MAP 2320 – MÉTODOS NUMÉRICOS EM EQUAÇÕES DIFERENCIAIS II

2º Semestre - 2020

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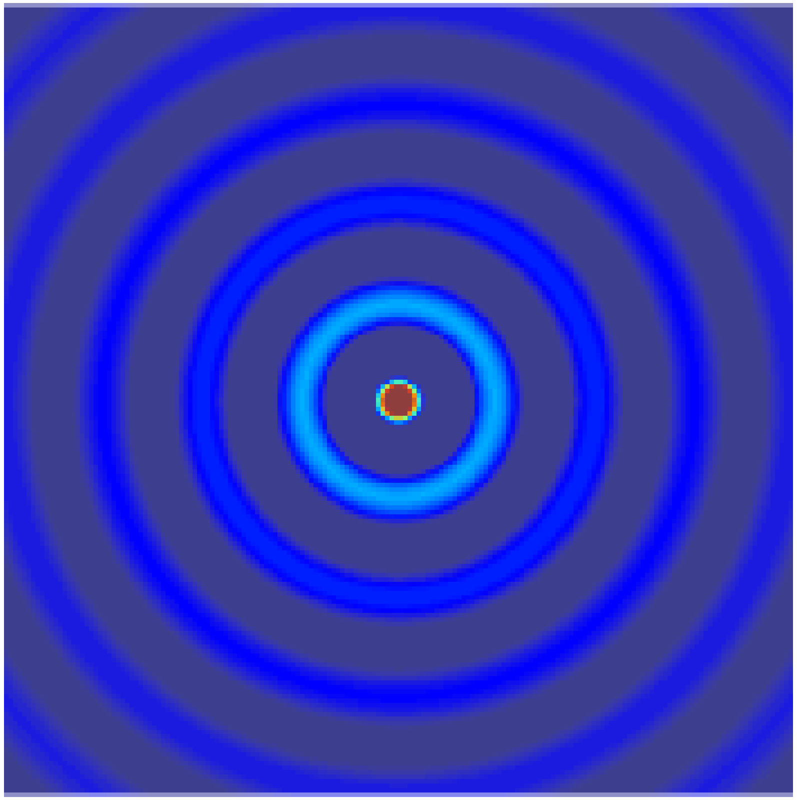
$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$



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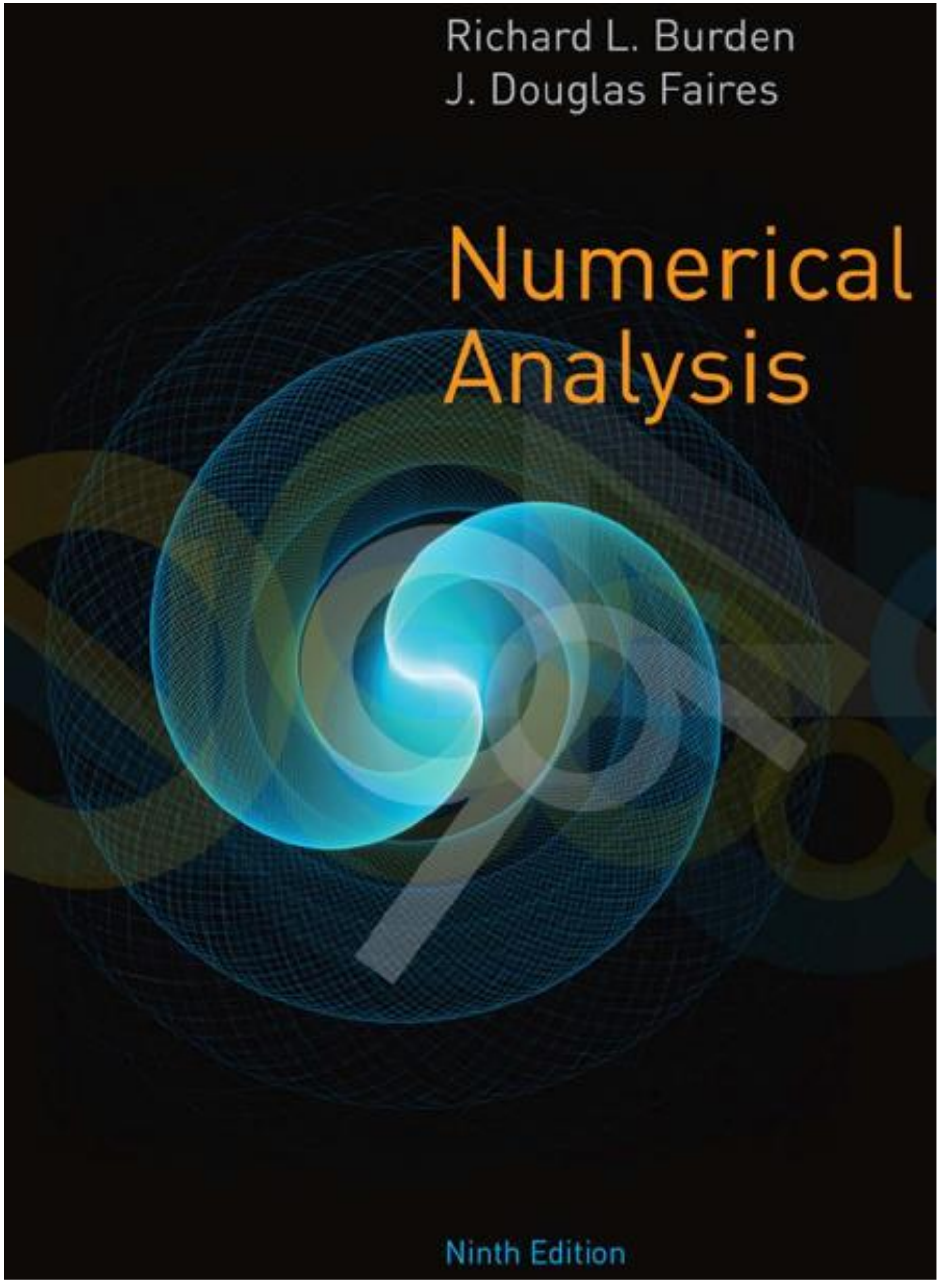
Richard L. Burden
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Numerical Analysis
Ninth Edition

Numerical Analysis

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12 Numerical Solutions to Partial Differential Equations 713

12.3 Hyperbolic Partial Differential Equations

In this section, we consider the numerical solution to the **wave equation**, an example of a *hyperbolic* partial differential equation. The wave equation is given by the differential equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad 0 < x < l, \quad t > 0, \tag{12.16}$$

subject to the conditions

extremidades fixas

$$u(0,t) = u(l,t) = 0, \quad \text{for} \quad t > 0,$$

$$u(x,0) = f(x), \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x), \quad \text{for} \quad 0 \le x \le l,$$

where α is a constant dependent on the physical conditions of the problem.

Select an integer $m > 0$ to define the x-axis grid points using $h = l/m$. In addition, select a time-step size $k > 0$. The mesh points (x_i, t_j) are defined by

$$x_i = ih \quad \text{and} \quad t_j = jk,$$

for each $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots$.

At any interior mesh point (x_i, t_j), the wave equation becomes

$$\frac{\partial^2 u}{\partial t^2}(x_i, t_j) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x_i, t_j) = 0. \tag{12.17}$$

$$h = \Delta x, \qquad k = \Delta t$$

$$D^2u(\bar{x}) = \frac{1}{h^2}[u(\bar{x}-h) - 2u(\bar{x}) + u(\bar{x}+h)]$$

The difference method is obtained using the centered-difference quotient for the second partial derivatives given by

$$\frac{\partial^2 u}{\partial t^2}(x_i, t_j) = \frac{u(x_i, t_{j+1}) - 2u(x_i, t_j) + u(x_i, t_{j-1})}{k^2} - \frac{k^2}{12}\frac{\partial^4 u}{\partial t^4}(x_i, \mu_j),$$

where $\mu_j \in (t_{j-1}, t_{j+1})$, and

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_j) = \frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{h^2} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i, t_j),$$

where $\xi_i \in (x_{i-1}, x_{i+1})$. Substituting these into Eq. (12.17) gives

$$\frac{u(x_i, t_{j+1}) - 2u(x_i, t_j) + u(x_i, t_{j-1})}{k^2} - \alpha^2 \frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{h^2}$$

$$= \frac{1}{12}\left[k^2 \frac{\partial^4 u}{\partial t^4}(x_i, \mu_j) - \alpha^2 h^2 \frac{\partial^4 u}{\partial x^4}(\xi_i, t_j)\right].$$

Neglecting the error term

$$\tau_{i,j} = \frac{1}{12}\left[k^2 \frac{\partial^4 u}{\partial t^4}(x_i, \mu_j) - \alpha^2 h^2 \frac{\partial^4 u}{\partial x^4}(\xi_i, t_j)\right], \qquad (12.18)$$

leads to the difference equation

$$\frac{w_{i,j+1} - 2w_{i,j} + w_{i,j-1}}{k^2} - \alpha^2 \frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{h^2} = 0.$$

Define $\lambda = \alpha k/h$. Then we can write the difference equation as

$$w_{i,j+1} - 2w_{i,j} + w_{i,j-1} - \lambda^2 w_{i+1,j} + 2\lambda^2 w_{i,j} - \lambda^2 w_{i-1,j} = 0$$

and solve for $w_{i,j+1}$, the most advanced time-step approximation, to obtain

$$w_{i,j+1} = 2(1 - \lambda^2)w_{i,j} + \lambda^2(w_{i+1,j} + w_{i-1,j}) - w_{i,j-1}. \tag{12.19}$$

This equation holds for each $i = 1, 2, \ldots, m-1$ and $j = 1, 2, \ldots$. The boundary conditions give

$$w_{0,j} = w_{m,j} = 0, \quad \text{for each } j = 1, 2, 3, \ldots, \tag{12.20}$$

and the initial condition implies that

$$w_{i,0} = f(x_i), \quad \text{for each } i = 1, 2, \ldots, m-1. \tag{12.21}$$

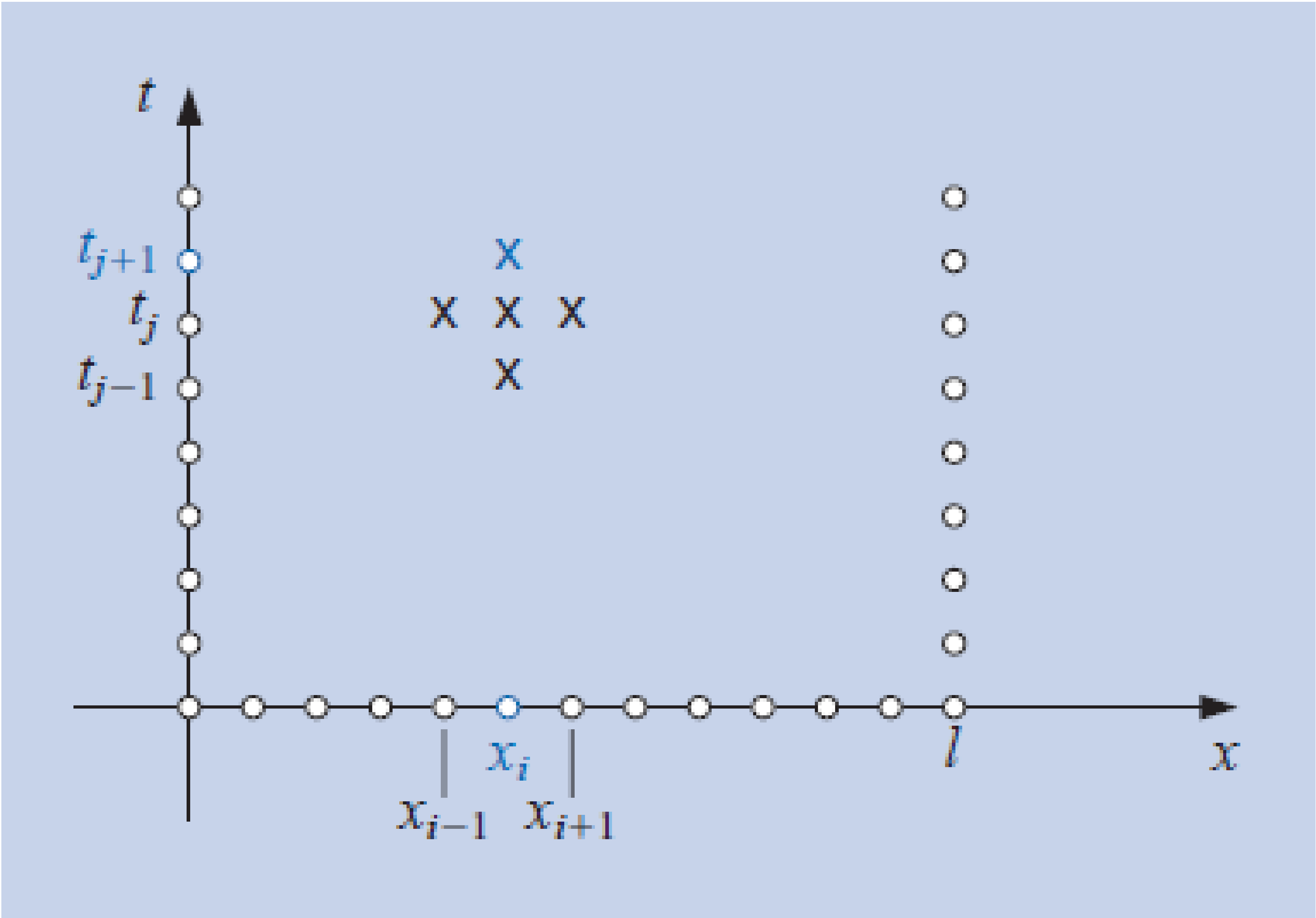
Writing this set of equations in matrix form gives

$$\begin{bmatrix} w_{1,j+1} \\ w_{2,j+1} \\ \vdots \\ w_{m-1,j+1} \end{bmatrix} = \begin{bmatrix} 2(1-\lambda^2) & \lambda^2 & 0 & \cdots & 0 \\ \lambda^2 & 2(1-\lambda^2) & \lambda^2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \lambda^2 \\ 0 & \cdots & 0 & \lambda^2 & 2(1-\lambda^2) \end{bmatrix} \begin{bmatrix} w_{1,j} \\ w_{2,j} \\ \vdots \\ w_{m-1,j} \end{bmatrix} - \begin{bmatrix} w_{1,j-1} \\ w_{2,j-1} \\ \vdots \\ w_{m-1,j-1} \end{bmatrix}. \tag{12.22}$$

Equations (12.18) and (12.19) imply that the $(j+1)$st time step requires values from the jth and $(j-1)$st time steps. (See Figure 12.12.) This produces a minor starting problem because values for $j = 0$ are given by Eq. (12.20), but values for $j = 1$, which are needed in Eq. (12.18) to compute $w_{i,2}$, must be obtained from the initial-velocity condition

$$\frac{\partial u}{\partial t}(x,0) = g(x), \quad 0 \le x \le l.$$

Figure 12.12



One approach is to replace $\partial u/\partial t$ by a forward-difference approximation,

$$\frac{\partial u}{\partial t}(x_i,0) = \frac{u(x_i,t_1) - u(x_i,0)}{k} - \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_i,\tilde{\mu}_i), \tag{12.23}$$

for some $\tilde{\mu}_i$ in $(0,t_1)$. Solving for $u(x_i,t_1)$ in the equation gives

$$\begin{aligned} u(x_i,t_1) &= u(x_i,0) + k\frac{\partial u}{\partial t}(x_i,0) + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}(x_i,\tilde{\mu}_i) \\ &= u(x_i,0) + kg(x_i) + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}(x_i,\tilde{\mu}_i). \end{aligned}$$

Deleting the truncation term gives the approximation,

$$w_{i,1} = w_{i,0} + kg(x_i), \quad \text{for each } i = 1,\ldots,m-1. \tag{12.24}$$

However, this approximation has truncation error of only $O(k)$ whereas the truncation error in Eq. (12.19) is $O(k^2)$.

To obtain a better approximation to $u(x_i, 0)$, expand $u(x_i, t_1)$ in a second Maclaurin polynomial in t. Then

$$u(x_i, t_1) = u(x_i, 0) + k\frac{\partial u}{\partial t}(x_i, 0) + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}(x_i, 0) + \frac{k^3}{6}\frac{\partial^3 u}{\partial t^3}(x_i, \hat{\mu}_i),$$

for some $\hat{\mu}_i$ in $(0, t_1)$. If f'' exists, then

$$\frac{\partial^2 u}{\partial t^2}(x_i, 0) = \alpha^2 \frac{\partial^2 u}{\partial x^2}(x_i, 0) = \alpha^2 \frac{d^2 f}{dx^2}(x_i) = \alpha^2 f''(x_i)$$

and

$$u(x_i, t_1) = u(x_i, 0) + kg(x_i) + \frac{\alpha^2 k^2}{2} f''(x_i) + \frac{k^3}{6}\frac{\partial^3 u}{\partial t^3}(x_i, \hat{\mu}_i).$$

This produces an approximation with error $O(k^3)$:

$$w_{i1} = w_{i0} + kg(x_i) + \frac{\alpha^2 k^2}{2} f''(x_i).$$

If $f \in C^4[0,1]$ but $f''(x_i)$ is not readily available, we can use the difference equation in Eq. (4.9) to write

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} - \frac{h^2}{12} f^{(4)}(\tilde{\xi}_i),$$

for some $\tilde{\xi}_i$ in (x_{i-1}, x_{i+1}). This implies that

$$u(x_i, t_1) = u(x_i, 0) + kg(x_i) + \frac{k^2\alpha^2}{2h^2}[f(x_{i+1}) - 2f(x_i) + f(x_{i-1})] + O(k^3 + h^2k^2).$$

Because $\lambda = k\alpha/h$, we can write this as

$$\begin{aligned} u(x_i, t_1) &= u(x_i, 0) + kg(x_i) + \frac{\lambda^2}{2}[f(x_{i+1}) - 2f(x_i) + f(x_{i-1})] + O(k^3 + h^2k^2) \\ &= (1 - \lambda^2) f(x_i) + \frac{\lambda^2}{2} f(x_{i+1}) + \frac{\lambda^2}{2} f(x_{i-1}) + kg(x_i) + O(k^3 + h^2k^2). \end{aligned}$$

Thus, the difference equation,

$$w_{i,1} = (1 - \lambda^2) f(x_i) + \frac{\lambda^2}{2} f(x_{i+1}) + \frac{\lambda^2}{2} f(x_{i-1}) + kg(x_i), \tag{12.25}$$

can be used to find $w_{i,1}$, for each $i = 1, 2, \ldots, m-1$. To determine subsequent approximates we use the system in (12.22).

Algorithm 12.4 uses Eq. (12.25) to approximate $w_{i,1}$, although Eq. (12.24) could also be used. It is assumed that there is an upper bound for the value of t to be used in the stopping technique, and that $k = T/N$, where N is also given.

ALGORITHM
12.4

Wave Equation Finite-Difference

To approximate the solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad 0 < x < l, \quad 0 < t < T,$$

subject to the boundary conditions

$$u(0,t) = u(l,t) = 0, \quad 0 < t < T,$$

and the initial conditions

$$u(x,0) = f(x), \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x), \quad \text{for} \quad 0 \le x \le l,$$

ALGORITHM 12.4

INPUT endpoint l; maximum time T; constant α; integers $m \geq 2$, $N \geq 2$.

OUTPUT approximations $w_{i,j}$ to $u(x_i, t_j)$ for each $i = 0, \ldots, m$ and $j = 0, \ldots, N$.

Step 1 Set $h = l/m$;
$k = T/N$;
$\lambda = k\alpha/h$.

Step 2 For $j = 1, \ldots, N$ set $w_{0,j} = 0$;
$w_{m,j} = 0$;

Step 3 Set $w_{0,0} = f(0)$;
$w_{m,0} = f(l)$.

Step 4 For $i = 1, \ldots, m-1$ (*Initialize for $t = 0$ and $t = k$.*)
set $w_{i,0} = f(ih)$;

$$w_{i,1} = (1-\lambda^2)f(ih) + \frac{\lambda^2}{2}[f((i+1)h) + f((i-1)h)] + kg(ih).$$

Step 5 For $j = 1, \ldots, N-1$ (*Perform matrix multiplication.*)
for $i = 1, \ldots, m-1$
set $w_{i,j+1} = 2(1-\lambda^2)w_{i,j} + \lambda^2(w_{i+1,j} + w_{i-1,j}) - w_{i,j-1}$.

Step 6 For $j = 0, \ldots, N$
set $t = jk$;
for $i = 0, \ldots, m$
set $x = ih$;
OUTPUT $(x, t, w_{i,j})$.

Step 7 STOP. (*The procedure is complete.*)

Example 1

Approximate the solution to the hyperbolic problem

$$\frac{\partial^2 u}{\partial t^2}(x,t) - 4\frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad 0 < x < 1, \quad 0 < t,$$

with boundary conditions

$$u(0,t) = u(1,t) = 0, \quad \text{for} \quad 0 < t,$$

and initial conditions

$$u(x,0) = \sin(\pi x), \quad 0 \le x \le 1, \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad 0 \le x \le 1,$$

using $h = 0.1$ and $k = 0.05$. Compare the results with the exact solution

$$u(x,t) = \sin \pi x \cos 2\pi t.$$

Example 1

Solution Choosing $h = 0.1$ and $k = 0.05$ gives $\lambda = 1$, $m = 10$, and $N = 20$. We will choose a maximum time $T = 1$ and apply the Finite-Difference Algorithm 12.4. This produces the approximations $w_{i,N}$ to $u(0.1i, 1)$ for $i = 0, 1, \ldots, 10$. These results are shown in Table 12.6 and are correct to the places given. ■

Console

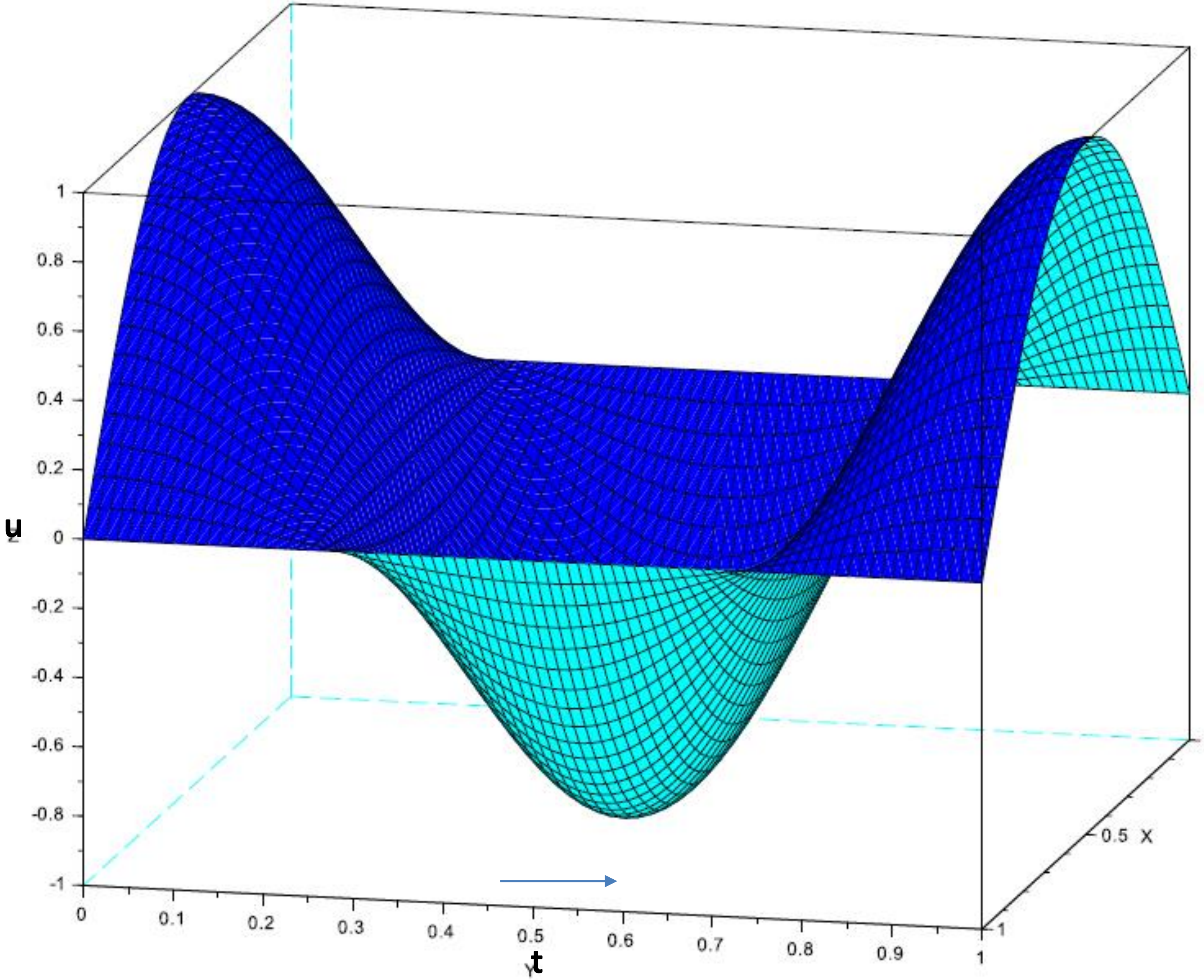
```
-->u(:,21)
 ans  =

     0.
   0.3090170
   0.5877853
   0.8090170
   0.9510565
     1.
   0.9510565
   0.8090170
   0.5877853
   0.3090170
     0.
```

Minha implementação em Scilab

Table 12.6

x_i	$w_{i,20}$
0.0	0.0000000000
0.1	0.3090169944
0.2	0.5877852523
0.3	0.8090169944
0.4	0.9510565163
0.5	1.0000000000
0.6	0.9510565163
0.7	0.8090169944
0.8	0.5877852523
0.9	0.3090169944
1.0	0.0000000000



Minha implementação em Scilab, m = 40

The results of the example were very accurate, more so than the truncation error $O(k^2 + h^2)$ would lead us to believe. This is because the true solution to the equation is infinitely differentiable. When this is the case, Taylor series gives

$$\frac{u(x_{i+1},t_j) - 2u(x_i,t_j) + u(x_{i-1},t_j)}{h^2}$$
$$= \frac{\partial^2 u}{\partial x^2}(x_i,t_j) + 2\left[\frac{h^2}{4!}\frac{\partial^4 u}{\partial x^4}(x_i,t_j) + \frac{h^4}{6!}\frac{\partial^6 u}{\partial x^6}(x_i,t_j) + \cdots\right]$$

and

$$\frac{u(x_i,t_{j+1}) - 2u(x_i,t_j) + u(x_i,t_{j-1})}{k^2}$$
$$= \frac{\partial^2 u}{\partial t^2}(x_i,t_j) + 2\left[\frac{k^2}{4!}\frac{\partial^4 u}{\partial t^4}(x_i,t_j) + \frac{h^4}{6!}\frac{\partial^6 u}{\partial t^6}(x_i,t_j) + \cdots\right].$$

Since $u(x,t)$ satisfies the partial differential equation,

$$\frac{u(x_i,t_{j+1}) - 2u(x_i,t_j) + u(x_i,t_{j-1})}{k^2} - \alpha^2\frac{u(x_{i+1},t_j) - 2u(x_i,t_j) + u(x_{i-1},t_j)}{h^2}$$
$$= 2\left[\frac{1}{4!}\left(k^2\frac{\partial^4 u}{\partial t^4}(x_i,t_j) - \alpha^2 h^2\frac{\partial^4 u}{\partial x^4}(x_i,t_j)\right)\right.$$
$$\left. + \frac{1}{6!}\left(k^4\frac{\partial^6 u}{\partial t^6}(x_i,t_j) - \alpha^2 h^4\frac{\partial^6 u}{\partial x^6}(x_i,t_j)\right) + \cdots\right]. \quad (12.26)$$

However, differentiating the wave equation gives

$$k^2\frac{\partial^4 u}{\partial t^4}(x_i, t_j) = k^2\frac{\partial^2}{\partial t^2}\left[\alpha^2\frac{\partial^2 u}{\partial x^2}(x_i, t_j)\right] = \alpha^2 k^2\frac{\partial^2}{\partial x^2}\left[\frac{\partial^2 u}{\partial t^2}(x_i, t_j)\right]$$
$$= \alpha^2 k^2\frac{\partial^2}{\partial x^2}\left[\alpha^2\frac{\partial^2 u}{\partial x^2}(x_i, t_j)\right] = \alpha^4 k^2\frac{\partial^4 u}{\partial x^4}(x_i, t_j),$$

and we see that since $\lambda^2 = (\alpha^2 k^2/h^2) = 1$, we have

$$\frac{1}{4!}\left[k^2\frac{\partial^4 u}{\partial t^4}(x_i, t_j) - \alpha^2 h^2\frac{\partial^4 u}{\partial x^4}(x_i, t_j)\right] = \frac{\alpha^2}{4!}[\alpha^2 k^2 - h^2]\frac{\partial^4 u}{\partial x^4}(x_i, t_j) = 0.$$

Continuing in this manner, all the terms on the right-hand side of (12.26) are 0, implying that the local truncation error is 0. The only errors in Example 1 are those due to the approximation of $w_{i,1}$ and to round-off.

As in the case of the Forward-Difference method for the heat equation, the Explicit Finite-Difference method for the wave equation has stability problems. In fact, it is necessary that $\lambda = \alpha k/h \leq 1$ for the method to be stable. (See [IK], p. 489.) The explicit method given in Algorithm 12.4, with $\lambda \leq 1$, is $O(h^2 + k^2)$ convergent if f and g are sufficiently differentiable. For verification of this, see [IK], p. 491.

Although we will not discuss them, there are implicit methods that are unconditionally stable. A discussion of these methods can be found in [Am], p. 199, [Mi], or [Sm,G].

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Roteiro do curso

- Introdução
- Séries de Fourier
- **Método de Diferenças Finitas**
- Equação do calor transiente (parabólica)
- Equação de Poisson (elíptica)
- **Equação da onda (hiperbólica)**