Hidden momentum of (possibly open) systems

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COVARIANT NEWTONIAN MECHANICS

In Newtonian mechanics, the distinct natures of mass and energy allow the definition of different kinds of "splitting" between a system of interest and the rest of the universe. Let us consider *closed*, *force-free*, and *isolated* systems. A system is said to be closed if it does not exchange *mass* with the rest of the universe; if it does, it is said to be *open*. A systems is said to be force-free if there are no *external* forces acting on it. The term "external" here has nothing to do with the location where the force is applied but rather refers to the "agent" responsible for the force: the rest of the universe. Finally, a system is said to be isolated if it is closed and force-free. It is easy to see that isolated systems also do not exchange *energy* with the rest of the universe (although the converse is not necessarily true).

A system is completely characterized (as far as its mechanical properties are concerned) by its mass-density distribution ρ, its momentum-density distribution π^i $(i = 1, 2, 3)$, and its stress tensor field $\widetilde{T}^{ij}$ $(i, j = 1, 2, 3)$. These quantities are *defined* such that for *any* (measurable) spatial region $\mathcal{V}$ and *any* (smooth, orientable) surface $\mathcal{S}$, the mass $M_{\mathcal{V}}$ and the momentum $P^i_{\mathcal{V}}$ of the system contained in $\mathcal{V}$ and the *internal* force $F^i_{int,\mathcal{S}}$ the system on one side of $\mathcal{S}$ exerts (across $\mathcal{S}$) on the other side are given (in a Cartesian-coordinate basis), respectively, by

$$M_{\mathcal{V}} = \int_{\mathcal{V}} d\Sigma\, \rho, \tag{1}$$

$$P^i_{\mathcal{V}} = \int_{\mathcal{V}} d\Sigma\, \pi^i, \tag{2}$$

$$F^i_{int,\mathcal{S}} = -\int_{\mathcal{S}} dS\, \widetilde{T}^{ij} n_j, \tag{3}$$

where $d\Sigma$ is the volume element in $\mathcal{V}$, dS is the area element on $\mathcal{S}$, and n_i is the unit vector orthogonal to $\mathcal{S}$ oriented towards the part of the system which *exerts* the force $F^i_{int,\mathcal{S}}$. Newton's second law can be stated for an arbitrary system as

$$\begin{pmatrix} \text{Rate of change of} \\ \text{total momentum of} \\ \text{the system} \end{pmatrix} = \begin{pmatrix} \text{Total force applied} \\ \text{on the system} \end{pmatrix} - \begin{pmatrix} \text{Flow of momentum} \\ \text{leaving the system} \end{pmatrix}, \tag{4}$$

which, when applied to the part of a system contained in a region $\mathcal{V}$ (whose boundary $\partial\mathcal{V}$ possibly changes with time, with u^i being its velocity field), leads to

$$\frac{dP^i_{\mathcal{V}}}{dt} = F^i_{\mathcal{V}} - \int_{\partial\mathcal{V}} dS\, \widetilde{T}^{ij} n_j - \int_{\partial\mathcal{V}} dS\, \pi^i (v^j - u^j) n_j, \tag{5}$$

or, in local terms [1],

$$f^i = \frac{\partial \pi^i}{\partial t} + \frac{\partial}{\partial x^j}(v^j \pi^i + \widetilde{T}^{ij}). \tag{6}$$

Here, f^i is the force density (which gives the "bulk" force $F^i_{\mathcal{V}}$ when integrated in $\mathcal{V}$) acting on the system of interest and v^i is the local velocity associated with the momentum flow. It is important to point out that f^i includes generic external forces as well as *non-local* internal forces. (An internal force is considered to be local if, and only if, for any volume it contributes only through boundary terms; i.e., its net result vanishes in any open set. Newton's gravitational *self-force* of an extended system, for instance, is a non-local internal force.)

One interesting (although simple) fact which is rarely stressed in textbooks is that mass conservation can be deduced as a consequence of the covariance of Newton's second law under Galilean "boosts." In fact, using the usual transformation properties $\rho \mapsto \rho' = \rho$, $\pi^i \mapsto \pi'^i = \pi^i - \rho V^i$, $\widetilde{T}^{ij} \mapsto \widetilde{T}'^{ij} = \widetilde{T}^{ij}$, and $f^i \mapsto f'^i = f^i$ under the transformation $x^i \mapsto x'^i = x^i - V^i t$, with V^i constant [2], we have:

$$
\begin{aligned}
f'^i &= \frac{\partial \pi'^i}{\partial t} + \frac{\partial}{\partial x'^j}(v'^j \pi'^i + \widetilde{T}'^{ij}) \Leftrightarrow \\
f^i &= \left(\frac{\partial}{\partial t} + V^j \frac{\partial}{\partial x^j}\right)(\pi^i - \rho V^i) + \frac{\partial}{\partial x^j}[(v^j - V^j)(\pi^i - \rho V^i) + \widetilde{T}^{ij}] \Leftrightarrow \\
f^i &= \frac{\partial \pi^i}{\partial t} + \frac{\partial}{\partial x^j}(v^j \pi^i + \widetilde{T}^{ij}) - V^i \left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v^j)}{\partial x^j}\right] \Leftrightarrow \\
0 &= \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v^j)}{\partial x^j},
\end{aligned}
\tag{7}
$$

which is the continuity equation expressing local mass conservation. Moreover, from the transformation property $\pi^i \mapsto \pi'^i = \pi^i - \rho V^i$ we can infer that v^i (the velocity associated with momentum flow) is given by $v^i = \pi^i/\rho$.

- **Exercise:** Illustrate, using a simple thought experiment, that violation of mass conservation would lead to violation of covariance of Newton's second law. (Suggestion: Think of an isolated system in which mass is not conserved and analyze it from different inertial perspectives.)

It is a common misconception that the four-dimensional view of the world is only useful when considering Einsteinian relativistic physics. This prejudice is probably fostered by the fact that in Newtonian mechanics *time* is an independent, absolute parameter, unaffected by the state of motion of the observer. However, this naive consideration neglects the fact that the *symmetry transformations* of Newtonian physics, the Galilean transformations, involve time in a nontrivial way (as in the Galilean "boosts"). In fact, the transformation properties of ρ, π^i, and $\widetilde{T}^{ij}$ mentioned above can be easily summarized by defining a stress-mass-momentum tensor field $T^{\mu\nu}$ $(\mu, \nu = 0, 1, 2, 3)$ on a four-dimensional spacetime $\mathcal{G}$ (topologically, $\mathbb{R}^4$), whose components in inertial Cartesian coordinates $\{(x^\mu)\}$ are given by

$$
T^{\mu\nu} = \begin{pmatrix} \rho & \pi^j \\ \pi^i & \widetilde{T}^{ij} + \pi^i v^j \end{pmatrix}, \tag{8}
$$

with the coordinate x^0 being the absolute Newtonian time t (determined up to an additive constant). One can easily check that such an object indeed transforms like a true tensor under coordinate transformations induced by the full ten-parameter Galilean group (space and time translations, spatial rotations, and Galilean "boosts"). This fact reinforces the convenience of the four-dimensional view. In terms of the stress-energy-momentum tensor, Newton's second law *and* mass-conservation law combine in a single tensorial equation:

$$
f^\mu = \partial_\nu T^{\mu\nu}, \tag{9}
$$

where $f^\mu = (0, f^i)$ (in inertial Cartesian coordinates) is also a true vector.

RELATIVISTIC MECHANICS AND HIDDEN MOMENTUM

In relativity, mass and energy are intertwined. This leads to the fact that one cannot distinguish, from a purely dynamical point of view, between exchange of momentum due to exchange of forces or exchange of matter carrying momentum. The object which characterizes an arbitrary system is its stress-energy-momentum tensor, which in an inertial Cartesian coordinate system has components

$$
T^{\mu\nu} = \begin{pmatrix} \rho c^2 & \pi^j c \\ \pi^i c & T^{ij} \end{pmatrix}. \tag{10}
$$

Notice that now it is not possible (or convenient) to split the space-space components of $T^{\mu\nu}$ in terms of exchange of forces of flows of momenta, for the reason explained just above.

Let $\mathcal{A}$ be a (possibly open) system characterized by the energy-momentum tensor $T^{\mu\nu}_{\mathcal{A}}$. Adopting an arbitrary inertial frame (with usual Lorentzian coordinates $\{(t, \vec{x})\}$), its center of mass/energy is given by

$$
\vec{X}_{\rm cme} := \frac{1}{Mc^2} \int_{\mathcal{V}} d^3x \, T^{00}_{\mathcal{A}} \, \vec{x}, \tag{11}
$$

where $M = \int_{\mathcal{V}} d^3x\, T_{\mathcal{A}}^{00}/c^2$ is the total (relativistic) mass of the system and $\mathcal{V}$ is the spatial region (possibly changing with time) where the system is defined. Multiplying Eq. (11) by M and taking the time derivative, we get:

$$\begin{aligned} \frac{dM}{dt}\vec{X}_{\rm cme} + M\frac{d\vec{X}_{\rm cme}}{dt} &= \int_{\mathcal{V}} d^3x\, \partial_0 T_{\mathcal{A}}^{00}\, \vec{x}/c + \int_{\partial\mathcal{V}} (d\vec{S}\cdot\vec{u})\, T_{\mathcal{A}}^{00}\, \vec{x}/c^2 \\ \Leftrightarrow M\frac{d\vec{X}_{\rm cme}}{dt} &= \int_{\mathcal{V}} d^3x\, \partial_0 T_{\mathcal{A}}^{00}\, (\vec{x}-\vec{X}_{\rm cme})/c + \int_{\partial\mathcal{V}} (d\vec{S}\cdot\vec{u})\, T_{\mathcal{A}}^{00}\, (\vec{x}-\vec{X}_{\rm cme})/c^2, \end{aligned} \tag{12}$$

where $\vec{u}$ is the velocity field *of the boundary* $\partial\mathcal{V}$ of the region $\mathcal{V}$.

Now, we leave it open to the possibility that the energy-momentum tensor of system $\mathcal{A}$ may fail to be conserved either by interaction with another system (i.e., $\mathcal{A}$ is possibly not *isolated*) or by exchanging mass/energy/momentum with the region "outside" $\mathcal{V}$ (i.e., system $\mathcal{A}$ is possibly not *closed* either). The fact that system $\mathcal{A}$ may interact with another system means that everywhere *in* $\mathcal{V}$ we have $\partial_\mu T_{\mathcal{A}}^{\mu\nu} = f^\nu$, where f^μ is the 4-force density acting on $\mathcal{A}$. Therefore, *in* $\mathcal{V}$, $\partial_0 T_{\mathcal{A}}^{00} = f^0 - \partial_j T_{\mathcal{A}}^{j0}$ and we have:

$$\begin{aligned} M\frac{d\vec{X}_{\rm cme}}{dt} - \int_{\mathcal{V}} d^3x\, f^0\, (\vec{x}-\vec{X}_{\rm cme})/c &= -\int_{\mathcal{V}} d^3x\, \partial_j T_{\mathcal{A}}^{j0}\, (\vec{x}-\vec{X}_{\rm cme})/c + \int_{\partial\mathcal{V}} (d\vec{S}\cdot\vec{u})\, T_{\mathcal{A}}^{00}\, (\vec{x}-\vec{X}_{\rm cme})/c^2 \\ &= -\int_{\mathcal{V}} d^3x\, \partial_j [T_{\mathcal{A}}^{j0}\, (\vec{x}-\vec{X}_{\rm cme})]/c + \int_{\mathcal{V}} d^3x\, T_{\mathcal{A}}^{j0}\, \partial_j\vec{x}/c \\ &\quad + \int_{\partial\mathcal{V}} (d\vec{S}\cdot\vec{u})\, T_{\mathcal{A}}^{00}\, (\vec{x}-\vec{X}_{\rm cme})/c^2 \\ &= -\int_{\partial\mathcal{V}} dS_j\, (T_{\mathcal{A}}^{j0} - T_{\mathcal{A}}^{00}\, u^j/c)\, (\vec{x}-\vec{X}_{\rm cme})/c + \int_{\mathcal{V}} d^3x\, \vec{\pi} \\ &= \vec{P} - \int_{\partial\mathcal{V}} d\vec{S}\cdot(\vec{\pi}-\rho\vec{u})\, (\vec{x}-\vec{X}_{\rm cme}), \end{aligned} \tag{13}$$

where we have used that $\rho = T_{\mathcal{A}}^{00}/c^2$ is the (relativistic) mass density and $(\vec{\pi})^i = T_{\mathcal{A}}^{i0}/c$ is the momentum density of the system. (Recall that $\vec{u}$ is the velocity associated with the possible movement of the boundary $\partial\mathcal{V}$; it has nothing to do with $T_{\mathcal{A}}^{\mu\nu}$.)

Rewriting the equation above for the total momentum of system $\mathcal{A}$,

$$\vec{P} = M\frac{d\vec{X}_{\rm cme}}{dt} + \int_{\partial\mathcal{V}} d\vec{S}\cdot(\vec{\pi}-\rho\vec{u})\, (\vec{x}-\vec{X}_{\rm cme}) - \int_{\mathcal{V}} d^3x\, f^0\, (\vec{x}-\vec{X}_{\rm cme})/c, \tag{14}$$

it is interesting to note that the last term is the *only* one which is purely relativistic. This means that in Newtonian mechanics, the total momentum of an arbitrary system can be completely assessed by simply keeping track of a very simple kinematic parameter, its center of mass, and of the asymmetric exchange of mass/momentum at its boundary (in case of open systems). In relativistic mechanics, on the other hand, part of the total momentum cannot be assessed by the motion of the center of mass nor by information at the boundary. This might motivate us to define the "hidden" part of the total momentum of a system as $\vec{P}_{hid} := \vec{P} - M d\vec{X}_{\rm cme}/dt - \int_{\partial\mathcal{V}} d\vec{S}\cdot(\vec{\pi}-\rho\vec{u})\, (\vec{x}-\vec{X}_{\rm cme})$, which would then be calculated by:

$$\vec{P}_{hid} = -\int_{\mathcal{V}} d^3x\, f^0\, (\vec{x}-\vec{X}_{\rm cme})/c. \tag{15}$$

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[1] Here we use that for any function f,

$$\frac{d}{dt}\int_{\mathcal{V}} d^3x\, f = \int_{\mathcal{V}} d^3x\, \frac{\partial f}{\partial t} + \int_{\partial\mathcal{V}} (d\vec{S}\cdot\vec{u})\, f\,,$$

where $\vec{u}$ is the velocity field *of the boundary* $\partial\mathcal{V}$.

[2] These transformations follow from the usual assumption that in Newtonian physics mass and forces are boost-invariants while momentum transforms like $P^i \mapsto P'^i = P^i - MV^i$.