}_{0} + \int_0^{\infty} (x+1) t^x e^{-t}\, dt$$

whence it comes that:

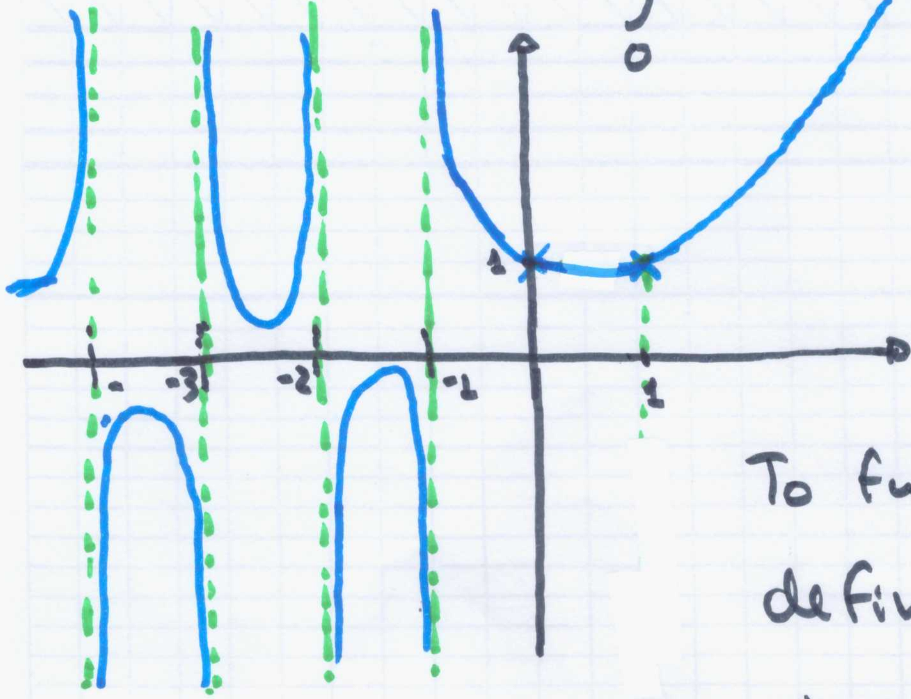
$$\Pi(x+1) = (x+1)\Pi(x) \qquad x \in \mathbb{R} \mid x > -1$$

For $x = n \in \mathbb{N} \Rightarrow \begin{cases} \Pi(n+1) = (n+1)\Pi(n) \\ \quad = (n+1)n\Pi(n-1) \\ \Pi(n+1) = (n+1)! \end{cases}$

$\Pi(1) = 1 = \Pi(1+0) = 1\,\Pi(0) \Rightarrow \Pi(0) = 1 = 0!$

Gamma Function:

$$\Gamma(x) = \Pi(x-1) = \int_0^\infty t^{x-1} e^{-t}\, dt \qquad x>0, \quad x \in \mathbb{R}$$

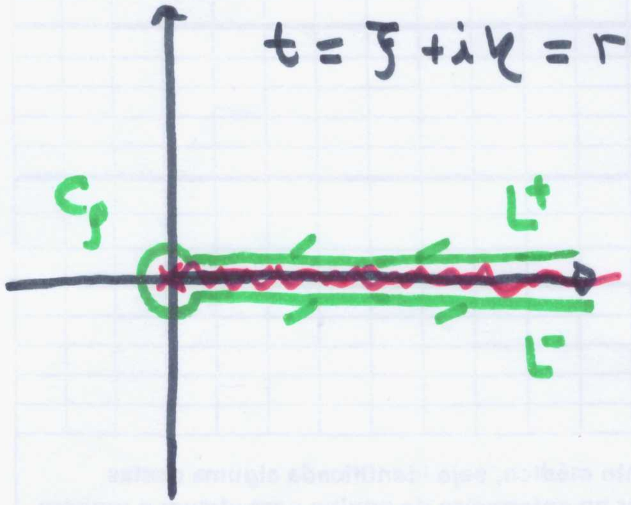


The above definitio
only applies to the
positive x portio

To further extend it, w
define: $f(z,t) = t^{z-1} e^{-t}$
where both $t, z \in \mathbb{C}$

$$f(z,t) = \exp\left[(z-1)\log t \;\; - t\right] \qquad 0 < \arg(t) < 2\pi$$

$$t = \xi + i\eta = re^{i\theta}, \quad z = x + iy$$



$$C' = L^+ \oplus C_\rho \oplus L^-$$

$$\begin{cases} L^+ \Rightarrow t = \xi + i\delta\eta \;;\; \rho < r < \infty \\ C_\rho \Rightarrow t = \rho e^{i\theta} \qquad 0 < \theta < 2\pi \\ L^- \Rightarrow t = \xi - i\delta\eta \;;\; \rho < r < \infty \end{cases}$$

$$\int_{C'} f(z,t)\, dt = \int_{L^+} f(z,t)\, dt + \int_{C_\rho} f(z,t)\, dt + \int_{L^-} f(z,t)\, dt$$

$$f(z,t) = \exp\left\{\left[\frac{(x-1)}{2}\ln(\xi^2+\eta^2) - y\tan^{-1}\left(\frac{\eta}{\xi}\right) - \xi\right] + i\left[\frac{y}{2}\ln(\xi^2+\eta^2) + (x-1)\tan^{-1}\left(\frac{\eta}{\xi}\right) - \eta\right]\right\}$$

$$\lim_{\substack{\delta\eta\to 0\\ \delta\theta\to 0}} f(z,t)\Big|_{L^+} = \xi^{(z-1)}\, e^{-\xi}$$

$$\lim_{\substack{\delta\eta\to 0\\ \delta\theta\to 2\pi}} f(z,t)\Big|_{L^-} = \xi^{(z-1)}\, e^{-\xi + \underline{i2\pi z}}$$

$$f(z,t)\Big|_{C_\rho} = \exp\left\{\left[(x-1)\ln\rho - y\theta - \rho\cos\theta\right] + i\left[y\ln\rho + (x-1)\theta + -\rho\sin\theta\right]\right\}$$

$$\lim_{\rho\to 0} f(z,t)\Big|_{C_\rho} = \lim_{\ln\rho\to -\infty} f(z,t)\Big|_{C_\rho} = 0$$

On substituting these for their counterparts in the integration yields:

$$\int_{C'} f(z,t)\,dt = -\int_0^\infty \xi^{(z-1)} e^{-\xi}\, d\xi + 0 + e^{i2\pi z}\int_0^\infty \xi^{(z-1)} e^{-\xi}\, d\xi$$

④

$$\int_{c'} t^{(z-1)} e^{-t}\, dt = \left(e^{i2\pi z} - 1\right) \underbrace{\int_0^{\infty} e^{-\xi}\, \xi^{(z-1)}\, d\xi}_{}$$

original definition of Γ, but now it has been extended over the complex plane

$$\left.\begin{matrix} z \in \mathbb{C} \\ \xi \in \mathbb{R} \end{matrix}\right\} \Longrightarrow$$

$$\Gamma(z) \equiv \int_0^{\infty} \xi^{(z-1)} e^{-\xi}\, d\xi = \frac{1}{\left(e^{-i2\pi z} - 1\right)} \int t^{(z-1)} e^{-t}\, dt$$

$\xi \in \mathbb{R}$; $z, t \in \mathbb{C}$

From Lebedev's book, and by making use of the 1st integral:

$$\Gamma(z) = \int_0^{\infty} e^{-t}\, t^{z-1}\, dt$$

but now for: $\left\{\begin{matrix} z \in \mathbb{C} \\ t \in \mathbb{R} \\ \mathrm{Re}\{z\} > 0 \end{matrix}\right.$ $(0 < t < +\infty)$

$$\Gamma(z) = \underbrace{\int_0^{1} e^{-t}\, t^{z-1}\, dt}_{P(z)} + \underbrace{\int_1^{\infty} e^{-t}\, t^{z-1}\, dt}_{Q(z)} = P(z) + Q(z)$$

the kernel $[e^{-t}\, t^{z-1}]$ is analytic in z

and t for $Re\{z\} > 0$ and $0 < t < \infty$

$P(z) \Rightarrow \quad 0 < \delta \leq R\{z\}$; $0 < t < 1 \Rightarrow \ln(t) < 0$

$$\left| \int_0^1 e^{-t}\, t^{z-1}\, dt \right| \leq \int_0^1 e^{-t}\, t^{\delta-1}\, dt < \infty$$

$Q(z) \Rightarrow \quad Re\{z\} \leq A < \infty$; $t \geq 1 \Rightarrow \ln(t) > 0$

$$\left| \int_1^\infty e^{-t}\, t^{z-1}\, dt \right| \leq \int_1^\infty e^{-t}\, t^{A-1}\, dt < \infty$$

this is for a finite z, it would blow up for $z = \infty$

$P(z)$ is analytic on the Right half plane $Re\{z\} > 0$, whereas $Q(z)$ is entire (it only blows up at $z = \infty$.

Therefore $\Gamma(z) = P(z) + Q(z)$ is analytic on the Right half plane $Re\{z\} > 0$

Under these circumstances, we make:

$\longrightarrow \longrightarrow$

$$P(z) = \int_0^1 t^{z-1} e^{-t}\, dt = \int_0^1 t^{z-1} \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^1 t^{k+z+1}\, dt$$

$$P(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^1 t^{k+z+1}\, dt \Rightarrow \boxed{P(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{(z+k)}}$$

Hence, $P(z)$ has simple poles at $z=-n$, $\forall n \in \mathbb{N}$, that is, it has simple poles at the negative integers.

The function $\Gamma(z)$, on its turn, can be written as:

$$\Gamma(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{(z+k)} + \int_1^{\infty} e^{-t} t^{z-1}\, dt$$

$$\forall z \in \mathbb{C} \mid z \neq -n \quad (n \in \mathbb{N})$$

Basic Properties of the Gamma function:

1) $\Gamma(z+1) = z\Gamma(z)$

$$\Gamma(z+1) = \int_0^{\infty} e^{-t} t^z\, dt = \cancelto{0}{\left. -e^{-t} t^z \right|_0^{\infty}} + z \int_0^{\infty} e^{-t} t^{z-1}\, dt = z\Gamma(z)$$

2) $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$

Proof:

$$\Gamma(z)\Gamma(1-z) = \int_0^\infty e^{-t} t^{z-1} dt \int_0^\infty e^{-s} s^{1-z-1} ds =$$

$$\int_0^\infty\int_0^\infty e^{-(t+s)} s^{-z} t^{z-1} ds dt \equiv \Delta$$

change of variables: $\begin{cases} u \equiv s+t \\ v \equiv \dfrac{t}{s} \end{cases}$

$$\left.\begin{aligned} s &= \frac{u}{(1+v)} \\ t &= \frac{uv}{(1+v)} \end{aligned}\right\} \quad \frac{\partial(s,t)}{\partial(u,v)} = \begin{vmatrix} \dfrac{1}{(1+v)} & \dfrac{-u}{(1+v)^2} \\ \dfrac{v}{(1+v)} & \dfrac{u(1+v)-uv}{(1+v)^2} \end{vmatrix} = \frac{u}{(1+v)^2} \neq 0$$

It is shown in the notes that

$$\Delta = -\int_0^\infty \frac{v^{z-1}}{(1+v)} dv = \frac{-\pi}{\sin[\pi(z-1)]} = \frac{\pi}{\sin[\pi z]}$$

$$\boxed{\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}} \quad \forall z \in \mathbb{C} \,\middle|\, z \neq 0, \pm 1, \pm 2, \ldots$$

3) $2^{2z-1}\,\Gamma(z)\Gamma(z+1/2) = \sqrt{\pi}\,\Gamma(2z)$

$$Ⓗ \equiv 2^{2z-1}\Gamma(z)\Gamma(z+1/2) = \int_0^\infty\int_0^\infty e^{-(s+t)}\,(2\sqrt{st})^{2z-1}\,t^{-1/2}\,ds\,dt$$

change of variables:

$\alpha \equiv \sqrt{s} \Rightarrow s = \alpha^2 \Rightarrow ds = 2\alpha\,d\alpha$

$\beta \equiv \sqrt{t} \Rightarrow t = \beta^2 \Rightarrow dt = 2\beta\,d\beta$

$$\frac{\partial(s,t)}{\partial(\alpha,\beta)} = 4\alpha\beta \neq 0$$

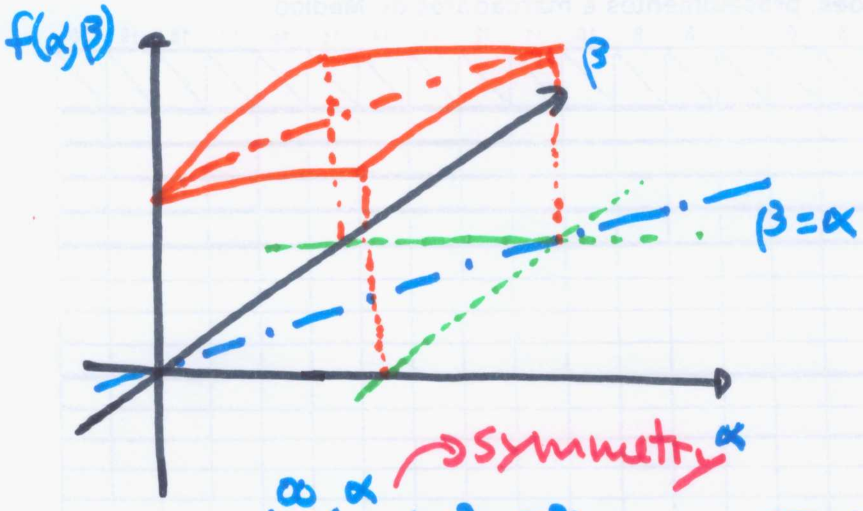
$$Ⓗ = 4\int_0^\infty\int_0^\infty e^{-(\alpha^2+\beta^2)}\,(2\alpha\beta)^{2z-1}\,\beta^{-1}\alpha\,\beta\,d\alpha\,d\beta =$$

$$= 4\int_0^\infty\int_0^\infty e^{-(\alpha^2+\beta^2)}\,(2\alpha\beta)^{2z-1}\,\alpha\,d\alpha\,d\beta$$

on permuting α and β roles in the above integral, we get a symmetric result, which can be added to the previous one, to give:

$$Ⓗ = \frac{4}{2}\int_0^\infty\int_0^\infty e^{-(\alpha^2+\beta^2)}\,(2\alpha\beta)^{2z-1}\,(\alpha+\beta)\,d\alpha\,d\beta$$

We can take further advantage of the symmetry by making:



$$\int_0^\infty\int_0^\alpha f(\alpha,\beta)\,d\beta\,d\alpha = \frac{1}{2}\int_0^\infty\int_0^\infty f(\alpha,\beta)\,d\beta$$

$$H = 4\int_0^\infty\int_0^\alpha e^{-(\alpha^2+\beta^2)}\,(2\alpha\beta)^{2z-1}\,(\alpha+\beta)\,d\beta\,d\alpha$$

Yet another change of variables:

$$\left.\begin{aligned} u &\equiv \alpha^2+\beta^2 \\ v &\equiv 2\alpha\beta \end{aligned}\right\} \quad \begin{aligned} (\alpha+\beta)^2 &= u+v \\ (\alpha+\beta) &= \sqrt{u+v} \end{aligned} \quad \left| \quad \begin{aligned} (\alpha-\beta)^2 &= u-v \\ (\alpha-\beta) &= \sqrt{u-v} \end{aligned}\right.$$

$$\frac{\partial(u,v)}{\partial(\alpha,\beta)} = 4(\alpha^2-\beta^2) = 4\sqrt{u^2-v^2}$$

$$\frac{\partial(\alpha,\beta)}{\partial(u,v)} = \frac{1}{4\sqrt{u^2-v^2}}$$

$$H = 4\int_0^\infty\int_0^\infty e^{-u}\,v^{2z-1}\,\frac{\sqrt{u+v}}{4\sqrt{(u^2-v^2)}}\,du\,dv = \int_0^\infty\int_0^\infty \frac{e^{-(u-v)}\,e^{-v}\,v^{2z-1}}{\sqrt{u-v}}\,du\,dv$$

$$w = +\sqrt{u-v} \quad = \quad dw = \frac{du}{2\sqrt{u-v}}$$

$$\textcircled{H} = \int_0^\infty \int_0^\infty e^{-v} \, v^{2z-1} \, e^{-w^2} \, dw \, dv =$$

$$= \int_0^\infty e^{-v} \, v^{2z-1} \, dv \int_0^\infty e^{-w^2} \, dw$$

$$\textcircled{H} = \Gamma(2z)\sqrt{\pi}$$

which finally leads to:

$$2^{2z-1} \, \Gamma(z)\,\Gamma(z+1/2) = \sqrt{\pi}\;\Gamma(2z)$$

$$\boxed{\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \, \Gamma(z)\,\Gamma(z+1/2)}$$

$$\int_0^\infty e^{-pt}\, t^{z-1}\, dt = \frac{\Gamma(z)}{p^z} \quad Re\{p\}, R\{z\} > 0$$

↳ that is shown at length in the notes.

The Beta Function is defined as:

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt \quad \begin{cases} Re(x) > 0 \\ Re(y) > 0 \end{cases}$$

Then, by changing variables in a way that is fully presented in the notes, one gets:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad x,y \in \mathbb{C} \quad \begin{cases} Re\{x\} > 0 \\ Re\{y\} > 0 \end{cases}$$

Frobenius Method to Solving ODEs:

$$W''(z) + p(z)W'(z) + q(z)W(z) = 0$$

Homogeneous ODE, $W' = \frac{dW}{dz}$, $W'' = \frac{d^2W}{dz^2}$

But the coefficients $P(z)$ and $Q(z)$ are ~~not~~ constant. On considering an arbitrary point $z_0 \in \mathcal{D}$ (domain), we consider 3 cases:

1. P and Q are analytic at z_0 ($z_0 \Rightarrow$ ordinary point)
2. P and Q have the so-called removable singularities at z_0. So $(z-z_0)P(z)$ and $(z-z_0)^2 Q(z)$ are analytic at z_0, which is called regular point
3. P and Q have any other kinds of singularities at z_0, which is called an irregular singularity of the ODE.

1. in the 1st case, we should try solutions like: $w = \sum_{k=0}^{\infty} a_k z^k$

$$w' = \sum_{k=1}^{\infty} a_k k z^{k-1}$$

$$w'' = \sum_{k=2}^{\infty} a_k k(k-1) z^{k-2}$$

Then we substitute these for their counterparts in the ODE and collect like powers

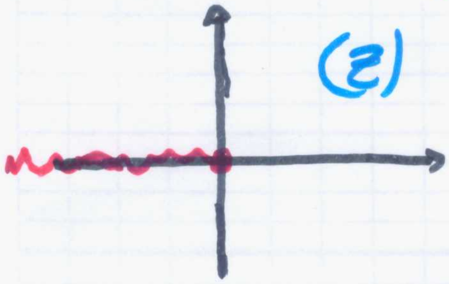
2. in the 2nd case, let's consider the example:

$$w'' + \frac{a}{z} w' + \frac{b}{z^2} w = 0$$

$P(z) = \dfrac{a}{z}$, $Q(z) = \dfrac{b}{z^2}$ $\Rightarrow$ two removable singularities at $z = 0$.

We'll try solutions of the form $w(z) = z^{\alpha}$ where $\alpha \in \mathbb{C}$: $w(z) = \mathrm{Exp}[\alpha \log(z)]$

$-\pi < \theta < \pi$, $z = re^{i\theta}$



$$\begin{cases} w(z) = z^{\alpha} \\ w'(z) = \alpha z^{\alpha-1} \\ w''(z) = \alpha(\alpha-1) z^{\alpha-2} \end{cases}$$

On introducing these into the ode, one ends up with:

$$\alpha(\alpha-1) z^{(\alpha-2)} + a\alpha z^{(\alpha-2)} + b z^{(\alpha-2)} = 0$$

$$[\alpha(\alpha-1) + a\alpha + b]\, z^{(\alpha-2)} = 0$$

Since z is not identically zero, we must have: $\alpha(\alpha-1) + \alpha a + b = 0$

Indicial equation: $\alpha(\alpha-1)+a\alpha+b=0$

2nd order equation: two roots (α_1, α_2)

$\alpha_1 \neq \alpha_2 \Rightarrow W(z) = c_1 z^{\alpha_1} + c_2 z^{\alpha_2}$

$\alpha_1 = \alpha_2 \Rightarrow W_1(z) = c_1 z^{\alpha_1}$ and another appropriate method is needed to find another independent solution.

3. In the third case, we'll try solutions of the form: $W(z) = z^{\alpha} \sum_{n=0}^{\infty} a_n z^n$

One of the tools we've learned for power series that may well come in handy is the so-called Cauchy Product (Churchill, page 162)

$$\left.\begin{aligned} f(z) &= \sum_{n=0}^{\infty} c_n (z-z_0)^n \\ g(z) &= \sum_{n=0}^{\infty} d_n (z-z_0)^n \end{aligned}\right\} \quad f(z).g(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{where } a_n = \sum_{k=0}^{\infty} c_k d_{n-k}$$

$$f(x).g(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c_k d_{n-k} (z-z_0)^n$$

So, in the more general case, we have:

$$W = \sum_{n=0}^{\infty} a_n Z^{n+\alpha}$$

$$W' = \sum_{n=0}^{\infty} a_n(n+\alpha) Z^{(n+\alpha-1)}$$

$$W'' = \sum_{n=0}^{\infty} a_n (n+\alpha)(n+\alpha-1) Z^{(n+\alpha-2)}$$

As an example, let's consider the ODE (Butkov, page 134)

$$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0$$

$$y = \sum_{n=0}^{\infty} c_n x^{s+n}$$

$$y' = \sum_{n=0}^{\infty} c_n (s+n) x^{s+n-1}$$

$$y'' = \sum_{n=0}^{\infty} c_n (s+n)(s+n-1) x^{s+n-2}$$

$$\sum_{n=0}^{\infty} \left\{ c_n (s+n)(s+n-1) + c_n (s+n) - \frac{c_n}{4} \right\} x^{s+n} +$$

$$+ \sum_{n=0}^{\infty} c_n x^{s+n+2} = 0$$

$$c_0\left[s(s-1)+s-\frac{1}{4}\right]x^s + c_1\left[(s+1)s+(s+1)-\frac{1}{4}\right]x^{s+1} +$$

$$+\sum_{n=2}^{\infty} c_n\left[(s+n)(s+n-1)+(s+n)-\frac{1}{4}\right]x^{s+n} + \sum_{n=0}^{\infty} c_n x^{s+n+2} = 0$$

$$c_0\left[s^2-\frac{1}{4}\right]x^s + c_1\left[s^2+2s+\frac{3}{4}\right]x^{s+1} + \sum_{k=0}^{\infty} c_{k+2}\left[(s+k+2)(s+k+1)+\right.$$

$$\left.+s+k+2-\frac{1}{4}\right]x^{s+k+2} + \sum_{n=0}^{\infty} c_n x^{s+n+2} = 0$$

where $\left.\begin{array}{l} k \equiv n-2 \\ n = k+2 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} n=2 \Rightarrow k=0 \\ n=\infty \Rightarrow k\to\infty \end{array}\right.$

$$c_0\left[s^2-\frac{1}{4}\right]x^s + c_1\left[s^2+2s+\frac{3}{4}\right]x^{s+1} + \sum_{k=0}^{\infty}\left\{c_k\left[(s+k+2)(s+k+1)+\right.\right.$$

$$\left.\left.+s+k+2-\frac{1}{4}\right]+c_k\right\}x^{s+k+2} = 0$$

From the above, we must have:
(for $x \neq 0$) (indicial equation)

$c_0\left[s^2-\frac{1}{4}\right]=0$ for $c_0 \neq 0 \Rightarrow s = \left\{\begin{array}{l} +1/2 \\ -1/2 \end{array}\right.$

Also,

$c_1\left[s^2+2s+\frac{3}{4}\right]=0$ for $c_1 \neq 0 \Rightarrow \left\{\begin{array}{l} s=\frac{1}{2} \Rightarrow c_1\left[\frac{1}{4}+1+\frac{3}{4}\right]=2c_1 \neq c_1 = 0 \\ s=-\frac{1}{2} \Rightarrow c_1\left[\frac{1}{4}-1+\frac{3}{4}\right]=0 \Rightarrow c_1 \neq 0 \end{array}\right.$

Recurrence relation:

$$C_{k+2} = \frac{C_k}{\left[(s+k+2)(s+k+1) + (s+k+2) - \frac{1}{4}\right]}$$

where C_0 and C_1 should be fully specified by two boundary conditions, to be imposed on the ODE.

In this particular case, even and odd coefficients are independent sets.

for $s = \frac{1}{2}$ and $C_1 = 0$, we get:

$$Y_1(x) = C_0\sqrt{x}\left[1 - \frac{x^2}{6} + \frac{x^4}{5!} - \dots\right]$$

for $s = -1/2$ we get:

$$Y_2(x) = \frac{C_0}{\sqrt{x}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots\right] + \frac{C_1}{\sqrt{x}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right]$$

$$Y_2(x) = \frac{C_0}{\sqrt{x}}\cos(x) + \frac{C_1}{\sqrt{x}}\sin(x)$$

These, in turn, correspond to:

$$\begin{cases} J_{-1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\cos(x) \\ J_{+1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\sin(x) \end{cases}$$

which are Bessel functions of the first kind, of orders $-\frac{1}{2}$ and $+\frac{1}{2}$, respectively.

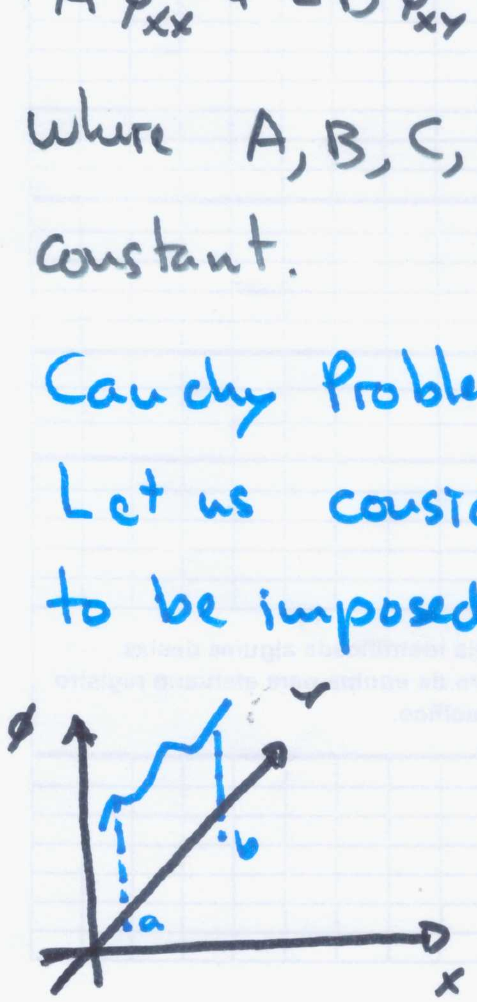
Partial Differential Equations — PDEs.

$$A\phi_{xx} + 2B\phi_{xy} + C\phi_{yy} + D\phi_x + E\phi_y + F\phi + G = 0$$

where A, B, C, D, E, F and G are <u>not</u> necessarily constant.

Cauchy Problem:

Let us consider possible boundary conditions to be imposed on the PDE



$\phi(0,y) = f(y)$ for $\forall y \in [a,b]$

↳ Cauchy data

we can get $\phi_y(0,y), \phi_{yy}(0,y)$ by repeated differentiation with respect to y.

where we have tacitly assumed that $\phi(0,y) = f(y)$ is at least $C^2 \to$ twice differentiable.

However, nothing can be said about $\phi_x(0,y)$, $\phi_{xx}(0,y)$ ....

Apart from the fact that the PDE itself imposes a certain relation between x and y partial derivatives.

However, if we were given instead the following Cauchy Data:

$$\begin{cases} \phi(0,y) = f(y) \\ \phi_x(0,y) = g(y) \end{cases} \quad \Bigg| \quad x=0 \text{ and } y \in [a,b]$$

Now, on differentiating $f(y)$ twice with respect to y and $g(y)$ once with respect to y too, we get:

$$\begin{array}{l|l} \phi(0,y) = f(y) & \phi_x(0,y) = g(y) \\ \phi_y(0,y) = f'(y) & \phi_{xy}(0,y) = g'(y) \\ \phi_{yy}(0,y) = f''(y) & \end{array}$$

on assuming that f and g differentiable C^2

Now it's easy to see that on substituthing them for their counter parts in the PDE, we get:

$$A\phi_{xx} + 2B\, g_y + Cf_{yy} + Dg + Ef_y + Ff + G = 0$$

from which we get ϕ_{xx} at $x = 0$
$y \in [a,b]$

Now, with this information and on assuming that, in principle, both $f(y)$ and $g(y)$ are $C^\infty \Rightarrow$ infinitely differentiable. We could expand $\phi(x,y)$ in a Taylor series about the plane $x=0$ for $y \in [a,b]$

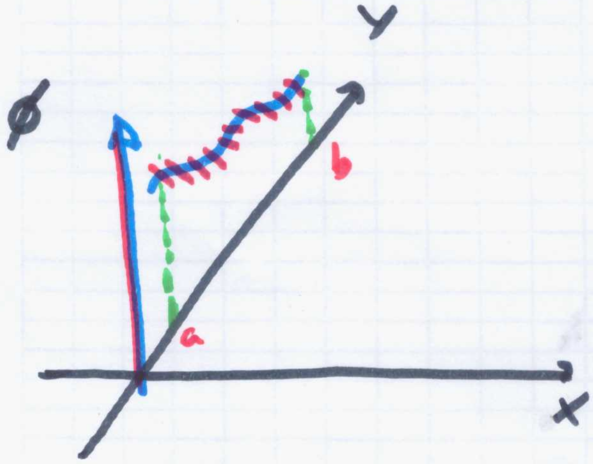
$$\phi(x,y) = \phi(0,y) + x\phi_x(0,y) + \frac{x^2}{2!}\phi_{xx}(0,y) + \dots$$

Now, it is true and must be borne in mind that it all depends on having $A(x,y) \neq 0$

There is a formal Theorem by Cauchy-Kowalski that ensure the above approach does generate a unique solution to

$\phi(x,y)$ which is analytic in some neighborhood of a point $(0, y_0)$. Provided that the functions $f(y), g(y)$, $\frac{B(x,y)}{A(x,y)}$; $\frac{C(x,y)}{A(x,y)}$; $\frac{D(x,y)}{A(x,y)}$ are all analytic in that neighborhood.

Hence, the necessary and sufficient Cauchy Data for this PDE is given by:



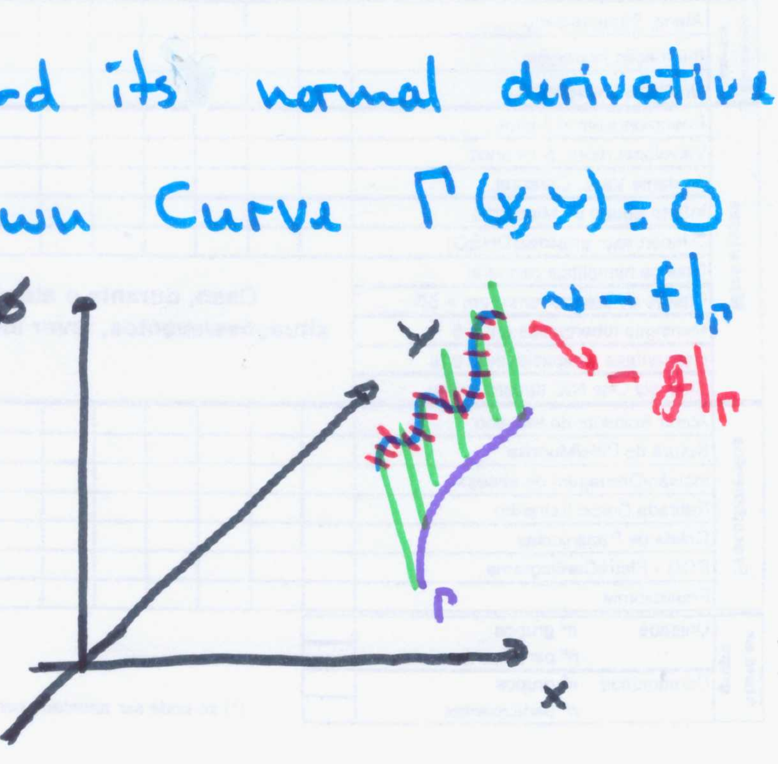
$$\begin{cases} \phi(0,y) = f(y) & \text{Dirichlet} \\ \phi_x(0,y) = g(y) & \text{Von Neumann} \end{cases}$$

It's like a "Ribbon"

More Generally we should have:

Given $\phi(x,y)$ and its normal derivative $\frac{\partial \phi}{\partial n}$ along a known Curve $\Gamma(x,y) = 0$

$$\begin{cases} \phi\Big|_\Gamma = f\Big|_\Gamma \\ \frac{\partial \phi}{\partial n}\Big|_\Gamma = g\Big|_\Gamma \end{cases}$$



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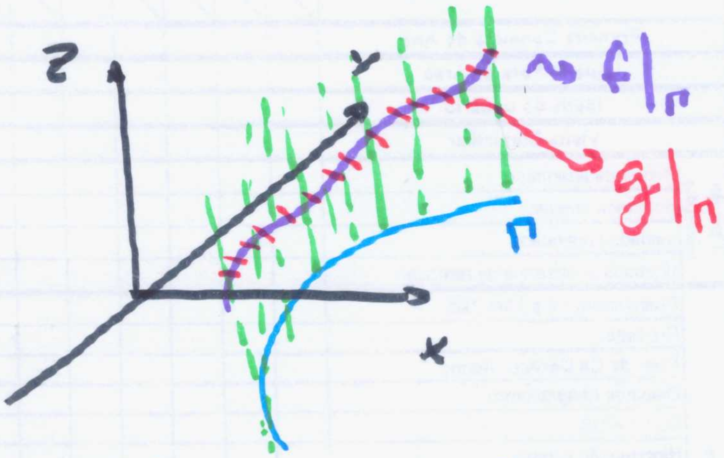
Cauchy Data: minimal set for a general 2nd order PDE

$$\begin{cases} \phi\big|_{\Gamma} = f\big|_{\Gamma} \\ \dfrac{\partial \phi}{\partial n}\Big|_{\Gamma} = g\big|_{\Gamma} \end{cases}$$



PDE: $A\phi_{xx} + 2B\phi_{xy} + C\phi_{yy} + F_L(x, y, \phi, \phi_x, \phi_y) = 0$

a) Linear with respect to the highest order derivatives if A, B and C depend on x and y only

b) If these coefficients also depend on ϕ, ϕ_x, ϕ_y, besides x and y, the eq. is termed quasi-linear.

c) The eq. is just linear if A, B and C depend only on x and y, and F_L can be written in the form

$F_L = D\phi_x + E\phi_y + F\phi + G$

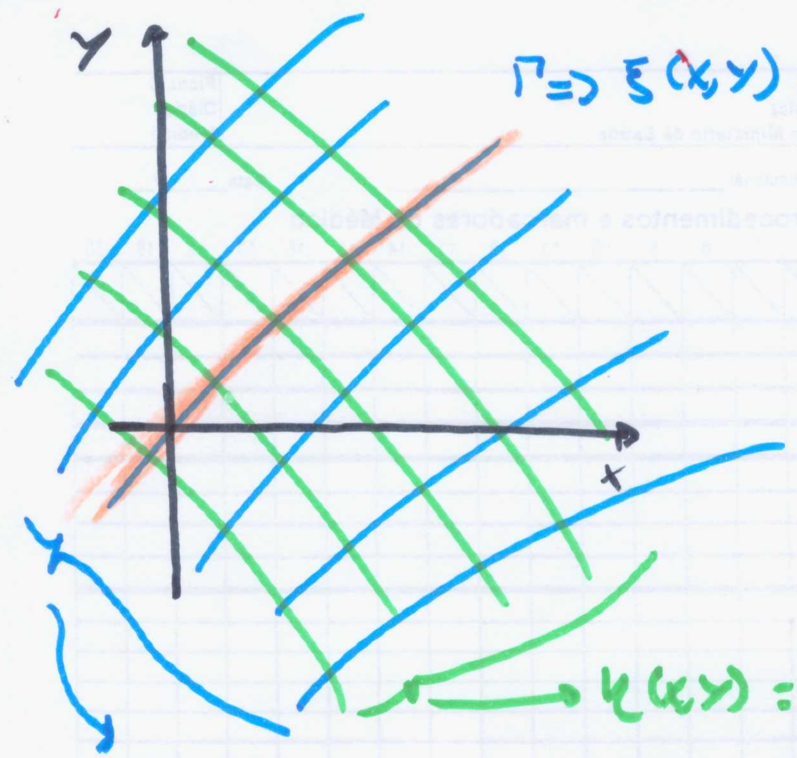
where all coefficients (D, E, F, G) only depend on x, y as well.

d) Other variations are non-linear

$\Gamma \Rightarrow \xi(x,y) = 0 \Rightarrow$ Cauchy Data

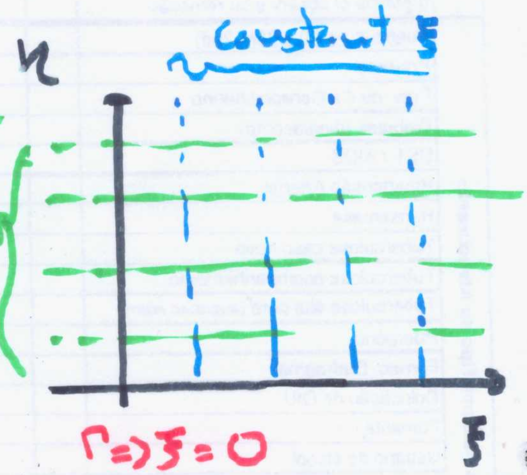
$$\begin{cases} \phi(0,\eta)_\Gamma = f(\eta) \\ \dfrac{\partial \phi}{\partial \xi}(0,\eta)_\Gamma = g(\eta) \end{cases}$$

(ξ, η) is the new coordinat system



$\eta(x,y)$: constant

$\xi(x,y) =$ constant



$$\left.\begin{matrix} \xi(x,y) \\ \eta(x,y) \end{matrix}\right\} \rightleftharpoons \begin{cases} x(\xi,\eta) \\ y(\xi,\eta) \end{cases}$$

Transformation Jacobian : $J = \dfrac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix}$

$J = \xi_x \eta_y - \xi_y \eta_x \neq 0$

As an illustrative side note, in the particular case of a mapping by Analytic functions:

$z = x + iy$, $\zeta = \xi + i\eta = \zeta(z)$

$$\frac{d\zeta}{dz} = \xi_x + i\xi_y = \eta_y - i\eta_x \Rightarrow \left|\frac{d\zeta}{dz}\right| = \frac{d\zeta}{dz}\overline{\left(\frac{d\zeta}{dz}\right)} =$$

$$\left|\frac{d\zeta}{dz}\right| = \xi_x^2 + \xi_y^2 = \xi_x \eta_y - \xi_y \eta_x = J$$

$J = 0$ at critical Points

$$\phi_x = \phi_\xi \xi_x + \phi_\eta \eta_x$$

$$\phi_{xx} = \phi_{\xi\xi}\xi_x^2 + 2\phi_{\xi\eta}\xi_x\eta_x + \phi_{\eta\eta}\eta_x^2 + \phi_\xi \xi_{xx} + \phi_\eta \eta_{xx}$$

⋮ (see Tikonov pag 18 - Mathematical Physics) - The Equations of Mathematical Physics, Mir Publishers, Moscow.

In the most General form, one gets:

$$\phi_{\xi\xi}\left(A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2\right) + 2\phi_{\xi\eta}\left[A\xi_x\eta_x + B(\xi_x\eta_y + \eta_x\xi_y) + C\xi_y\eta_y\right] + \phi_{\eta\eta}\left(A\eta_x^2 + 2B\eta_x\eta_y + C\eta_y^2\right) +$$

$$+ \phi_\xi\left(A\xi_{xx} + 2B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y\right) +$$

$$+ \phi_\eta\left(A\eta_{xx} + 2B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y\right) + F\phi + G = 0$$

Just as in the previous case, the reconstructio of the solution $\phi(x,y)$ from the Cauchy date specified on Γ hinges on the coefficien of the higher order derivatives being non-zero

Let us consider the case where

$$A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2 = 0 \qquad \text{on } \Gamma$$

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + 2B\left(\frac{\xi_x}{\xi_y}\right) + C = 0 \quad \text{for } \xi_y \neq 0$$

$$\left(\frac{\xi_x}{\xi_y}\right) = \frac{-2B \pm \sqrt{4B^2 - 4AC}}{2A} = \frac{-B \pm \sqrt{B^2 - AC}}{A}$$

$$\boxed{\Delta \equiv B^2 - AC} \qquad \boxed{A \neq 0}$$

There are two possibilities regarding the existence of curves $\xi(x,y) = \text{constant}$ that meet the above condition:

1. there is no real function $\xi(x,y)$ that satisfies the above eq. Hence one can specify Cauchy Data along any Curve Γ in the Domain

2. there are some real Functions $\xi(x,y)$ which satisfy the above eq. and, hence, they cannot be picked to specify Cauchy Data on.

It all depends on the sign of the discriminant:

$$\Delta \Rightarrow \begin{cases} >0 \\ =0 \\ <0 \end{cases}$$

Case 1 $\Delta > 0$, $A \neq 0$

$$\frac{\xi_x}{\xi_y} = \frac{-B \pm \sqrt{\Delta}}{A}$$ $\Bigg\{$ there are two families of curves that meet the condition:

along a ξ constant curve, we have:

$$d\xi \Big|_{\xi = \text{constant}} = \xi_x dx + \xi_y dy = 0$$

Whence it comes that: $\dfrac{dy}{dx} = -\dfrac{\xi_x}{\xi_y} = \dfrac{+B \mp \sqrt{\Delta}}{A}$

These two family of curves are nowhere tangent. Moreover, Cauchy Data that is specified along any of them would be insufficient.

Since these families $\xi(x,y) = \text{constant}$ and $\eta(x,y) = \text{constant}$ only depend on the particular PDE, they are called characteristic Curves.

$$\begin{cases} A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2 = 0 \\ A\eta_x^2 + 2B\eta_x\eta_y + C\eta_y^2 = 0 \end{cases} \qquad \frac{\xi_x}{\xi_y} = \frac{-B+\sqrt{\Delta}}{A} \qquad \frac{\eta_x}{\eta_y} = \frac{-B-\sqrt{\Delta}}{A}$$

So, if we use those two families of characteristics to define a coordinate system and to write the PDE in terms of (ξ, η), the two coefficients of $\phi_{\xi\xi}$ and $\phi_{\eta\eta}$ vanish, and we are left with

$$\phi_{\xi\eta} + \alpha \phi_{\xi} + \beta \phi_{\eta} + \gamma \phi + \delta = 0$$

where α, β, γ and δ are all functions of (ξ, η)

This is the so-called canonical form of the PDE. It is called hyperbolic PDE because this form resembles the expression of a hyperbola:

$$\alpha x^2 + 2\beta xy + \gamma y^2 + \delta x + \varepsilon y = \text{constant}.$$

for $\beta^2 - \alpha\gamma > 0$

Important: the sign of Δ CANNOT be changed by coordinate changes.

To show this, let's assume for the moment that the equation is in (x, y) and we want to cast it in the system (ξ, η), which doesn't need to be characteristic. In the latter, the coefficient of $\phi_{\xi\xi}$, $\phi_{\eta\xi}$ and $\phi_{\eta\eta}$ are a, b and c, respectively. It can be shown that: $b^2 - ac = (B^2 - AC)(\xi_x \eta_y - \xi_y \eta_x)^2$

$$\boxed{b^2 - ac = \Delta J^2}$$

Therefore the sign of Δ cannot be changed by coordinate transformations.

Case II : $\Delta = 0$ in a region of the domain R. It follows that the coefficients of $\phi_{\xi\xi}$ and $\phi_{\eta\eta}$ cannot both vanish throughout that region. Hence, we assume that $A \neq 0$ in R, then we get a single characteristic family:

$$\frac{\xi_x}{\xi_y} = -\frac{B}{A} \Rightarrow \boxed{\frac{dy}{dx} = \frac{B}{A}}$$

This is the only family of curves along which Cauchy Data is insufficient.

If we take $\xi(x,y) = \text{constant}$ to be the characteristic family, and choose any other family $\eta(x,y) = \text{constant}$ as the other coordinate in (ξ, η) — as long as $J = \frac{\partial(\xi,\eta)}{\partial(x,y)} \neq 0$. Then the coefficient of $\phi_{\xi\xi}$ will vanish. It also means that the coefficient of $\phi_{\xi\eta}$ must also vanish,

$$\cancel{b^2}^{\,\to 0} - \underbrace{\cancel{a}}_{0}c = 0 \quad \Big\{ \begin{array}{l} a \Rightarrow \text{coef. } \phi_{\xi\xi} \Rightarrow a = 0, \; c \text{ is not} \\ \text{constrained, but } b = 0 \text{ so as to} \\ \text{have } \Delta = 0 \end{array}$$

The end result is that both coefficients of $\phi_{\xi\xi}$ (a) and of $\phi_{\xi\eta}$ (b) vanish, and one is left with $c\phi_{\eta\eta}$, where $c \neq 0$ for $b^2 - ac = \Delta J^2 = 0$ ⑧

the canonical form then is:

$$\phi_{\eta\eta} + \alpha\phi_{\xi} + \beta\phi_{\eta} + \gamma\phi + \delta = 0$$

which is similar to the equation of a parabola and, thus, these are known as parabolic equations.

Case $\underline{\text{III}}$, $\Delta < 0$

when $\Delta < 0$ in R, it means that there are <u>no</u> real curves or families thereof where Cauchy Data is insufficient. Any curve can be used to impose boundary conditions.

Then, we'll look for the canonical form by driving the coefficient b of $\phi_{\xi\eta}$ to zero

$$\xi_x(A\eta_x + B\eta_y) + \xi_y(B\eta_x + C\eta_y) = 0$$

Under these circumstances, if we choose any reasonable family $\eta(x,y) = $ constant, we would get another one by making:

$$\left.\frac{dy}{dx}\right|_{\xi=\text{constant}} = -\frac{\xi_x}{\xi_y} = \frac{B\eta_x + C\eta_y}{A\eta_x + B\eta_y}$$

and thus getting $\xi(x,y) = $ constant.

And these two families would be independent.

Then we would have:

$$\frac{\xi_x}{\xi_y} = \frac{\eta_x}{\eta_y} = \rho \Rightarrow -\rho = \frac{(B\rho + C)}{(A\rho + B)}$$

which has <u>no</u> real roots.

(consistent with the $\Delta < 0$)

So we get that $b = 0$ in $b\phi_{\xi\eta}$ and we get:

$$a\phi_{\xi\xi} + c\phi_{\eta\eta} + d\phi_\xi + e\phi_\eta + f\phi + g = 0$$

which is termed elliptic equation for its similitude with the eq. of an ellipse.

Laplace Equation is a particular case:

$$\phi_{\xi\xi} + \phi_{\eta\eta} = 0$$

A further simplification is possible, one can make the coefficients of $\phi_{\xi\xi}$ and $\phi_{\eta\eta}$, "a" and "c", equal — while still keeping the coefficient of $\phi_{\xi\eta}$, $b = 0$

To show this, let's assume we start from a case where $B = 0$. Then the formula for the coefficients of $\phi_{\xi\xi}, \phi_{\xi\eta}$ and $\phi_{\eta\eta}$ become:

we make:

$$\left\{\begin{aligned} a &= A\xi_x^2 + C\xi_y^2 \\ b &= A\xi_x\eta_x + C\xi_y\eta_y \\ c &= A\eta_x^2 + C\eta_y^2 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} & A\xi_x^2 + C\xi_y^2 = A\eta_x^2 + C\eta_y^2 \\ & A\xi_x\eta_x + C\xi_y\eta_y = 0 \end{aligned}\right.$$

Since $\Delta = B^2 - AC < 0 \Rightarrow B^2 < AC$ then A and C must have the same sign and neither can vanish. Then, one way to solve the above set is to make:

$$\boxed{\sqrt{A}\,\xi_x = \sqrt{C}\,\eta_y} \quad \text{and} \quad \boxed{\sqrt{C}\,\xi_y = -\sqrt{A}\,\eta_y}$$

Within this framework, the particular case where $a = c = 1$ yields the C-R conditions $\Box$