MAP 2320 – MÉTODOS NUMÉRICOS EM EQUAÇÕES DIFERENCIAIS II

2º Semestre - 2020

Prof. Dr. Luis Carlos de Castro Santos

lsantos@ime.usp.br

A Guide to Numerical Methods for Transport Equations

2010

Dmitri Kuzmin

Friedrich-Alexander-Universität
Erlangen-Nürnberg

http://www.mathematik.uni-dortmund.de/~kuzmin/cfdbook.html

Mathematics of Transport Phenomena

$$\frac{\partial}{\partial t}\int_V u(\mathbf{x},t)\,\mathrm{d}\mathbf{x} + \int_S \mathbf{f}\cdot\mathbf{n}\,\mathrm{d}s = \int_V s(\mathbf{x},t)\,\mathrm{d}\mathbf{x}.$$

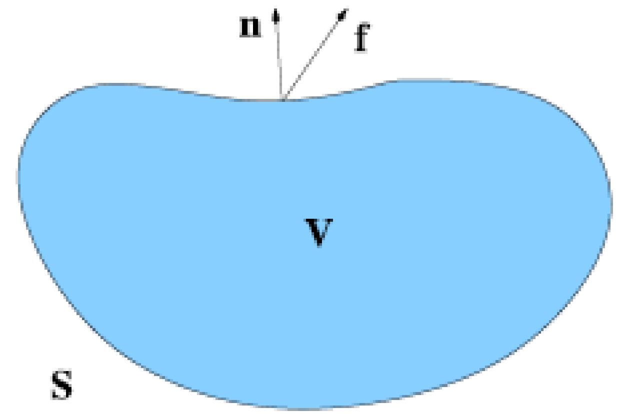


Fig. 1.1 A fixed control volume V bounded by the control surface S.

Quantidade conservada

Fonte (ou sumidouro) da quantidade conservada

Fluxo da quantidade conservada

Massa – Quantidade de Movimento - Energia

If the functions $u(\mathbf{x},t)$ and $\mathbf{f}(\mathbf{x},t)$ are differentiable, then the divergence theorem, as applied to the surface integral in (1.2), yields the identity

$$\int_V \left[\frac{\partial u(\mathbf{x},t)}{\partial t} + \nabla \cdot \mathbf{f}(\mathbf{x},t) - s(\mathbf{x},t)\right] \mathrm{d}\mathbf{x} = 0.$$

Since the choice of V is arbitrary, the expression in the square brackets must vanish, so the evolution of $u(\mathbf{x},t)$ is governed by the partial differential equation (PDE)

$$\frac{\partial u(\mathbf{x},t)}{\partial t} + \nabla \cdot \mathbf{f}(\mathbf{x},t) = s(\mathbf{x},t). \tag{1.3}$$

massa transportada

Gradiente de concentração

In general, both convective and diffusive effects must be taken into account, so

$$\mathbf{f}(\mathbf{x},t) = \mathbf{v}(\mathbf{x},t)u - \mathscr{D}(\mathbf{x},t)\rho\nabla c. \tag{1.7}$$

However, the rates of convective and diffusive transport may be quite different. For example, the transport of pollutants in a river is dominated by convection, whereas the spreading of pollutants in a lake is dominated by diffusion (dispersion).

Velocidade de transporte

Coeficiente de difusão

Convecção
Ou
Advecção

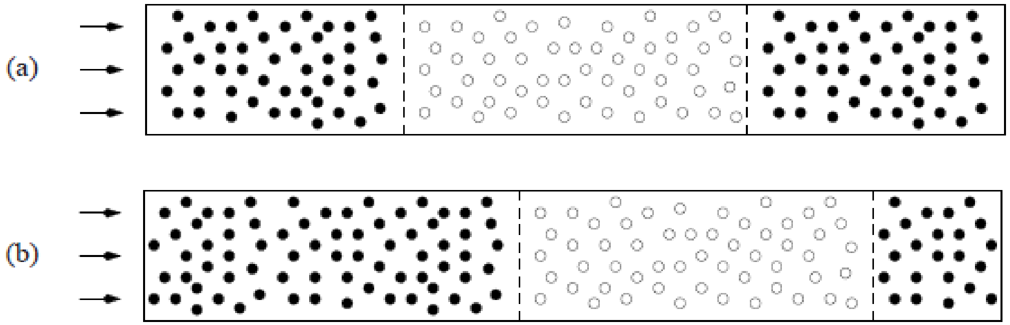


Fig. 1.2 Transport of tracer particles in a pipe filled with moving water.

Difusão

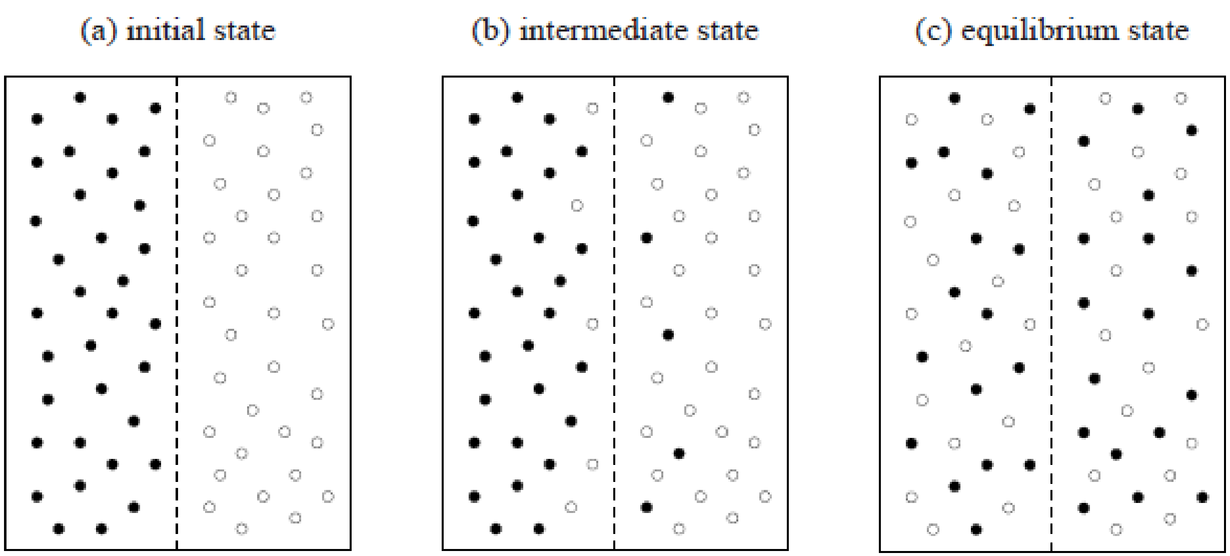


Fig. 1.3 Random motion of molecules across an interface in a stationary liquid.

The Generic Transport Equation

$$\frac{\partial \rho c}{\partial t} + \nabla \cdot (\mathbf{v}\rho c) - \nabla \cdot (\mathscr{D}\rho \nabla c) = s.$$

- the rate-of-change term $\frac{\partial \rho c}{\partial t}$ is the net gain/loss of mass per unit volume and time;
- the convective term $\nabla \cdot (\mathbf{v}\rho c)$ is due to the downstream transport with velocity $\mathbf{v}$;
- the diffusive term $-\nabla \cdot (\mathscr{D}\rho \nabla c)$ is due to a nonuniform spatial distribution of c;
- the source or sink term s combines all other effects that create or destroy ρc.

$$\frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{v}u) - \nabla \cdot (\mathscr{D}\nabla u) = s.$$

convection-diffusion-reaction (CDR) equation

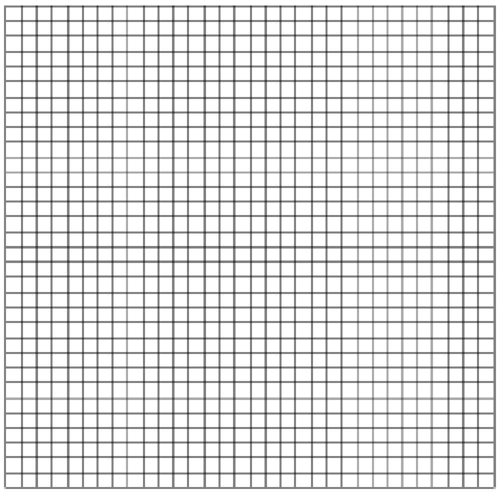
Summary of Model Problems

Table 1.1 Summary of models for convection, diffusion, and reaction processes.

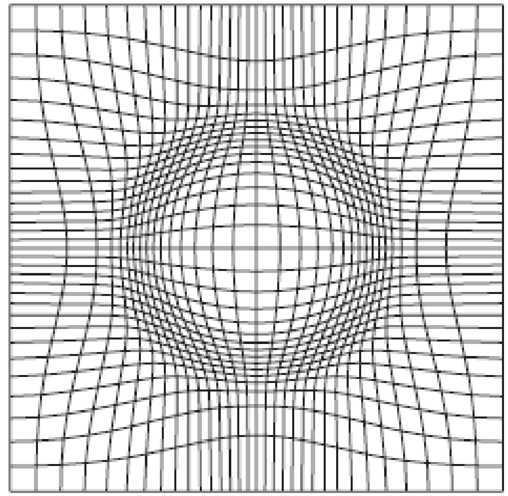
PDE type	multidimensional	one-dimensional
elliptic	$\nabla \cdot (\mathbf{v}u - \mathscr{D}\nabla u) = s$ $-\nabla \cdot (\mathscr{D}\nabla u) = s$	$v\frac{\partial u}{\partial x} - d\frac{\partial^2 u}{\partial x^2} = s$ $-d\frac{\partial^2 u}{\partial x^2} = s$
hyperbolic	$\nabla \cdot (\mathbf{v}u) = s$ $\frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{v}u) = s$	$v\frac{\partial u}{\partial x} = s$ $\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = s$
parabolic	$\frac{\partial u}{\partial t} - \nabla \cdot (\mathscr{D}\nabla u) = s$ $\frac{\partial u}{\partial t} + \nabla \cdot (\mathbf{v}u - \mathscr{D}\nabla u) = s$	$\frac{\partial u}{\partial t} - d\frac{\partial^2 u}{\partial x^2} = s$ $\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} - d\frac{\partial^2 u}{\partial x^2} = s$

Space Discretization Techniques ***Computational Meshes*** MAP2320

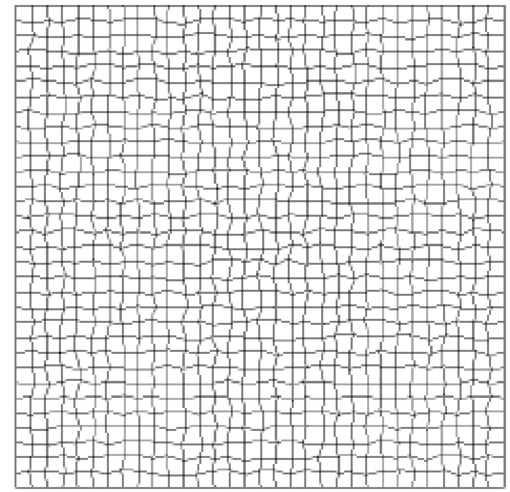
(a) structured, uniform



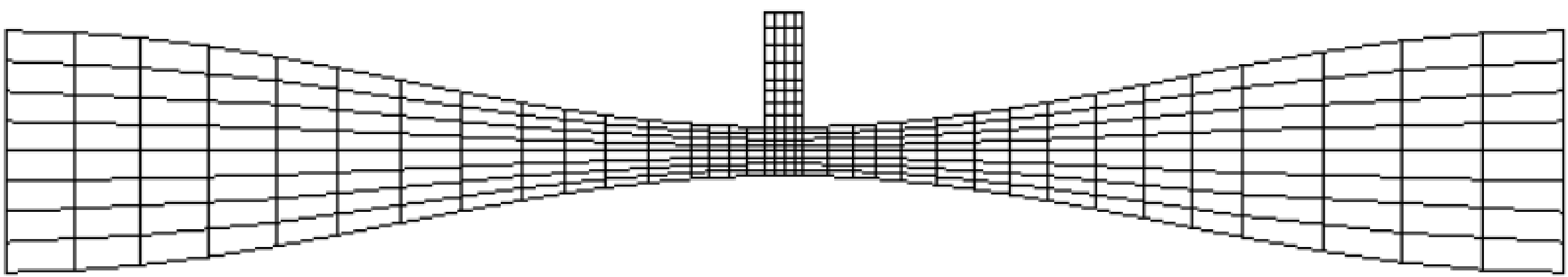
(b) structured, deformed



(c) structured, perturbed

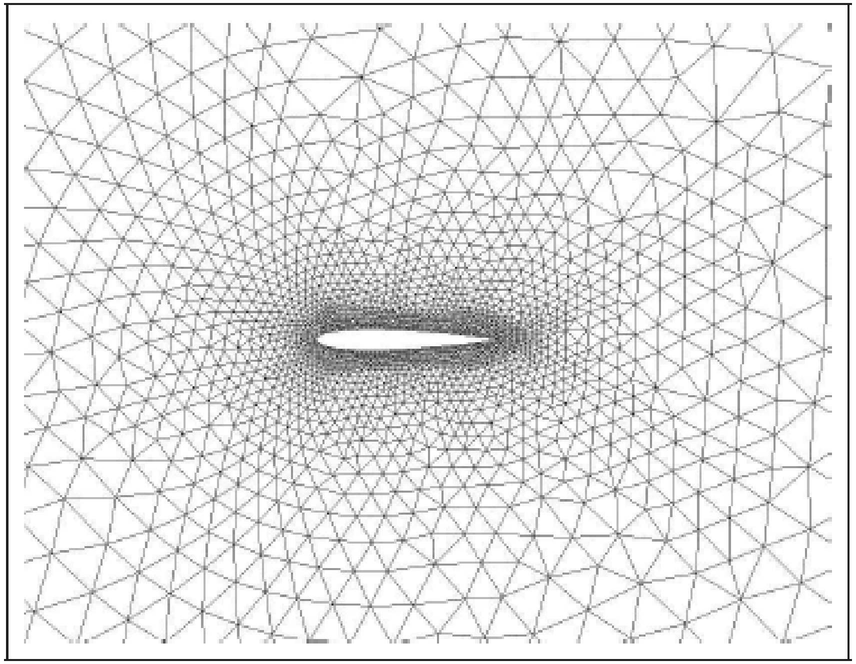


(d) block-structured, 2 subdomains

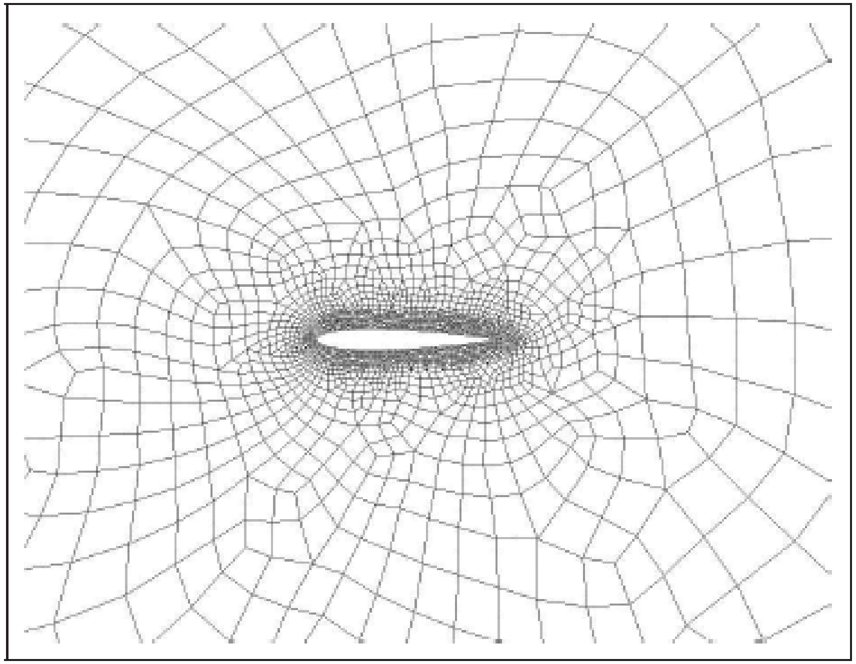


Domínios que podem ser mais facilmente mapeados no quadrado são mais propensos a aplicação do métodos de diferenças finitas.

(e) unstructured, triangular



(f) unstructured, quadrilateral



Domínios mais complexos não podem facilmente ser discretizados de forma a aplicar o método de diferenças finitas. Formulações mais gerais como o método dos Volumes Finitos e o método dos Elementos Finitos são capazes de lidar com discretizações em polígonos.

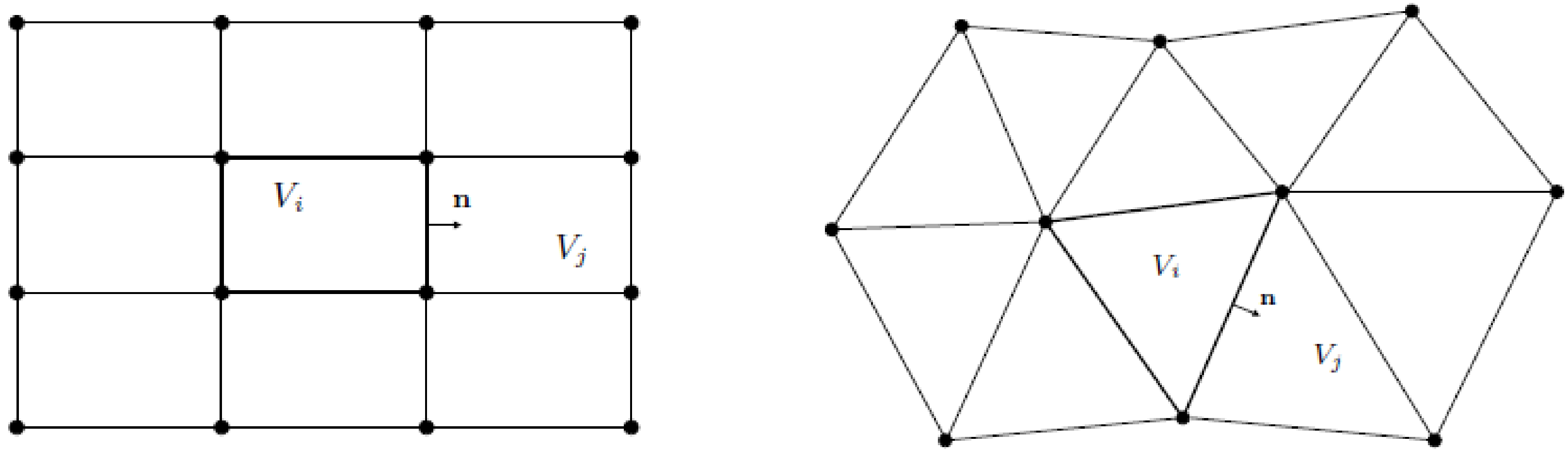


Fig. 1.5 Control volumes for a cell-centered FVM in two dimensions.

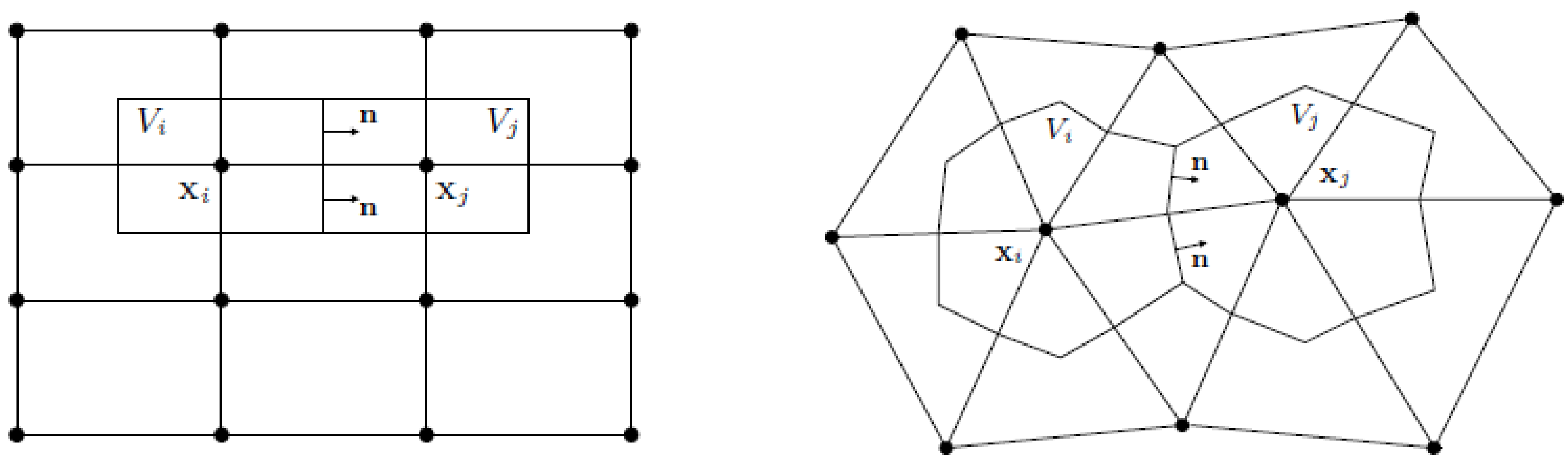
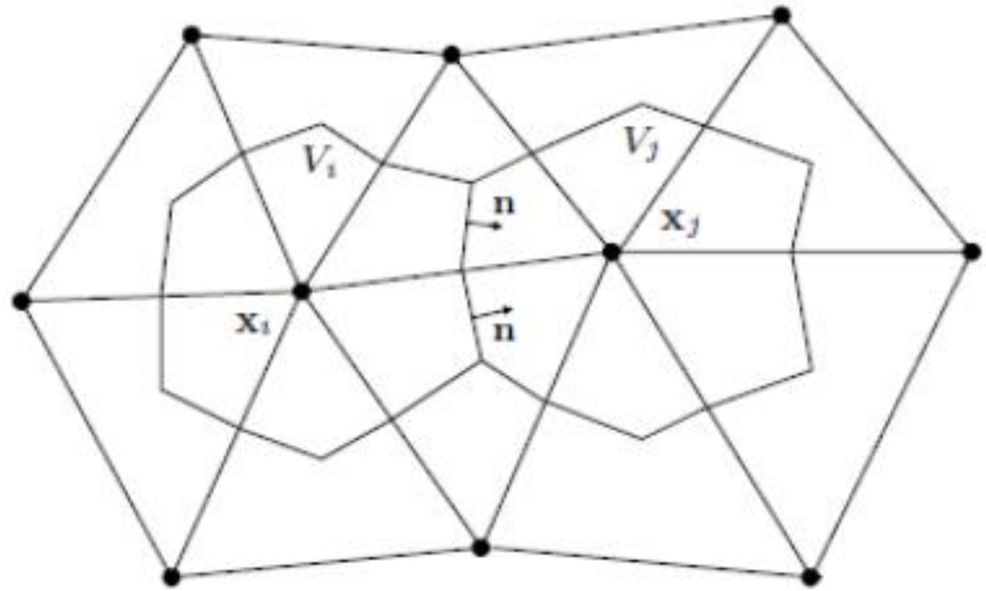


Fig. 1.6 Control volumes for a vertex-centered FVM in two dimensions.

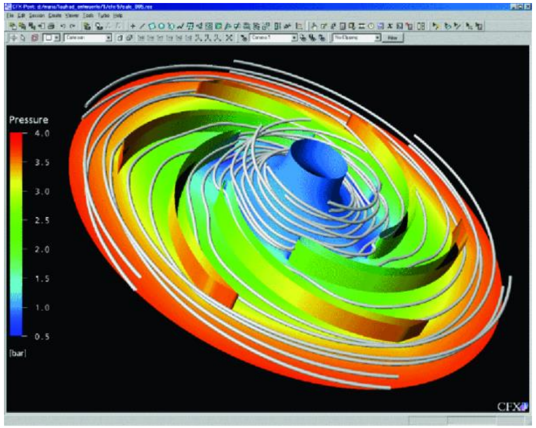


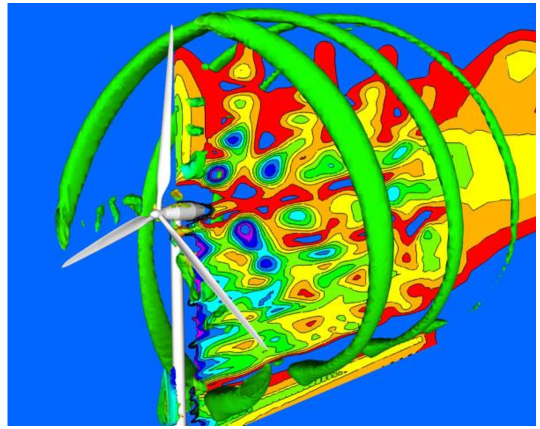
$$\frac{\partial}{\partial t}\int_{V_i} u(\mathbf{x},t)\,\mathrm{d}\mathbf{x} + \int_{S_i} (\mathbf{v}u - \mathscr{D}\nabla u)\cdot \mathbf{n}\,\mathrm{d}s = \int_{V_i} s(\mathbf{x},t)\,\mathrm{d}\mathbf{x},$$

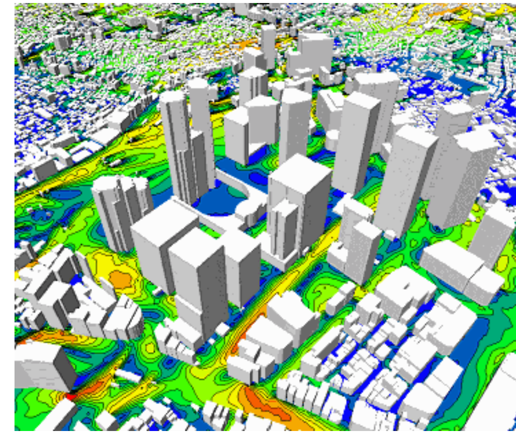
$$|V_i|\frac{\mathrm{d}u_i}{\mathrm{d}t} + \sum_j \int_{S_{ij}} (\mathbf{v}u - \mathscr{D}\nabla u)\cdot \mathbf{n}\,\mathrm{d}s = |V_i| s_i.$$

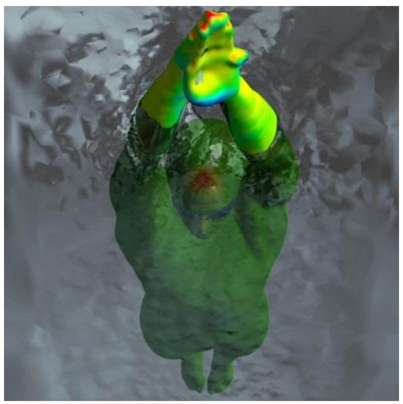
$$|V_i|\frac{\mathrm{d}u_i}{\mathrm{d}t} + \sum_j (c_{ij} + d_{ij}) u_j = |V_i| s_i.$$

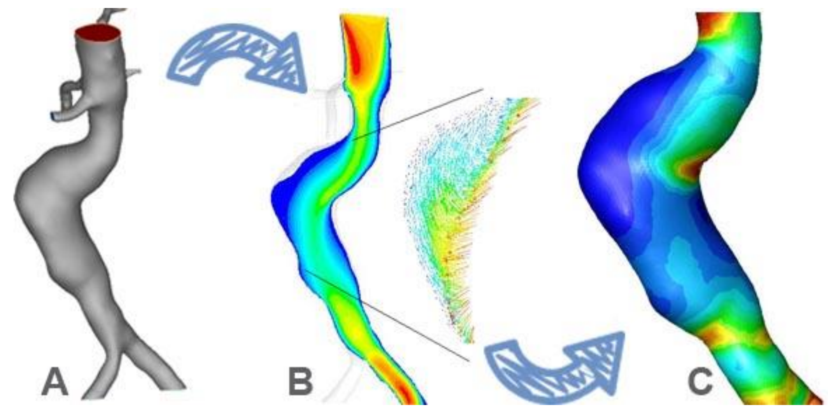
MAP2320

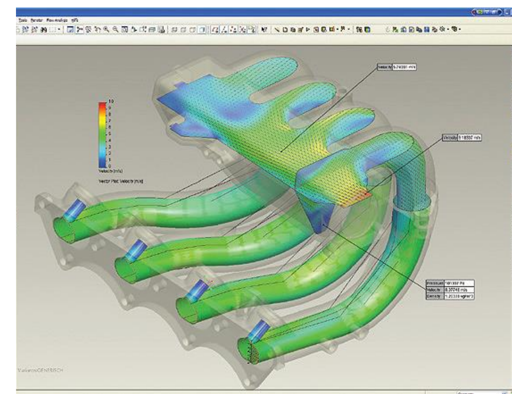

Pressure
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
[bar]
CFX

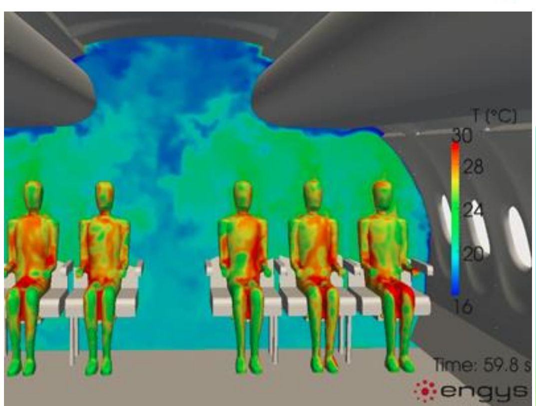


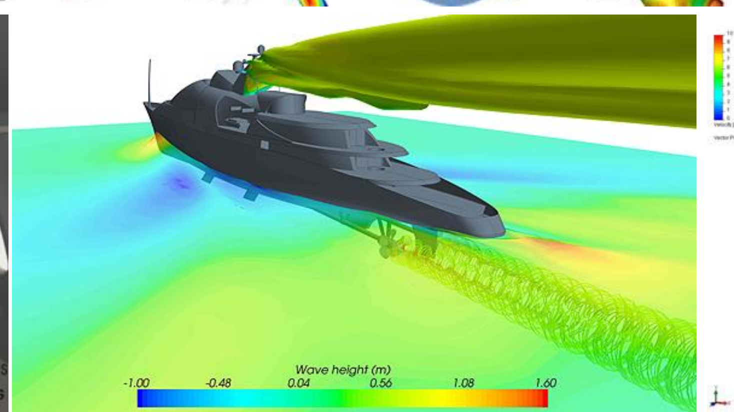


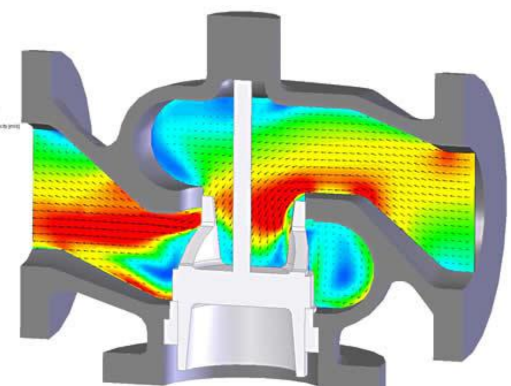


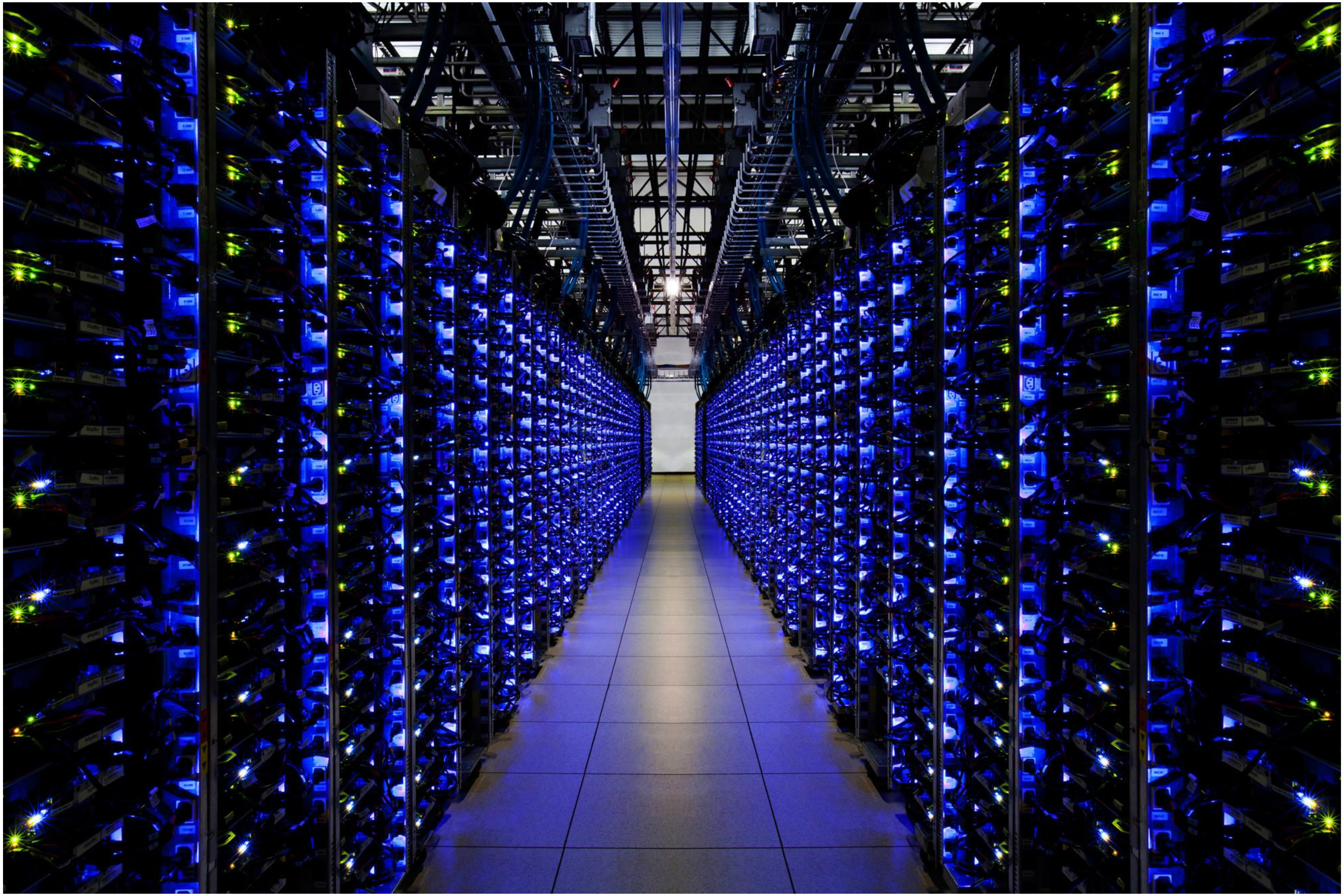

A
B
C



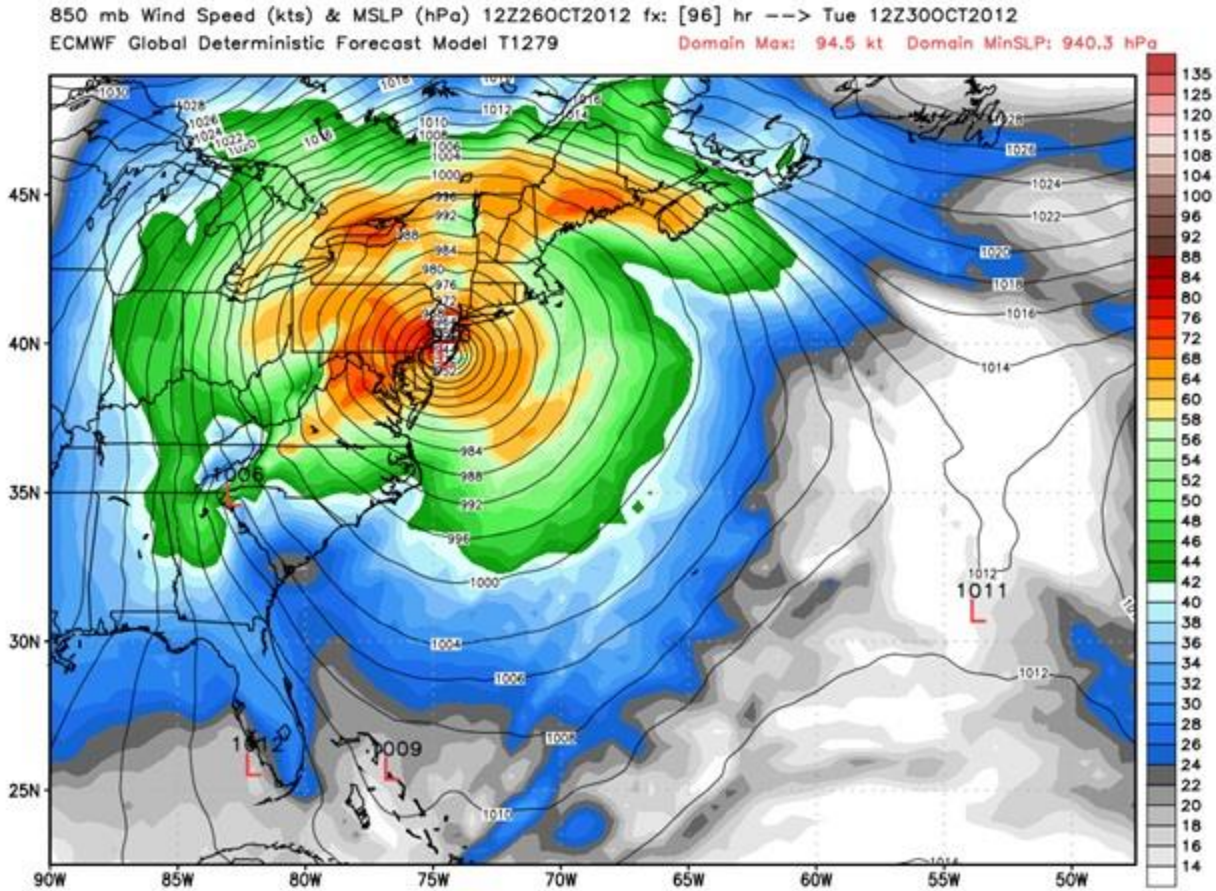

T (°C)
30
28
24
20
16
Time: 59.8 s
engys

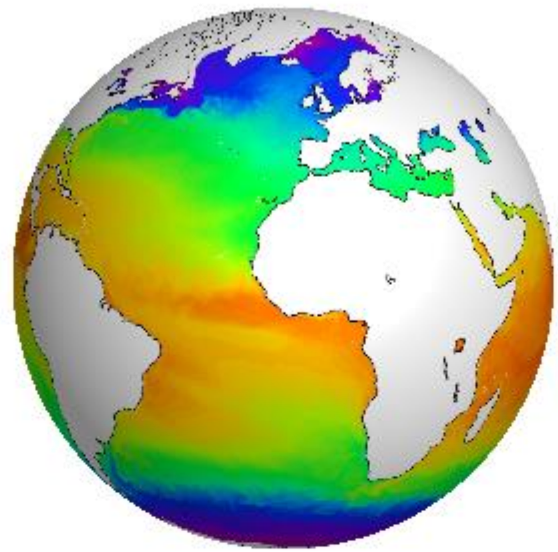

Wave height (m)
-1.00
-0.48
0.04
0.56
1.08
1.60

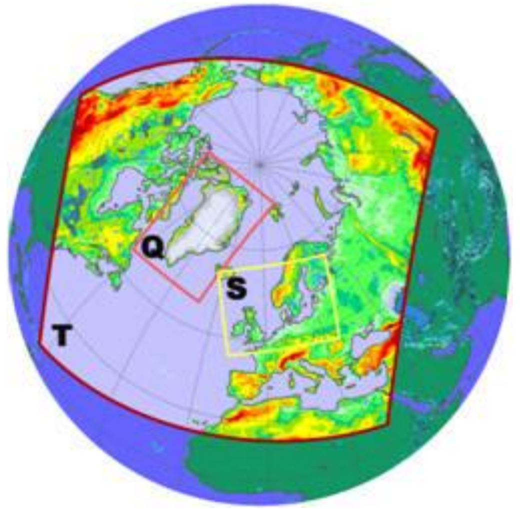




MAP2320







$$\frac{\partial u}{\partial t}+\dot{\sigma}\,\frac{\partial u}{\partial \sigma}+u\,\frac{\partial u}{\partial x}+v\,\frac{\partial u}{\partial y}-fv-\frac{uv}{r}\tan\phi+g\,\frac{\partial z}{\partial x}+c_p\theta\,\frac{\partial \pi}{\partial x}+F_x=0$$

$$\frac{\partial v}{\partial t}+\dot{\sigma}\,\frac{\partial v}{\partial \sigma}+u\,\frac{\partial v}{\partial x}+v\,\frac{\partial v}{\partial y}+fu+\frac{u^2}{r}\tan\phi+g\,\frac{\partial z}{\partial x}+c_p\theta\,\frac{\partial \pi}{\partial y}+F_y=0$$

$$\frac{\partial (gz)}{\partial \sigma}+c_p\theta\,\frac{\partial \pi}{\partial \sigma}=0,$$

$$\frac{\partial \theta}{\partial t}+\dot{\sigma}\,\frac{\partial \theta}{\partial \sigma}+u\,\frac{\partial \theta}{\partial x}+v\,\frac{\partial \theta}{\partial y}+H=0,$$

$$\frac{\partial p_\sigma}{\partial t}+\frac{\partial}{\partial \sigma}\,(\dot{\sigma}p_\sigma)+\frac{\partial}{\partial x}\,(up_\sigma)+\frac{\partial}{\partial y}\,(vp_\sigma)-\frac{vp_\sigma}{r}\tan\phi=0, \qquad \pi=\left(\frac{p}{P}\right)^{\kappa}.$$

MAP 2320 – MÉTODOS NUMÉRICOS EM EQUAÇÕES DIFERENCIAIS II

2º Semestre - 2020

Roteiro do curso

- Introdução
- Séries de Fourier
- **Método de Diferenças Finitas**
- Equação do calor transiente (parabólica)
- Equação de Poisson (elíptica)
- **Equação da onda (hiperbólica)**