Electric and magnetic dipole moments of a moving current loop which is neutral in its rest frame

(Dated:)

Let us consider a neutral conducting ring with radius R_0 at rest in an inertial frame S' (with usual Lorentzian coordinates $\{(ct', \vec{x}')\}$), carrying a stationary electric current I_0. This system can be characterized by the current density:

$$\vec{j}'(\vec{x}') = I_0 \, \delta(r' - R_0)\delta(z') \, \hat{e}'_\theta, \tag{1}$$

where $\vec{x}' = (x', y', z')$, $r' = \sqrt{x'^2 + y'^2}$, and $\hat{e}'_\theta = (-y'/r', x'/r', 0)$. Therefore, the components of the 4-current density in S' are given by:

$$j'^\mu = \frac{I_0}{R_0} \, (0, -y', x', 0) \, \delta(r' - R_0)\delta(z'). \tag{2}$$

Performing a Lorentz transformation to another inertial frame S (with Lorentzian coordinates $\{(ct, \vec{x})\}$) moving with velocity $-\vec{V} = (-V, 0, 0)$ with respect to S', we find the components of the same 4-current density in S to be:

$$j^\mu \;=\; \frac{\gamma I_0}{R_0} \, (-Vy/c, -y, x - Vt, 0) \, \delta(\sqrt{\gamma^2 (x - Vt)^2 + y^2} - R_0)\delta(z), \tag{3}$$

where $\gamma := 1/\sqrt{1 - V^2/c^2}$ is the usual Lorentz factor.

We can simplify the argument of the Dirac-delta "function" by defining variables r and θ through

$$x - Vt \;=\; r\cos\theta, \tag{4}$$

$$y \;=\; r\sin\theta, \tag{5}$$

so that the center of the now moving ring is always at $r = 0$. Using some properties of the Dirac-delta "function", we have:

$$\begin{aligned}
\delta(\sqrt{\gamma^2 (x - Vt)^2 + y^2} - R_0) &= 2R_0 \, \delta(\gamma^2 (x - Vt)^2 + y^2 - R_0^2) \\
&= \frac{2R_0}{\gamma^2} \, \delta((x - Vt)^2 + y^2/\gamma^2 - R_0^2/\gamma^2) \\
&= \frac{2R_0}{\gamma^2} \, \delta(r^2 - r^2 \sin\theta^2 V^2/c^2 - R_0^2/\gamma^2) \\
&= \frac{2R_0}{\gamma^2 (1 - V^2 \sin\theta^2/c^2)} \, \delta(r^2 - R_0^2/(\gamma^2 (1 - V^2 \sin\theta^2/c^2))) \\
&= \frac{1}{\gamma\sqrt{1 - V^2 \sin\theta^2/c^2}} \, \delta(r - R(\theta)),
\end{aligned} \tag{6}$$

where

$$R(\theta) := \frac{R_0}{\gamma\sqrt{1 - V^2 \sin\theta^2/c^2}} \tag{7}$$

is the angle-dependent radius of the (elliptical) ring. Note that the ring is Lorentz-contracted in the direction of its motion. Substituting Eqs. (4-7) into Eq. (3), we have:

$$j^\mu \;=\; \frac{I_0}{\gamma (1 - V^2 \sin\theta^2/c^2)} \, (-V\sin\theta/c, -\sin\theta, \cos\theta, 0) \, \delta(r - R(\theta))\delta(z). \tag{8}$$

This 4-current density represents a charged elliptical ring with electric charge density ρ and conducting a stationary current I (the part of the total current which is not due to motion of the charge density ρ) given respectively by

$$\rho \;=\; \frac{-I_0 V \sin\theta/c^2}{\gamma (1 - V^2 \sin\theta^2/c^2)} \, \delta(r - R(\theta))\delta(z), \tag{9}$$

$$I \;=\; \frac{I_0}{\gamma}. \tag{10}$$

From Eq. (9) we can obtain the electric dipole moment through

$$
\begin{aligned}
\vec{d} &= \int_\Sigma d^3x\, \rho\, \vec{x} \\
&= -\int_{\mathbb{R}} dz \int_{\mathbb{R}_+} dr \int_{[0,2\pi)} d\theta\, \frac{rI_0 V \sin\theta/c^2}{\gamma(1 - V^2 \sin\theta^2/c^2)}\, (r\cos\theta + Vt, r\sin\theta, z)\, \delta(r - R(\theta))\delta(z) \\
&= -\frac{I_0 V}{c^2\gamma} \int_{[0,2\pi)} d\theta\, \frac{R(\theta)\sin\theta}{(1 - V^2\sin\theta^2/c^2)}\, (R(\theta)\cos\theta + Vt, R(\theta)\sin\theta, 0) \\
&= -\frac{I_0 V R_0^2\, \hat{\mathrm{e}}_y}{c^2\gamma^3} \int_{[0,2\pi)} d\theta\, \frac{\sin\theta^2}{(1 - V^2\sin\theta^2/c^2)^2} \\
&= -\frac{I_0 V \pi R_0^2\, \hat{\mathrm{e}}_y}{c^2} = \frac{\vec{V} \times \vec{m}_0}{c^2},
\end{aligned} \tag{11}
$$

where $\vec{m}_0 = I_0 \pi R_0^2 \hat{\mathrm{e}}'_z$ is the *proper* magnetic moment of the ring (i.e., its magnetic moment in the rest frame S').

Now, let us calculate the magnetic moment of the ring in the frame S:

$$
\begin{aligned}
\vec{m} &= \frac{1}{2}\int_\Sigma d^3x\, (\vec{x} \times \vec{j}) \\
&= \int_{\mathbb{R}} dz \int_{\mathbb{R}_+} dr \int_{[0,2\pi)} d\theta\, \frac{rI_0}{2\gamma(1 - V^2\sin\theta^2/c^2)}\, (-z\cos\theta, -z\sin\theta, r + Vt\cos\theta)\, \delta(r - R(\theta))\delta(z) \\
&= \frac{I_0\, \hat{\mathrm{e}}_z}{2\gamma} \int_{[0,2\pi)} d\theta\, \frac{R(\theta)^2}{(1 - V^2\sin\theta^2/c^2)} \\
&= \frac{I_0 R_0^2\, \hat{\mathrm{e}}_z}{2\gamma^3} \int_{[0,2\pi)} d\theta\, \frac{1}{(1 - V^2\sin\theta^2/c^2)^2} \\
&= I_0 \pi R_0^2 \left(1 - \frac{V^2}{2c^2}\right) \hat{\mathrm{e}}_z \\
&= \frac{\vec{m}_0}{\gamma^2} + \frac{\vec{d} \times \vec{V}}{2}.
\end{aligned} \tag{12}
$$

This result is quite easy to understand. The first term in the right-hand-side of Eq. (12) comes from the fact that in the frame S the ring has its area Lorentz-contracted, (Area) = (Area$_0$)/γ, and is conducting a stationary current $I = I_0/\gamma$ [see Eq. (10)]: $I(\text{Area}) = I_0(\text{Area}_0)/\gamma^2 = m_0/\gamma^2$. The second term is simply due to the fact that the moving ring is polarized in S; that is the usual magnetic moment of a moving electric dipole.